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Extremal properties of the bipartite vertex frustration of graphs

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ABSTRACT

The smallest number of vertices that have to be deleted from a graph G to obtain a bipartite subgraph is called the bipartite vertex frustration of G and denoted by $\psi(G)$. In this paper, some extremal properties of this graph invariant are presented. Moreover, we present an exact formula for the bipartite vertex frustration of the corona product of graphs.

1. Introduction

Let G be a graph on n vertices. The vertex and edge sets of G are denoted by $V(G)$ and $E(G)$, respectively. Molecular graphs represent the constitution of molecules. They are generated using the following rule: vertices stand for atoms and edges for bonds. It is clear that the degree of each vertex in a molecular graph is at most four.

A topological index is a numeric quantity derived from the structure of a graph which is invariant under automorphisms of the considered graph. The smallest number of edges that have to be deleted from a graph to obtain a bipartite spanning subgraph is called the bipartite edge frustration of G and denoted by $\varphi(G)$. This topological index has important applications in computing stability of fullerenes [1,2]. Because of this success it is natural to study its vertex version.

The first result regarding the large bipartite spanning subgraphs of a given non-bipartite graph published in a paper by Erdös [3] and Edwards [4]. They proved that every graph G contains a bipartite subgraph with at least $\frac{|E(G)|}{2} + \frac{|V(G)|-1}{4}$ edges. After publishing the mentioned paper, some authors presented better lower bounds for various classes of graphs; see [5–7] for details. To investigate the large bipartite subgraphs of a given graph G, it is possible to find a smallest set of edges that must be deleted from G in order to make the remaining graph bipartite. The cardinality of such small set of edges is called the *bipartite edge frustration* of G and denoted by $\varphi(G)$. In [1,8,9], some mathematical properties of this new graph invariant was obtained. In [10], the authors computed the bipartite edge frustration index for some classes of nanotubes.

The *bipartite vertex frustration* of G, $\psi(G)$ is defined as the minimum number of vertices that have to be deleted from G to obtain a bipartite subgraph H of G. Obviously, if G is not bipartite then H is not a spanning subgraph of G and so, H is not in general a large bipartite subgraph of G. It seems that it is possible to find an algorithm for constructing a large bipartite spanning subgraph from H.

Suppose G is a graph. A subset $A \subseteq V(G)$ such that $G - A$ is bipartite is called a vertex bipartization for G. The vertex bipartization problem for the graph G is to find the minimum number of vertices whose removal makes the graph bipartite which is equivalent to the problem of computing $\psi(G)$. Similar to the edge version, the vertex bipartization problem has a rich history. The problem has numerous applications, for instance, in computational biology [11], and register allocation [12].

Given a graph G, a matching M in G is a set of pairwise non-adjacent edges; that is, no two edges share a common vertex. We say that a vertex is matched if it is incident to an edge in the matching. Otherwise the vertex is unmatched. The *line*

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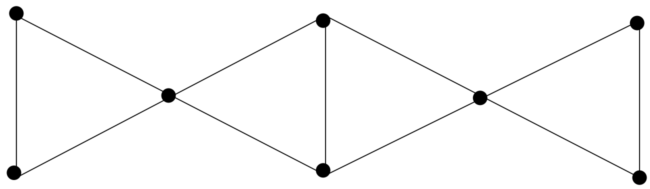

Fig. 1. The graph G.

graph, $L(G)$, is the graph whose vertices are the edges of G and two edges of G are adjacent in $L(G)$ if and only if they are incident to the same vertex.

All graphs in this paper are assumed to be finite, simple and connected. For terms and concepts not defined here, we refer the reader to any of several standard monographs such as, e.g., [13,14].

2. Main results

It is clear that G is bipartite if and only if $\varphi(G) = \psi(G) = 0$. Since the quantity $\psi(G)$ is, in general, difficult to compute; it makes sense to search for classes of graphs that allow its efficient computation. It is also worthwhile to investigate how the bipartite vertex frustration of some composite graphs is related to the bipartite vertex frustrations of their components. The aim of this section is to study this problem for suspension and corona product of graphs which will be defined later.

Let K_n, C_n and W_n denote the complete, cycle and wheel on n vertices. Then $\psi(K_n) = n - 2$, $\psi(C_n) = \begin{cases} 0 & 2 \mid n \\ 1 & 2 \nmid n \end{cases}$ and $\psi(W_n) = \begin{cases} 2 & 2 \mid n \\ 1 & 2 \nmid n \end{cases}$.

