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Existence and Uniqueness of a Global Smooth Solution for the Vlasov–Poisson–Fokker–Planck System in Three Dimensions

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Studying precisely the regularity of the force field created by the solution of the linear Vlasov–Fokker–Planck equation, we prove existence and uniqueness of a smooth solution to the Vlasov–Poisson–Fokker–Planck system in three dimensions. The attractive and repulsive cases are treated. The method involves an analysis of the effect of the Fokker–Planck kernel, without any force field, on non-linear expressions coming from the Vlasov term. © 1993 Academic Press, Inc.

I. INTRODUCTION

The linear Vlasov–Fokker–Planck equation is

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f + \operatorname{div}_v (E - \beta v) f - \sigma \Delta_v f = 0 \qquad \text{in} \quad]0, \infty[\times \mathbb{R}^{2N}, \tag{1}$$

$f(t, x, v)$, $t \geqslant 0$, $x, v \in \mathbb{R}^N$ is the density of particles of a fluid (x is the position and v the velocity), $E(t, x)$ is the force field acting on particles. The strictly positive parameters β and σ model a certain type of interaction between particles. Equation (1) is completed by an initial datum

$$f(0, \cdot) = f_0. \tag{2}$$

Given a field E with some specific regularity, there exists a unique solution f of (1) and (2), from which we define the induced field $\tilde{E}$ created by the particles themselves

$$\tilde{E}(t, x) = \frac{1}{|S^{N-1}|} \frac{x}{|x|^N} \underset{x}{*} \rho(t, x), \qquad \text{where} \quad \rho(t, x) = \int_{\mathbb{R}^N} f(t, x, v)\, dv. \tag{3}$$

To solve the Vlasov–Poisson–Fokker–Planck system means to find

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simultaneously f and E satisfying (1), (2), (3), and the supplementary relation

$$E = \omega \tilde{E}. \tag{4}$$

The sign $\omega = +1$ corresponds to the case of repulsive charged particles, $\omega = -1$ corresponds to the case of attractive massive particles. The equations (1), (2), (3), (4) form a non-linear system.

For this system, we are interested in the existence, the uniqueness, and the regularity of the solution (for f, E, and ρ), essentially in terms of Sobolev spaces. The first kind of regularity studied is the time conservation of some bounds on f, assuming that they hold for f_0. For example, it is easy to prove that the total mass $\iint f(t, x, v)\, dx\, dv$ is conserved, as well as the sign of f_0 which is assumed non-negative. The maximum principle gives also $\|f(t, \cdot)\|_\infty \leqslant \|f_0\|_\infty e^{N\beta t}$. We prove in Proposition 2 that, roughly speaking, $\rho(t, \cdot)$ belongs to L^p if $\rho(0, \cdot)$ does, as soon as we know some bound on $E(t, \cdot)$ in L^r for some $r > 2N$. The second kind of regularity we are interested in is the boundedness of E (in L^∞_x), locally in time. For these two studies, we take as a reference the case of the free transport equation instead of (1). In this perspective we define the field

$$E_0(t, x) = \frac{1}{|S^{N-1}|} \frac{x}{|x|^N} \ast_x \int_{\mathbb{R}^N} f_0(x - tv, v)\, dv. \tag{5}$$

The formulae we establish in Section II can be useful for some different purposes, as obtaining some gained regularity on ρ, E for positive time, consequence of the velocity Laplacian term. This study will be the object of another paper [2].

As soon as we know that E is bounded in L^∞_x locally in time, we deduce readily many other bounds on the solution. This is classical for similar systems as the Vlasov–Poisson system (i.e., when $\sigma = \beta = 0$), and was proved by E. Hörst [10]. Because of this property, we call "strong solutions" solutions such that E is bounded locally in time, and "weak solutions" all other solutions (in some cases we only know that $E \in L^\infty_{\mathrm{loc}}([0, \infty[, L^2(\mathbb{R}^N)))$. H. D. Victory and B. P. O'Dwyer proved in [15] the local existence of a strong solution for (1), (2), (3), (4). One can also refer to K. Dressler [7], and to H. Neunzert, M. Pulvirenti, and L. Triolo [12]. On the other hand the methods introduced by E. Hörst [10] and R. J. DiPerna and P. L. Lions [4, 5] allow us to prove the existence of weak but global in time solutions, under very weak assumptions on f_0 (when $N > 3$, $\omega = +1$ is dictated). P. Degond proved in [3] the global existence of strong solutions in dimension $N = 1$ or 2. Stationary solutions have been studied by K. Dressler in [6].

