An Omega Theorem on the General Asymmetric Divisor Problem*

Manfred Kühleitner

Institut für Mathematik, Universität für Bodenkultur, Gregor Mendel Strasse 33, A-1180 Vienna, Austria

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This paper deals with a lower estimate for the general asymmetric divisor problem. Continuing on a paper of Nowak, a lower bound for the error term in the asymptotic formula for the corresponding Dirichlet summatory function is established. © 1997 Academic Press

1. INTRODUCTION

For a given integer $p \geq 2$ and fixed natural numbers $a = a_1 \leq a_2 \leq \cdots \leq a_p$, consider the general asymmetric divisor function

$$d(a_1, \ldots, a_p; n) = d(a; n) = \# \left\{ (m_1, \ldots, m_p) \in \mathbb{N}^p : m_1^{a_1} \cdots m_p^{a_p} = n \right\} (n \in \mathbb{N}).$$

For a large real variable $x$, we consider the remainder term $E(a; x)$ in the asymptotic formula

$$D(a; x) = \sum_{n \leq x} d(a; n) = H(a; x) + E(a; x),$$

(1)

where

$$H(a; x) = \sum_{\alpha = 0, 1/a_1, \ldots, 1/a_p} \text{Res}_{s = \alpha} \left( \prod_{j=1}^{p} \zeta(a_j) \frac{x^s}{s} \right).$$

A survey on results concerning upper estimates for the remainder function $E(a; x)$ is given in the textbook of Krätzel [10], supplemented by the recent papers of Menzer [13] and Krätzel [11].

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Lower estimates, for the special case $p = 2$ were established by Krätzel [9], Schierwagen [17, 18], and Hafner [7]. Concerning lower estimates for the general case $p \geq 2$, a classic theorem of Landau [12] contains the estimate

$$E(a; x) = \Omega_\pm(x^{(p-1)/2(a_1 + \cdots + a_p)}).$$

This result was improved by Ivić [8] and recently by Nowak [16], who showed that

$$E(a; x) = \Omega_\pm(x^{\theta}(\log x)^{\theta(p-1)}(\log \log x)^{p-1}),$$

where $\theta = \pm$ in the case $p \geq 4$, $\theta = +$ in the case $p = 2, 3$ and

$$\theta = \frac{p-1}{2(a_1 + \cdots + a_p)},$$

(2)

2. SUBJECT AND RESULT OF THE PAPER

Continuing on the paper of Nowak [16], we improve the $\Omega$-theorem for $E(a; x)$ using Hafner’s ingenious refinement of Hardy’s technique. (See, e.g., [6, 7].)

**Theorem.**

$$E(a; x) = \Omega_\pm(x^{\theta}(\log x)^{\theta(p-1)}(\log \log x)^{p-1} + c\theta \exp(-A \sqrt{\log \log x})), $$

where $\theta$ is defined in (2), $A$ is a positive constant,

$$c = \max\{p \log 2 - 1, p \log (a+1) - a\},$$

(3)

and $\theta = \pm$ for $p \geq 4$, whereas $\theta = +$ in the case $p = 2$ and 3.

**Remark.** The special case $a_1 = a_2 = 1, p = 2$ yields the celebrated result established in Hafner [5].

3. SOME LEMMAS

Let $a$ be as above. We say that $n$ is “$a$-full” if for any prime $q$ which divides $n$, $q^n$ divides $n$, too. For large positive real $x$ we define

$$A(x) = \{n \in \mathbb{N} \mid n \leq x \text{ and } n \text{ “}a\text{-full”}\},$$

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and
\[ H(x) = \{ n \in \mathbb{A}(x) \mid \omega(n) \geq p \log \log x - A \sqrt{\log \log x} \} , \]

where \( A \) is a positive constant and \( \omega(n) \) is the number of distinct prime divisors of \( n \).

**Lemma 1.**

(a) \( \# H(x) \ll x^{1/\omega}(\log x)^{1 - p \log^2} \exp(A \log 2 \sqrt{\log \log x}) \).

(b) \( \# H(x) \ll x^{1/\omega}(\log x)^{a - p \log(a + 1)} \exp(A \log(a + 1) \sqrt{\log \log x}) \).

