EXPONENTIAL MEAN SQUARE STABILITY
OF PARTIALLY LINEAR STOCHASTIC SYSTEMS

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Abstract—The purpose of this paper is to state sufficient conditions for the existence of linear feedback laws which render the equilibrium solution of a composite partially linear stochastic system (the linear part of which is deterministic) exponentially stable in mean square.

1. INTRODUCTION

In this paper, we propose sufficient conditions for the exponential stabilization in mean square, by means of linear feedback laws, of composite partially linear stochastic systems.

The global stabilization, by means of state feedback, of deterministic composite partially linear systems has been studied by many authors (see [1-4]). In [5], Saberi, Kokotovic and Sussmann give sufficient conditions for the stabilization, by means of linear state feedback, of systems in the form

\[ \dot{x} = f(x, 0) + G(x, \xi)\xi, \quad x \in \mathbb{R}^n, \quad \xi \in \mathbb{R}^q \]

\[ \dot{\xi} = A\xi + Bu, \quad u \in \mathbb{R}^q. \]

Under suitable assumptions, they prove, if the nonlinear zero dynamics

\[ \dot{x} = f(x, 0) \]

are globally exponentially stable, that the equilibrium solution \((x, \xi) \equiv (0, 0)\) of the composite system (1), (2) is globally exponentially stable for every linear feedback law \(u = K\xi\) such that the matrix \(A + BK\) is asymptotically stable.

The aim of this paper is to extend the above result when equation (1) is corrupted by noise. Actually, only few results on the feedback stabilization of nonlinear stochastic differential systems can be found in the literature (see [6-9]). The procedure used in these papers is based on the stochastic Lyapunov machinery developed by Khasminskii [10].

This paper is divided in three sections organized as follows. In Section 2, we recall some results on the stochastic Lyapunov machinery that we need to prove the exponential stability in mean square of the equilibrium state of a stochastic differential equation. In Section 3, we introduce the class of composite partially linear stochastic systems we are dealing with in this paper. In Section 4, we state and prove the main result of this paper on the stabilization, by means of linear feedback laws, of the class of stochastic systems introduced in Section 3.

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2. EXPONENTIAL STABILITY IN MEAN SQUARE

The aim of this section is to summarize the main definitions and results on the exponential stability in mean square of the equilibrium solution of a stochastic differential equation that we need in the sequel. For a complete presentation of this topic, we refer the reader to [10] or [11] for example.

Consider a complete probability space \((\Omega, \mathcal{F}, P)\) and \(\{w_t, t \in \mathbb{R}_+\}\) a standard \(\mathbb{R}^m\)-valued Wiener process defined on this space. Denote by \((\mathcal{F}_t)_{t \in \mathbb{R}_+}\) the complete right-continuous filtration generated by the standard Wiener process \(w\).

Let \(z_t \in \mathbb{R}^n\) be the stochastic process solution of the stochastic differential equation written in the sense of Itô,

\[
x_t = x_0 + \int_0^t b(x_s) \, ds + \sum_{k=1}^m \int_0^t \sigma_k(x_s) \, dw_s^k
\]

and assume that the coefficients \(b\) and \(\sigma_k, 1 \leq k \leq m\), are Lipschitz functionals mapping \(\mathbb{R}^n\) into \(\mathbb{R}^{n \times n}\) such that

1. \(b(0) = 0\) and \(\sigma_k(0) = 0\) for all \(k \in \{1, \ldots, m\}\).
2. There exists a non-negative constant \(K\) such that

\[
|b(x)| + \sum_{k=1}^m |\sigma_k(x)| \leq K(1 + |x|)
\]

for every \(x\) in \(\mathbb{R}^n\).

Furthermore, for any \(s \geq 0\) and \(x \in \mathbb{R}^n\), denote by \(x_t^{s, x}\), \(s \leq t\), the solution at time \(t\) of the stochastic differential equation (4) starting from the state \(x\) at time \(s\).

Then, one can introduce the notion of stochastic stability we are dealing with as follows.

**Definition 2.1.** The equilibrium solution \(z_t \equiv 0\) of the stochastic differential equation (4) is said to be exponentially stable in mean square if there exist positive constants \(c_1\) and \(c_2\) such that

\[
E|z_t^{s, x}|^2 \leq c_1|x|^2 e^{-c_2(t-s)}.
\]

**Remark 2.2.** Note that in the case \(\sigma_k \equiv 0\), \(1 \leq k \leq m\), Definition 2.1 reduces to the usual definition of global exponential stability.

Denote by \(L\) the infinitesimal generator associated with the stochastic differential equation (4), that is, \(L\) is the second order differential operator defined for any function \(\psi\) in \(C^2(\mathbb{R}^n)\) by

\[
L\psi(x) = \sum_{i=1}^n b^i(x) \frac{\partial \psi}{\partial x_i}(x) + \frac{1}{2} \sum_{i,j=1}^n a^{ij}(x) \frac{\partial^2 \psi}{\partial x_i \partial x_j}(x),
\]

where \(a^{ij}(x) = \sum_{k=1}^m \sigma_k^i(x)\sigma_k^j(x), 1 \leq i, j \leq n\).

