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A Formula for the General Solution of a Constant-coefficient Difference Equation

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We give a formula for the general solution of a dth-order linear difference equation with constant coefficients in terms of one of the solutions of its associated homogeneous equation. The formula neither uses the roots of the characteristic equation nor their multiplicities. It can be readily generalized to the case where the domain of the difference equation is the real numbers, and the initial values are given by a function defined on the interval $[0, d)$. In both cases, we express the general solution of the difference equation in terms of a single solution of its associated homogeneous equation at integer arguments.

1. Constant-coefficient Difference Equations

Usually, a dth-order linear difference equation with constant coefficients is defined as an equation of the form

$$f(n) = r(n) + \sum_{1 \le l \le d} a_{d-l} f(n-l) \tag{1.1}$$

where $f : \mathbb{Z} \to \mathbb{C}$ is an unknown function and a_{d-l} is a given constant coefficient, $1 \le l \le d$, and $a_0 \ne 0$. The function $r : \mathbb{Z} \to \mathbb{C}$ is an arbitrary given function, and there are d arbitrary initial values at consecutive arguments, such as $f(0), f(1), \ldots, f(d-1)$.

2. Solution

We express f in terms of a function F_i on the integers whose initial values are

$$F_i(n) = \begin{cases} 0 & \text{if } n \ne i \\ 1 & \text{if } n = i \end{cases}$$

where $0 \le i$, $n \le d-1$, and which satisfies the homogeneous form of equation (1.1):

$$f(n) = \sum_{1 \le l \le d} a_{d-l} f(n-l) \tag{2.1}$$

at integer arguments.

Theorem 2.1. *Equation (2.2) below gives the general solution of equation (1.1) for all* $n \in \mathbb{Z}$

$$f(n) = \begin{cases} \sum_{0 \le i < d} f(i)F_i(n) + \sum_{0 \le i \le n-d} r(n-i)F_{d-1}(d-1+i) & \text{if } n \ge 0 \\ \sum_{0 \le i < d} f(i)F_i(n) - \sum_{1 \le i \le -n} r(n+d+i-1)F_{d-1}(-i) & \text{if } n \le 0. \end{cases} \tag{2.2}$$

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PROOF. The sum $\sum_{0\le i<d} f(i)F_i(n)$ is the complementary solution of equation (2.1). This follows because the F_i are d linearly independent solutions of equation (2.1).

We can show that

$$\sum_{0\le i\le n-d} r(n-i)F_{d-1}(d-1+i)$$

is a particular solution of equation (1.1) when $n\ge 0$. In this particular solution $f(i)=0$ where $0\le i<d$. This convolution is the coefficient of the term in x^n where $n\ge d$ of the product of two generating functions:

$$G_1(x)=\sum_{n\ge 0} F_{d-1}(d-1+n)x^n \tag{2.3}$$

and

$$G_2(x)=\sum_{n\ge d} r(n)x^n. \tag{2.4}$$

We can verify that

$$\sum_{n\ge 0} f(n)x^n = G_1(x)G_2(x)$$

where f is this particular solution of equation (1.1) as required.

Similarly, we can show that $-\sum_{1\le i\le -n} r(n+d+i-1)F_{d-1}(-i)$ is a particular solution of equation (1.1) when $n\le 0$.

The result follows because it is the sum of the complementary solution and the particular solution when $n\ge 0$, and when $n\le 0$. □

The following lemma expresses the function F_i in terms of the single complementary solution F_{d-1}.

LEMMA 2.2.

$$F_i(n)=\sum_{0\le j\le i} a_{i-j}F_{d-1}(n-j-1). \tag{2.5}$$

PROOF. From equation (1.1), the ordinary generating function of $F_i(n)$ is

$$G(F_i(n))=x^i\left(1+\sum_{0\le j\le i}\frac{a_{i-j}x^{d-i+j}}{1-\sum_{1\le l\le d} a_{d-l}x^l}\right). \tag{2.6}$$

In particular,

$$G(F_{d-1}(n))=\frac{x^{d-1}}{1-\sum_{1\le l\le d} a_{d-l}x^l}. \tag{2.7}$$

From equations (2.6) and (2.7), we have

$$\begin{aligned}
G(F_i(n)) &= x^i+\sum_{0\le j\le i} a_{i-j}x^{j+1}G(F_{d-1}(n))\\
&= x^i+\sum_{0\le j\le i} a_{i-j}G(F_{d-1}(n-j-1))-\sum_{0\le j\le i} a_{i-j}\sum_{0\le l\le j} x^l F_{d-1}(l-j-1)\\
&= \sum_{0\le j\le i} a_{i-j}G(F_{d-1}(n-j-1))
\end{aligned}$$

as required. The last step uses

$$x^i = \sum_{0\le j\le i} a_{i-j} \sum_{0\le l\le j} x^l F_{d-1}(l-j-1)$$

which can be shown by changing the order of summations on the right side and using the definition of F_{d-1}. There is a similar proof for the case when $n < 0$. □