**Proof.** By the definition of \( H(x) \), we have
\[
\# H(x)(\log x)^{p \log(a + 1)} \exp(-A \log(a + 1) \sqrt{\log \log x}) \ll \sum_{n \in \mathbb{A}(x)} 2^{\omega(n)}.
\]

Define
\[
f(n) = \begin{cases} 
2^{\omega(n)} & \text{if } n \text{ "a-full" } \\
0 & \text{else.}
\end{cases}
\]

Consider the generating function
\[
\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \zeta^2(as) G(s),
\]

where \( G(s) \) has a Dirichlet series absolutely convergent for \( \Re s > 1/(a + 1) \). From this the first part of Lemma 1 follows by standard techniques. In quite the same way we obtain
\[
\# H(x)(\log x)^{p \log(a + 1)} \exp(-A \log(a + 1) \sqrt{\log \log x}) \sum_{n \in \mathbb{A}(x)} (a + 1)^{\omega(n)}
\]
\[
\ll \sum_{n \in \mathbb{A}(x)} d(n) \ll x^{1/\omega}(\log x)^a.
\]

**Lemma 2.** For any integer \( m \),
\[
\sum_{\substack{n \leq x \\ (n, m) = 1}} p(n) n^\theta = K \prod_{q \mid m} \left( 1 - \frac{1}{q} \right) x^\theta (\log x)^{\rho - 1} + O(x^\eta (\log x)^{\rho - 2} ).
\]
where
\[\rho(n) \overset{\text{def}}{=} \sum_{n_1 \cdots n_p = n} \frac{1}{n_1 \cdots n_p},\]
\[K = \frac{1}{(p-1)!} \prod_{i=1}^{p} \frac{1}{a_i},\]
and the implied \(O\)-constant depends on \(a_1, \ldots, a_p\) and \(m\).

**Proof.** The result follows by standard techniques, since the generating function,
\[\sum_{\substack{n \geq 1 \\ (n, m) = 1}} \frac{\rho(n)}{n^s} = \prod_{i=1}^{p} \sum_{d \mid m} \mu(d) d^{\alpha_i(\theta - s) - 1} \zeta(a_i(s - \theta) + 1), \quad (\Re s \geq \theta),\]
has a pole of order \(p\) at \(s = \theta\).

**Lemma 3.**
\[\sum_{n \in \mathcal{A}(x)} \rho(n)n^{\omega(x)} w(n) = O(x^{\omega(\log x)^{p-1} \log \log x}).\]

**Proof.** Let \(q\) and \(r\) denote prime numbers. First we observe that
\[\sum_{n \in \mathcal{A}(x)} \rho(n)n^{\omega(x)} w(n) = \sum_{n \leq x} \rho(n)n^{\omega} \sum_{q^s \mid n} 1 \]
\[= \sum_{q^s \leq x} \sum_{s \geq 0} \rho(q^s) q^{s\omega} \]
\[= \sum_{q^s \leq x} \sum_{s \geq 0} \rho(q^{s+x}) q^{(s+x)\omega} \sum_{k \leq x/(q^{s+x})} \rho(k) k^{\omega} \]
\[= \sum_{q^s \leq x} \sum_{s \geq 0} \rho(q^{s+x}) \left\{ K \left(1 \frac{1}{q}\right)^{p} x^{\omega(\log x)^{p-1}} \right. \]
\[+ O(x(\log x + \log q)^{p-2}) \left. \right\} \]
\[= Kp x^{\omega(\log x)^{p-1} \log \log x + O(x^{\omega(\log x)^{p-1}})}.\]
(Use Lemma 2 and \(\sum_{s \geq 0} \rho(q^{s+x}) = p/q + O(1/q^2).\))
In the same way we conclude that
\[
\sum_{n \leq x} \rho(n) n^{\alpha} \omega(n) = \sum_{n \leq x} \rho(n) n^{\alpha} \sum_{q^r} \sum_{n \leq x} 1 \\
= \sum_{q^r \leq x} \sum_{n \leq x} \rho(n) n^{\alpha} \\
= \left\{ \sum_{q^r, r^s \leq x} + \sum_{q^r, r^s \leq x} \right\} \sum_{n \leq x} \rho(n) n^{\alpha}.
\]