Lemma 1. *Let G be a graph with components $G_1, G_2, \ldots, G_n$. Then $\psi(G) = \sum_{i=1}^{n} \psi(G_i)$.*

Proof. The proof is straightforward. □

To compute the bipartite vertex frustration under graph operations, it is enough to consider connected graphs and then apply Lemma 1. From now on, all graphs are assumed to be connected.

Lemma 2. *$\psi(G) \le \varphi(G)$. Moreover, if $\psi(G) = \varphi(G)$ then every minimal edge bipartization of G is a matching.*

Proof. Suppose $M = \{e_1, \ldots, e_{\varphi(G)}\}$ is a subset of $E(G)$ such that $G - M$ is bipartite. If M is a matching of G then we define A to be a set of vertices such that each vertex of A is incident to one and only one edge of M. Since $G - A$ is a subgraph of $G - M$, it is bipartite and so $\psi(G) \le \varphi(G)$. If M is not a matching of G then there are two edges e and f of M containing a common vertex v. Therefore, $e, f \notin E(G - v)$ and we can find a set of vertices, say B, such that $G - B$ is bipartite and $|B| < \varphi(G)$. Thus $\psi(G) \le |B| < \varphi(G)$, as desired.

We now assume that $\psi(G) = \varphi(G)$ and choose $M \subseteq E(G)$ such that $G - M$ is bipartite and $|M| = \varphi(G)$. We claim that M is a matching of G. Otherwise, there are edges $e, f \in M$ with a common vertex v. Choose v and one vertex of each edge of $M - \{e, f\}$. Then we obtain a subset X of $V(G)$ of size less than $|M|$ such that $G - X$ is bipartite, contradicts by equality of $\psi(G)$ and $\varphi(G)$. □

The converse of Lemma 2 is not generally correct. To do this, we consider the graph G depicted in Fig. 1. Then $\varphi(G) = 3$, $\psi(G) = 2$ and each minimal bipartization of G is a matching.

Lemma 3. *Suppose G is a non-empty graph. Then for each minimal vertex bipartization F of G, $G - F$ is non-empty.*

Proof. Suppose $F = \{v_1, \ldots, v_{\psi(G)}\}$ is a vertex bipartization of G such that $G - F$ is an empty graph. Choose a vertex $v_i \in F$ and construct a maximal subgraph T of G with the vertex set $(G - F) \cup \{v_i\}$ such that v_i is incident to all edges of T. This shows that by choosing $F_i = F - \{v_i\}$, $G - F_i$ is also bipartite, which is impossible. □

The suspension of a graph G is the graph obtained from G by adding a new vertex and connecting this vertex to each vertex of G. In the next theorem, the bipartite vertex frustration of the suspension of a graph G is computed.

Theorem 4. *$\psi(\nabla G)$ is equal to zero if G is empty and $\psi(G) + 1$ otherwise.*

Proof. Suppose G has exactly n vertices. If G is empty then $\nabla G \cong S_{n+1}$, the star graph on $n + 1$ vertices. So, ∇G is bipartite, as desired. Suppose G is not empty and $V(\nabla G) - V(G) = \{v\}$. Define F to be a minimal set of vertices that have to be deleted from G to obtain a bipartite subgraph H of G. It is clear that H is a subgraph of ∇G and $\nabla G - (F \cup \{v\})$ is bipartite. Thus,

$$\psi(\nabla G) \le \psi(G) + 1. \tag{1}$$

We claim that each minimal vertex bipartization of ∇G is containing the vertex v. If so, then the other vertices of a minimal vertex bipartization of ∇G will be a vertex bipartization of G, which completes our argument. To prove our claim,

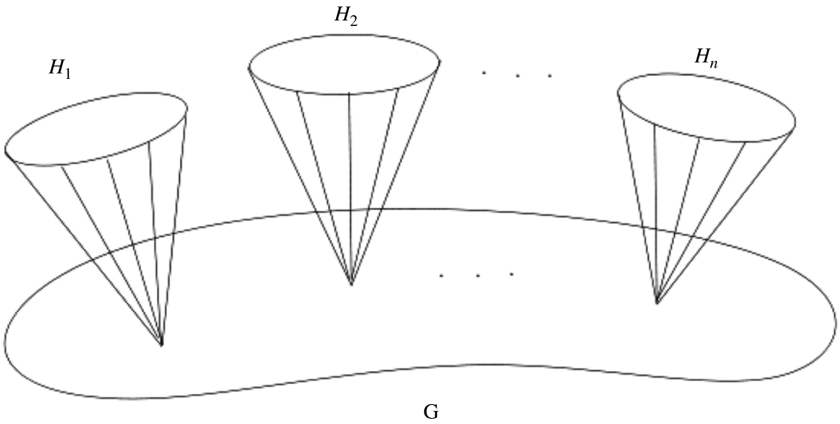

Fig. 2. The Corona product $G \circ H$.