Corollary 2.3.

$$F_0(n) = a_0 F_{d-1}(n-1).$$

We now use Lemma 2.2 and Corollary 2.3 to express the general solution of equation (1.1) in terms of F_0, or F_{d-1}. Let

$$s_1 = \sum_{0\le i<d} f(i) \sum_{0\le j\le i} a_{i-j} F_0(n-j)$$

and

$$s_2 = \sum_{0\le i<d} f(i) \sum_{0\le j\le i} a_{i-j} F_{d-1}(n-j-1).$$

We have

$$f(n) = \begin{cases} \frac{1}{a_0}\big(s_1 + \sum_{0\le i\le n-d} r(n-i)F_0(d+i)\big) & \text{if } n \ge 0 \\ \frac{1}{a_0}\big(s_1 - \sum_{1\le i\le -n} r(n+d+i-1)F_0(1-i)\big) & \text{if } n \le 0 \end{cases} \tag{2.8}$$

and

$$f(n) = \begin{cases} s_2 + \sum_{0\le i\le n-d} r(n-i)F_{d-1}(d-1+i) & \text{if } n \ge 0 \\ s_2 - \sum_{1\le i\le -n} r(n+d+i-1)F_{d-1}(-i) & \text{if } n \le 0. \end{cases} \tag{2.9}$$

By using equations (2.8) and (2.9), we can find the value of f for any $n \in \mathbb{Z}$ merely from the initial values, the function r, and the values of F_0, or F_{d-1} at integer arguments. Any applicable method can be used to solve equation (2.1) for F_0 or F_{d-1}.

If we use the method involving exponential generating functions and the Laplace transform (Doetsch, 1974), the initial values of F_{d-1} result in a simpler solution than for the other F_i. The transform of the associated differential equation for F_{d-1} is the reciprocal of the characteristic polynomial of equation (2.1).

3. A Generalization

We consider the generalization of finding a formula for the function $f : \mathbb{R} \to \mathbb{C}$ which satisfies equation (1.1) for all $x \in \mathbb{R}$. In this case, $r : \mathbb{R} \to \mathbb{C}$ is an arbitrary given function, and $g : [0, d) \to \mathbb{C}$ is a given *initial function* that is equal to f over this interval.

Given an initial function g whose domain is the interval $[0, d)$, we can compute every value of the function $f(x)$ where $x \in \mathbb{R}$. Equation (2.8) becomes

$$f(n+\epsilon) = \begin{cases} \frac{1}{a_0}\big(s_1' + \sum_{0\le i\le n-d} r(n+\epsilon-i)F_0(d+i)\big) & \text{if } n \ge 0 \\ \frac{1}{a_0}\big(s_1' - \sum_{1\le i\le -n} r(n+\epsilon+d+i-1)F_0(1-i)\big) & \text{if } n \le 0 \end{cases} \tag{3.1}$$

where $\epsilon \in [0, 1)$ and $s_1' = \sum_{0\le i<d} g(i+\epsilon) \sum_{0\le j\le i} a_{i-j} F_0(n-j)$.

Similarly, equation (2.9) has the following generalization

$$f(n+\epsilon) = \begin{cases} s_2' + \sum_{0\le i\le n-d} r(n+\epsilon-i)F_{d-1}(d-1+i) & \text{if } n \ge 0 \\ s_2' - \sum_{1\le i\le -n} r(n+\epsilon+d+i-1)F_{d-1}(-i) & \text{if } n \le 0 \end{cases} \tag{3.2}$$

where $\epsilon \in [0,1)$ and $s_2' = \sum_{0\le i<d} g(i+\epsilon) \sum_{0\le j\le i} a_{i-j}F_{d-1}(n-j-1)$.

3.1. RELATED GENERALIZATIONS

Kuczma *et al.* (1990, Section 3.1) discuss finding continuous solutions of linear equations of order 1 in the general setting of iterative functional equations. The dth-order linear equation is discussed in Section 6.7. They describe a method that can sometimes reduce it to a system of d linear equations of order 1.

Milne-Thomson (1933, Section 13.1) presents a method for finding the general solution of equation (1.1) when its domain is $\mathbb{C}$ and an initial function is not defined. This method solves a different problem than the one we consider.

The method uses periodic functions, the roots of the characteristic equation associated with equation (1.1), and their multiplicities. To apply it in our context, we would need to find these three things first. Determining the periodic functions can be done by a method that is closely related to one for finding the coefficients of the complementary solution in the discrete case.

Instead, we have given a formula for the general solution of the difference equation (1.1) when its domain is $\mathbb{R}$. This formula uses an initial function directly, and does not use periodic functions or the roots of the characteristic equation.

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