Obviously
\[
\sum_{q^r, r^s \leq x} \sum_{n \leq x} \rho(n) n^{\alpha} \ll x^\beta (\log x)^{\rho - 1} \log \log x.
\]

The main term is given by
\[
\sum_{q^r, r^s \leq x \atop q \neq r} \sum_{l \leq x (q^{\alpha + r} + r^{\alpha + s} \beta)} \rho(l) l^{\alpha} \\
= \sum_{q^r, r^s \leq x \atop q \neq r} \sum_{l \leq x (q^{\alpha + r} + r^{\alpha + s} \beta)} \rho(l) l^{\alpha} \\
= x^{\alpha \beta} K \left( \frac{1}{q} + \frac{1}{r} \right)^{\rho - 1} x^{\beta} (\log x)^{\rho - 1} \\
+ O(x^{\rho - 2} (\log x)^{\rho - 1} \log \log x).
\]

From these two formulas Lemma 3 follows.

**Lemma 4.**
\[
\sum_{n \leq N} \rho(n) n^{\alpha} \ll \frac{1}{A} N^{\rho} (\log N)^{\rho - 1}.
\]

**Proof.** This is an immediate consequence of Lemma 3.
Lemma 5 (Dirichlet’s approximation principle). Let $\mathbf{a} = (x_1, \ldots, x_s) \in \mathbb{R}^s$, $q \in \mathbb{N}$, $t_0 \in \mathbb{R}^+$, then there exist $t \in \mathbb{R}$ with $\|\mathbf{a}t\| < 1/q$ and $t_0 < t < t_0q$, where $\|\cdot\|$ denotes the distance from the nearest integer.

4. PROOF OF THE THEOREM

We start with the Borel mean-value

$$B(t) = \frac{1}{\Gamma(k + 1)} \int_0^\infty u^k e^{-u} E(a; Xu) \, du,$$

where $a = a_1 + \cdots + a_p$,

$$X = X(t) = K_1 (\log t)^{-a} (\log \log t)^{\omega} \exp(\lambda \sqrt{\log \log t}),$$

$$k = k(t) = K_3 (\zeta + tX^{-1/2})^2$$

with $\zeta$ as defined in (3), positive constants $K_1, K_2$, and real $\zeta$ to be specified later. We take formula (4.12) from Nowak [16],

$$B(t) \approx X^k (p-1)/4 \sum_{n=1}^\infty \rho(n) n^\gamma e^{-\gamma_1 (nX)^{\gamma/2}}$$

$$\times \cos(\zeta (nX)^{1/2} + n^{1/2} + \gamma_0) + O(k^{p/4 - 3/8}),$$

where $\gamma_0$ is some constant depending only on $p$.

We decompose this representation

$$B(t) \approx X^k (p-1)/4 \left\{ \sum_{n \in \mathbb{H}} + \sum_{n \notin \mathbb{H}} \right\} \rho(n) n^\gamma e^{-\gamma_1 (nX)^{\gamma/2}}$$

$$\times \cos(\zeta (nX)^{1/2} + n^{1/2} + \gamma_0),$$

and apply Dirichlet’s approximation theorem to the first sum. In Lemma 5 we choose $\gamma_1 = n^{1/2}$. Let $B_1$ be a large positive integer, $H_1 = \#H(B_1)$ and $q \in \mathbb{N}$, $q \geq 2$ a parameter to be fixed later. Then there exists a number $t$ in the interval