Then, the following theorem gives sufficient conditions in terms of Lyapunov function for the exponential stability in mean square for the equilibrium solution of the stochastic differential equation (4).

**Theorem 2.3.** (c.f. [10]) The equilibrium solution \(z_t \equiv 0\) of the stochastic differential equation (4) is exponentially stable in mean square if there exist a Lyapunov function \(V\) defined in \(\mathbb{R}^n\) (i.e., a proper function \(V\) of class \(C^2\) mapping \(\mathbb{R}^n\) into \(\mathbb{R}\) which is positive definite) and three positive constants \(\alpha_1, \alpha_2\) and \(\alpha_3\) such that

\[
\alpha_1|x|^2 \leq V(x) \leq \alpha_2|x|^2
\]

\[
LV(x) \leq -\alpha_3|x|^2
\]

for any \(x \in \mathbb{R}^n\).
Moreover, one can prove the following converse theorem which gives necessary conditions for exponential stability in mean square of the equilibrium solution of the stochastic differential equation (4).

**THEOREM 2.4.** (c.f. [10]) If the equilibrium solution $x_t = 0$ of the stochastic differential equation (4) is exponentially stable in mean square and the coefficients $b$ and $\sigma_k$, $1 \leq k \leq m$, have continuous bounded derivatives up to order two, then there exist a Lyapunov function $V$ defined in $\mathbb{R}^n$ and five positive constants $\alpha_i$, $1 \leq i \leq 5$, such that

$$
\alpha_1|x|^2 \leq V(x) \leq \alpha_2|x|^2 \\
LV(x) \leq -\alpha_3|x|^2 \\
|\nabla V(x)| \leq \alpha_4|x| \text{ and } |\nabla^2 V(x)| \leq \alpha_5
$$

for any $x \in \mathbb{R}^n$.

**REMARK 2.5.** In the case $\sigma_k \equiv 0$, $1 \leq k \leq m$, these two theorems reduce to the well-known Lyapunov theorem for global exponential stability.

In the following, we propose sufficient conditions for the stabilization, by means of feedback law, of composite linear stochastic systems.

### 3. STATEMENT OF THE PROBLEM

The aim of this section is to introduce the class of composite linear stochastic systems we are dealing with in this paper. Consider the pair of stochastic processes $(x_t, \xi_t) \in \mathbb{R}^n \times \mathbb{R}^p$ solution of the multi-inputs composite partially linear stochastic differential system written in the sense of Itô,

$$
\begin{align*}
\dot{x}_t &= x_0 + \int_0^t (f(x_s) + G(x_s, \xi_s)\xi_s) \, ds + \int_0^t g(x_s) \, dw_s \\
\xi_t &= \xi_0 + \int_0^t (A\xi_s + Bu) \, ds
\end{align*}
$$

where

1. $x_0$ and $\xi_0$ are given in $\mathbb{R}^n$ and $\mathbb{R}^p$, respectively.
2. $f$ and $g$ are $C^2$ functionals mapping $\mathbb{R}^n$ into $\mathbb{R}^n$ and $\mathbb{R}^{n \times p}$, respectively, with bounded derivatives and such that $f(0) = g(0) = 0$.
3. $G$ is a functional mapping $\mathbb{R}^n \times \mathbb{R}^p$ into $\mathbb{R}^{n \times p}$ such that there exists a non-decreasing scalar function $\gamma(|\xi|) \geq 0$ bounded for all bounded $\xi$ such that

$$|G(x, \xi)| \leq \gamma(|\xi|)|x|, \quad \forall (x, \xi) \in \mathbb{R}^n \times \mathbb{R}^p.$$

4. $A$ and $B$ are matrices in $\mathcal{M}_{p \times p}(\mathbb{R})$ and $\mathcal{M}_{p \times q}(\mathbb{R})$, respectively, such that the pair $(A, B)$ is stabilizable.
5. $u$ is a $\mathbb{R}^q$-valued control law.

Furthermore, introduce the notion of stabilizing feedback law for the stochastic differential system (6) as follows.

**DEFINITION 3.1.** A function $u$ mapping $\mathbb{R}^p$ into $\mathbb{R}^q$ such that the equilibrium solution of the closed-loop system

$$
\begin{align*}
\dot{x}_t &= x_0 + \int_0^t (f(x_s) + G(x_s, \xi_s)\xi_s) \, ds + \int_0^t g(x_s) \, dw_s \\
\dot{\xi}_t &= \xi_0 + \int_0^t (A\xi_s + Bu(\xi_s)) \, ds
\end{align*}
$$

is exponentially stable in mean square is said to be a stabilizing feedback law for the stochastic differential system (6).

In the following, we state sufficient conditions on the coefficients for which there exists a linear stabilizing feedback law for the stochastic differential system (6).
4. THE MAIN RESULT

In this section, we state and prove the main result on the stabilization by means of linear feedback laws of the composite partially linear stochastic systems introduced in Section 3. The main result of the paper is stated in the following theorem.