we assume that $\psi(\nabla G) = k+l$, where k is the number of vertices that have to be deleted from G and $l = 0, 1$. This implies that $\psi(G) \le k$. We first assume that $\psi(G) < k$. Then $\psi(G) + 1 \le k \le k+l = \psi(\nabla G)$. Therefore, by Eq. (1), $\psi(G) + 1 = \psi(\nabla G)$, as desired. Next we suppose that $\psi(G) = k$ and F is a set of vertices of size k such that $G - F$ is bipartite. By Lemma 3, $G - F$ is non-empty and so it has at least one edge e. Thus we have at least one triangle Δ in ∇G such that $v \in V(\Delta)$. This shows that for vertex bipartization of ∇G, the vertex v must be deleted, i.e. $l = 1$. Therefore, $\psi(\nabla G) = k + l = \psi(G) + 1$, which completes our proof. □

Corollary 5. *Let G be an n-vertex graph, $S = \{v_1, \ldots, v_k\} \subseteq V(G)$ such that $\deg(v_i) = n - 1$, $1 \le i \le k$, and $G - S$ is non-empty. Then $\psi(G) = k + \psi(G - S)$.*

Proof. Since $\deg(v_1) = n - 1$, $G = \nabla(G - \{v_1\})$, and since $\deg_{G-\{v_1\}}(v_2) = n - 2$, $G - v_1 = \nabla(G - \{v_1, v_2\})$. Therefore,

$$\begin{aligned} G &= \nabla(G - \{v_1\}) \\ &= \nabla(\nabla(G - \{v_1, v_2\})) \\ &= \vdots \\ &= \nabla(\nabla(\cdots \nabla(G - \{v_1, v_2, \ldots, v_k\}) \cdots)). \end{aligned}$$

Now, by applying Theorem 4, we have

$$\begin{aligned} \psi(G) &= \psi(\nabla(G - \{v_1\})) = \psi(G - \{v_1\}) + 1 \\ &= \psi(\nabla(\nabla(G - \{v_1, v_2\}))) = \psi(G - \{v_1, v_2\}) + 2 \\ &= \vdots \\ &= \psi(\nabla(\nabla(\cdots \nabla(G - \{v_1, v_2, \ldots, v_k\}) \cdots))) = \psi(G - \{v_1, \ldots, v_k\}) + k. \end{aligned}$$

This completes the proof. □

Corollary 6. *Suppose G is an n-vertex graph. Then $0 \le \psi(G) \le n - 2$ with the left equality if and only G is bipartite, with the right equality if and only $G \cong K_n$.*

Proof. It is enough to prove that if G is an n-vertex graph such that $\psi(G) = n - 2$ then $G \cong K_n$. On the contrary, we assume that there exists an n-vertex graph G such that $G \ncong K_n$ and $\psi(G) = n - 2$. Choose an edge e of G such that $K_{n-1} \le G \le K_n - \{e\}$. Since ψ is an order preserving function between the set of n-vertex graphs and natural numbers, $n - 3 = \psi(K_{n-1}) \le \psi(G) \le \psi(K_n - \{e\}) \le n - 3$. So, $\psi(G) = n - 3$, a contradiction. □

Theorem 7. *Suppose G is an n-vertex graph. Then*

(a) $\psi(G) = n - 3$ *if and only if* $G \cong K_n - \{e_1, \ldots, e_k\}$ *such that* $1 \le k \le n - 1$ *and* $\{e_1, \ldots, e_k\}$ *is a star in G.*
(b) $\psi(G) = 1$ *if and only if all odd cycles of G have a common vertex.*

Proof. (a) Suppose G is an n-vertex graph and $\psi(G) = n - 3$. By Corollary 6, $G \ncong K_n$ and so there are edges $e_1, \ldots, e_k$ such that $G = K_n - \{e_1, \ldots, e_k\}$. We claim that these edges constitute a star in G. On the contrary, we assume that there are edges $e_i = u_iv_i$ and $e_j = u_jv_j$ without common vertices. Consider the induced subgraph H generated by $X = \{u_i, v_i, u_j, v_j\}$. Clearly, H is bipartite and by deleting $n - 4$ vertices of $V(G) - X$ the subgraph H is obtained. Thus $\psi(G) \le n - 4$, a contradiction.