$$B_1 < t < B_1 q^{H_1},$$

such that

$$\left| \frac{1}{2\pi n^{1/2} t} \right| < \frac{1}{q}$$
for all \( \sigma_n \in H(B_1) \). From (6) we see that
\[
H_1 \gg (\log t)(\log q)^{-1}. \tag{7}
\]
Combining (7) and Lemma 3, a short calculation shows that
\[
B_1 \gg (\log q)^{-a} (\log t)^{a'} (\log \log t)^{-a''} \exp(-A \sqrt{\log \log t}).
\]
We define
\[
B_0 = c_0 (\log q)^{-a} (\log t)^{a_0} (\log \log t)^{-a} \exp(-A \sqrt{\log \log t})
\]
with \( c_0 \) sufficiently small such that \( B_0 < B_1 \). We conclude that
\[
|\cos(\zeta(nX)^{1/2} + n^{1/2} t + \gamma_0) - \cos(\zeta(nX)^{1/2} + \gamma_0)| < \frac{1}{q}
\]
for all \( n \in H(B_0) \). Therefore
\[
\left| \sum_{n \in H_0} \rho(n) n^\theta e^{-c_i(nX)^{1/2}} \{ \cos(\zeta(nX)^{1/2} + n^{1/2} t + \gamma_0) - \cos(\zeta(nX)^{1/2} + \gamma_0) \} \right| \\
\ll \sum_{n \leq B_0} \rho(n) n^\theta e^{-c_i(nX)^{1/2}} = \frac{1}{q} \int_{B_0}^{B_0} \exp(-c_i(uX)^{1/2}) \, dS(u) \\
\ll \frac{1}{q} X^{-\theta} (\log B_0)^{\theta - 1},
\]
where
\[
S(u) = \sum_{n \leq u} \rho(n) n^\theta = u^\theta (\log u)^{\theta - 1}. \tag{8}
\]
Analogously,
\[
\sum_{n \in H_0 \atop n \geq B_0} \rho(n) n^\theta e^{-c_i(nX)^{1/2}} \ll \frac{1}{A} X^{-\theta} (\log B_0)^{\theta - 1}.
\]
Those \( n \) with \( n \geq B_0 \) contribute
\[
\ll \sum_{n \geq B_0} \rho(n) n^\theta e^{-c_i(nX)^{1/2}} \\
\ll B_0 (\log B_0)^{\theta - 1} \exp(-c_i(B_0 X)^{1/2}) \\
+ \int_{B_0}^{\infty} \exp(-c_i(uX)^{1/2})(uX)^{1/2 - 1} X S(u) \, du.
\]
We split up the last integral in \( \int_{B_0}^{\infty} + \int_{B_0}^{\infty} \). The first integral contributes
\[
\ll (\log B_0)^{p-1} \int_{B_0}^{\infty} \exp(-c_1(uX)^{2/5})(uX)^{2/5} u^{\theta-1} \, du
\]
\[
\ll B_0^p (\log B_0)^{p-1} \exp(-c_1(B_0 X)^{2/5}).
\]
In a similar way one verifies that the contribution of the second integral is \( o(1) \) as \( t \to \infty \). We arrive at
\[
B(t) \asymp k^{(p-1)/4} \left\{ \sum_{n \leq B_0} \rho(n) n^\theta \exp(-c_1(nX)^{2/5}) \cos(\zeta(nX)^{1/2} + \gamma_0) + o(1) \right\}.
\]
We extend the range of summation up to infinity in this last series remarking that the error is estimated in exactly the same way as above. Our next step is an asymptotic formula for this last series, as \( X \to 0^+ \), \( \zeta \) some real constant, in the spirit of Berndt [2].

**Lemma 6.** For \( X \to 0^+ \),
\[
\sum_{n = 1}^{\infty} \rho(n) n^\theta \exp(-c_1(nX)^{2/5}) \cos(\zeta(nX)^{1/2} + \gamma_0)
\]
\[
= c_2 X^{-\theta} (\log X)^{p-1} (G(\zeta) + o(1)),
\]
with
\[
G(\zeta) = \int_{0}^{\infty} e^{-\sqrt{v}^4 X^{p-3/4}} \cos(c_1^{-1/2} \sqrt{v} + \gamma_0) \, dv.
\]

**Proof.** Using the Stieltjes integral notation,
\[
\sum_{n = 1}^{\infty} \rho(n) n^\theta \exp(-c_1(nX)^{2/5}) \cos(\zeta(nX)^{1/2} + \gamma_0)
\]
\[
= \int_{0}^{\infty} \exp(-c_1(uX)^{2/5}) \cos(\zeta(uX)^{1/2} + \gamma_0) \, dS(u).
\]
Integrating by parts and inserting the asymptotic expansion given in (8), we estimate the contribution of the error to be less than
\[
\ll \exp(-c_1(uX)^{3/2}) u'(1 + |\log u|)^{p-1} \big| u=0 \\
+ \int_0^\infty ((uX)^{2/3} + (uX)^{1/3}) \exp(-c_1(uX)^{1/3}) u^{\theta-1}(1 + |\log u|)^{p-2} \, du
\]
\[
\ll X^{-\theta} \int_0^\infty (e^{2/3} + v^{1/3}) \exp(-c_1 v^{2/3}) v^{\theta-1}(|\log X| + |\log v|)^{p-2} \, dv
\]
\[
\ll X^{-\theta} |\log X|^{p-2}.
\]