**Theorem 4.1.** If the equilibrium solution $x_t \equiv 0$ of the nonlinear stochastic differential equation

$$x_t = x_0 + \int_0^t f(x_s) \, ds + \int_0^t g(x_s) \, dw_s$$

(8)

is exponentially stable in mean square, then the control law $u$ defined on $\mathbb{R}^p$ by

$$u(\xi) = K\xi,$$

where $K$ is a matrix in $\mathcal{M}_{q \times p}(\mathbb{R})$ such that the matrix $A + BK$ is asymptotically stable (i.e., all the eigenvalues of the matrix $A + BK$ have negative real parts), is a stabilizing feedback law for the stochastic differential system (6).

**Proof of Theorem 4.1.** Since the pair of matrices $(A, B)$ is stabilizable, one knows that there exists a matrix $K$ in $\mathcal{M}_{q \times p}(\mathbb{R})$ such that the matrix $A + BK$ is asymptotically stable.

Then, the closed-loop system deduced from (6) when the control law $u$ is defined on $\mathbb{R}^p$ by

$$u(\xi) = K\xi$$

(9)

reads

$$\begin{cases}
x_t = x_0 + \int_0^t f(x_s) + G(x_s, \xi_s) \xi_s \, ds + \int_0^t g(x_s) \, dw_s \\
\xi_t = \xi_0 + \int_0^t (A\xi_s + BK\xi_s) \, ds.
\end{cases}$$

(10)

In the following, we prove that the equilibrium solution $(x_t, \xi_t) \equiv 0$ of the closed-loop system (10) is exponentially stable in mean square.

Since the equilibrium solution $x_t \equiv 0$ of the stochastic differential equation (8) is exponentially stable in mean square, one can deduce from the converse Lyapunov theorem (Theorem 2.4) that there exist a Lyapunov function $V$ defined on $\mathbb{R}^n$ and five positive constants $\alpha_i$, $1 \leq i \leq 5$, such that

$$\alpha_1|x|^2 \leq V(x) \leq \alpha_2|x|^2$$

(11)

$$LV(x) \leq -\alpha_3|x|^2$$

(12)

$$|\nabla V(x)| \leq \alpha_4|x|$$

(13)

$$|\nabla^2 V(x)| \leq \alpha_5$$

(14)

for all $x \in \mathbb{R}^n$ (in (12), $L$ denotes the infinitesimal generator associated with the stochastic differential equation (8)).

By application of Itô's formula to $V(x_t)$, where $x_t$ is the solution of the stochastic differential equation,

$$x_t = x_0 + \int_0^t (f(x_s) + G(x_s, \xi_s)\xi_s) \, ds + \int_0^t g(x_s) \, dw_s$$

yields

$$dV(x_t) = (LV(x_t) + \nabla V(x_t)G(x_t, \xi_t)) \, dt + M_t$$

(15)

where $M_t$ denotes an $\mathcal{F}_t$-martingale.

Taking the expectation on both sides of equality (15), one has

$$\frac{d}{dt}E(V(x_t)) = E(LV(x_t)) + E(\nabla V(x_t)G(x_t, \xi_t))$$

(16)

where the process $\xi_t$ satisfies

$$|\xi_t| \leq \beta_1 e^{-\beta_2 t}|\xi_0|$$

(17)

where $\beta_1$ and $\beta_2$ are some positive constants.
Taking into account Assumption 3, equations (11), (12), (13) and (17), one obtains from (16)

$$\frac{d}{dt} E(V(x_t)) \leq -\frac{\alpha_3}{\alpha_2} E(V(x_t)) + \frac{\alpha_4}{\alpha_1} \beta_1 \gamma(||\xi_t||) |\xi_0| e^{-\beta_1 t} E(V(x_t)).$$

(18)

Therefore, by Gronwall's lemma, one has

$$E(V(x_t)) \leq K(\xi_0)e^{-\alpha_3/\alpha_2}V(x_0)$$

where

$$K(\xi_0) = \exp\left(\frac{(\alpha_2 \alpha_4 \beta_1^2)}{(\alpha_3 \beta_2)} \gamma(\beta_1 ||\xi_0||) ||\xi_0||\right).$$

Thus, by means of Theorem 2.3, one can deduce that the equilibrium solution \((x_t, \xi_t) \equiv (0,0)\) of the closed-loop system (10) is exponentially stable in mean square. Then, the control law \(u\) defined by (9) is a stabilizing feedback law for the stochastic differential system (6). This concludes the proof of Theorem 4.1.

5. CONCLUSION

Actually, we do not know how to extend this result in the case where both equations (1) and (2) are corrupted by noise. In this case, if one uses the same scheme of proof, equation (18) reads

$$\frac{d}{dt} E(V(x_t)) \leq -\alpha E(V(x_t)) + \beta e^{-\alpha t} \sqrt{E(V(x_t))}$$

where \(\alpha, \beta\) and \(\alpha\) are positive constants, and the latter equation does not lead to estimates as in (19).

REFERENCES