To prove (b), we assume that $\psi(G) = 1$. If G have two odd cycles O_1 and O_2 without a common vertex then for vertex bipartization of G, we must at least delete two vertices of G, contradicts by our assumption. □

Let G and H be two graphs. Their corona product GoH is defined as the graph obtained by taking one copy of G and joining the ith vertex of G to every vertex in ith copy of H. An example is shown in Fig. 2.

Theorem 8. $\psi(GoH) = \begin{cases} |V(G)|(\psi(H)+1) & H \text{ is non-empty} \\ \psi(G) & \text{Otherwise} \end{cases}.$

Proof. If H is empty then from Fig. 1, one can easily seen that $\psi(GoH) = \psi(G)$. Suppose H is non-empty. Then $\psi(GoH) = \sum_{i=1}^{|V(G)|} \psi(\nabla H) = \sum_{i=1}^{|V(G)|} (\psi(H)+1) = |V(G)|(\psi(H)+1)$. □

The bipartite vertex frustration of line graphs can be computed from the following theorem:

Theorem 9. $\psi(L(G)) = \begin{cases} 2(|E(G)| - |V(G)|) & G \text{ is not an odd Cycle} \\ 1 & \text{Otherwise} \end{cases}.$

Proof. If G is an odd cycle then $L(G)$ will be an odd cycle of the same length. So, $\psi(L(G)) = 1$. We assume that G is not an odd cycle. From the definition of the line graph, $L(G)$ is not an odd cycle. By [14, Theorem 7.1.16], $L(G)$ decomposes into complete subgraphs, with each vertex of G appearing in at most two of these complete subgraphs. By this theorem, for each $v_i \in V(G)$ of degree $d_i = \deg_G(v_i)$ we have a complete subgraph K_{d_i} in $L(G)$. Since $\psi(K_{d_i}) = d_i - 2$, $\psi(L(G)) = \sum_{i=1}^{|V(G)|} \psi(K_{d_i}) = \sum_{i=1}^{|V(G)|} (d_i - 2) = 2(|E(G)| - |V(G)|)$. □

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References

[1] T. Došlić, D. Vukičević, Computing the bipartite edge frustration of fullerene graphs, Discrete Appl. Math. 155 (2007) 1294–1301.
[2] T. Došlić, Bipartivity of fullerene graphs and fullerene stability, Chem. Phys. Lett. 412 (2005) 336–340.
[3] P. Erdös, On some extremal problems in graph theory, Israel J. Math. 3 (1965) 113–116.
[4] C.S. Edwards, Some extremal properties of bipartite subgraphs, Canad. J. Math. 25 (1973) 475–485.
[5] Q. Cui, J. Wang, Maximum bipartite subgraphs of cubic triangle-free planar graphs, Discrete Math. 309 (2009) 1091–1111.
[6] G. Hopkins, W. Staton, Extremal bipartite subgraphs of cubic triangle-free graphs, J. Graph Theory 6 (1982) 115–121.
[7] J.A. Bondy, S.C. Locke, Largest bipartite subgraphs in triangle-free graphs with maximum degree three, J. Graph Theory 10 (1986) 477–504.
[8] Z. Yarahmadi, T. Došlić, A.R. Ashrafi, The bipartite edge frustration of composite graphs, Discrete Appl. Math. 158 (2010) 1551–1558.
[9] Z. Yarahmadi, The bipartite edge frustration of extension of splice and link graphs, Appl. Math. Lett. 23 (2010) 1077–1081.
[10] M. Ghojavand, A.R. Ashrafi, Computing the bipartite edge frustration of some nanotubes, Digest J. Nanomaterials and Biostructures 3 (2008) 209–214.
[11] F. Fomin, S. Gaspers, A. Pyatkin, Finding a minimum feedback vertex set in time O(1.7548n), IWPEC, in: Lecture Notes in Comput. Sci., vol. 4169, Springer, 2006, pp. 184–191.
[12] V. Raman, S. Saurabh, C. Subramanian, Faster fixed parameter tractable algorithms for finding feedback vertex sets, ACM Trans. Algorithms 2 (2006) 403–415.
[13] F. Harary, Graph Theory, Addison-Wesley, Reading MA, 1969.
[14] D.B. West, Introduction to Graph Theory, Prentice-Hall, Upper Saddle River, NJ, 1996.