We obtain the order term by quite a similar reasoning and a change of variable \( v = \sqrt{c_1(nX)^{1/2}} \). Using Lemma 6, we arrive at our desired asymptotic expansion
\[
B(t) = c_3 k^{(p-1)/4} |\log X|^{p-1} (G(\zeta) + o(k^{(p-1)/4}))
\]
with a positive constant \( c_3 \).

Now we make use of a deep result due to Steinig [19] which provides necessary and sufficient conditions for functions like \( G(\zeta) \) to have a change of sign.

**Lemma 7.** For \( \zeta, B, \gamma \in \mathbb{R}, \gamma > -1, \) let
\[
G_{\gamma, B}(\zeta) \overset{def}{=} \int_0^\infty e^{-\nu^2} \cos(\nu \zeta + \gamma B) \, d\nu.
\]

Then \( G_{\gamma, B}(\zeta) \) as a function of \( \zeta \) has a sign change if and only if
\[
\gamma > -2 |B - [B + 1/2]|. \tag{9}
\]
Otherwise, \( G_{\gamma, B}(\zeta) \neq 0 \) for all real values of \( \zeta \).

For \( p \geq 4 \), (9) is trivially satisfied. Thus there exist real numbers \( \zeta_1, \zeta_2 \) and a positive constant \( c_4 \) such that \( G(\zeta_1) \leq -c_4, G(\zeta_2) \geq c_4 \). First we take \( \zeta = \zeta_1 \), then \( \zeta = \zeta_2 \) in the definition (5); i.e., we put
\[
k_i(t) = k_i(t) = K_2(\zeta_i + tX(t)^{-1/2})^2 \quad (i = 1, 2),
\]
define \( B_i(t) \) like \( B(t) \) before, with \( k \) replaced by \( k_i \), and infer from the above argument that there exists an unbounded sequence of real numbers \( t \) with
\[
B_1(t) \leq -c_4 k^{(p-1)/4} (|\log t|)^{p-1}
\]
\[
B_2(t) \leq -c_4 k^{(p-1)/4} (|\log t|)^{p-1}.
\]
In order to complete the proof, let us suppose that, for some small positive constant \( K \),

\[
\pm \epsilon(a; T) \leq K(T(\log T)^{\alpha}(\log \log T)^{p-1+\epsilon})
\times \exp(A \sqrt{\log \log \log T})
\quad (i = 1 \text{ or } 2)
\]

for all sufficiently large \( T \). By the definition of \( B_i(t) \), this would imply that, for every large real \( t \),

\[
(\pm 1)^i B_i(t) \leq \frac{K}{T(k_i(t) + 1)} \int_0^\infty e^{-u(k_i(t) X(t) u^{2 \Sigma_2})^\theta} L(X(t) u^{2 \Sigma_2}) \, du,
\]

where

\[
L(w) = (\log w)^{\alpha_0} (\log \log w)^{p-1+\epsilon} \exp(-A \sqrt{\log \log \log w})
\]

for \( w \geq 10 \) and \( L(w) = L(10) \) else. Estimating this integral by Hafner’s Lemma 2.3.6 in [6, p. 51], we would obtain

\[
(\pm 1)^i B_i(t) \leq c_i(k_i(t))^{(p-1)/4} (\log \log t)^{p-1}.
\]

Together this would yield a positive lower bound for \( K \) (for both \( i = 1, 2 \)). This completes the proof of our theorem in the case \( p \geq 4 \).

It remains to deal with the case that \( p = 2, 3 \). In this case (9) is not satisfied. Consequently, \( G(\zeta) \) has always the same sign. The explicit evaluation for \( i = 1, 2 \) shows that \( G(\zeta) \geq 0 \), both for \( p = 2 \) and \( p = 3 \). This completes the proof of our theorem.

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