The first aim of this paper is to prove the existence and the uniqueness of a strong and global in time solution for (1), (2), (3), (4) in dimension

$N=3$ (in each of the repulsive and attractive cases). Our method has been strongly inspired by the paper of P. L. Lions and B. Perthame [11] concerning regularity for the Vlasov–Poisson system in three dimensions. Here the estimates are rougher, although formulae are more complicated. We also obtain uniqueness with weak assumptions on f_0 (no derivative of f_0 is needed).

K. Pfaffelmöser has also studied regularity for the Vlasov–Poisson system and has obtained in [13] some results with a method differing from the one introduced by P. L. Lions and B. Perthame, but which seems not adjustable to our case.

Our main result is the following, obtained combining Propositions 5 and 6.

THEOREM 1. *Assume $N=3$ and $\omega=\pm 1$.*
Given $f_0 \in L^1 \cap L^\infty(\mathbb{R}^6)$, $f_0 \geqslant 0$ satisfying

$$\exists m > 6 \iint_{\mathbb{R}^6} |v|^m f_0(x, v)\, dx\, dv < \infty, \tag{6}$$

there exists one and only one couple (f, E) with

$$f \in C([0, \infty[, L^1(\mathbb{R}^6)),$$

$$\sup_{0 \leqslant t \leqslant T} \|E(t, \cdot)\|_\infty < \infty \qquad \textit{for every} \quad T > 0,$$

solution of (1), (2), (3), (4).

Let us point out that with the indicated regularity, all equations make sense (for example, (1) holds in $\mathscr{D}'(]0, \infty[\times \mathbb{R}^6)$). Note also that it is easy to construct classical solutions, using the estimates developed in the proof of Theorem 1.

The main results of this paper were announced in F. Bouchut [1].

The paper is organized as follows. In Section II we study regularity for the linear Vlasov–Fokker–Planck equation (1), (2); in Section III we prove the existence part of Theorem 1; in Section IV we prove uniqueness; and in Section V we present generalizations, additional properties of solutions, and open problems.

II. REGULARITY OF THE DENSITY AND THE INDUCED FIELD FOR THE VLASOV–FOKKER–PLANCK EQUATION

In this section we fix a force field $E(t, x)$ and an initial datum f_0 such that

$$E \in L^\infty(]0, T[\times \mathbb{R}^N) \qquad \text{for every} \quad T > 0, \tag{7}$$

$$f_0 \in L^1(\mathbb{R}^{2N}), \qquad f_0 \geqslant 0. \tag{8}$$

For these data there exists a unique $f \in C([0, \infty[, L^1(\mathbb{R}^{2N}))$ solution of (1), (2). In addition, f is non-negative. We are going to extend the regularity of ρ and $\tilde{E}$ defined in (3). Of course, every result about ρ gives another one about $\tilde{E}$ (roughly speaking, $\tilde{E}$ has one more derivative than ρ). But using integral formulae, and the representation method introduced by P. L. Lions and B. Perthame in [11], which is here extended by introducing the "ahead of time" averages

$$\rho_\lambda(t, x) = \int_{\mathbb{R}^N} f(t, x - \lambda v, v)\, dv, \qquad \lambda \geqslant 0, \tag{9}$$

we are going to prove that $\tilde{E}$ is the sum of a term $\tilde{E}^1$ exclusively depending upon f_0 and term $\tilde{E}^2$ which has a better L^p regularity than the one obtained by the Sobolev injection relative to the known L^p regularity of ρ. More precisely, $\tilde{E}^2$ has nearly "$\frac{4}{3}$ more derivatives" than ρ.

II.1. *Some Convenient Formulation for* (1), (2), (3)

In order to write as simply as possible the formulae we need, we are going to introduce the Green function $G(t, x, v, \xi, \gamma)$ associated to (1), solution of

$$\begin{aligned} &\frac{\partial G}{\partial t} + v \cdot \nabla_x G - \beta\, \mathrm{div}_v(vG) - \sigma\, \Delta_v G = 0, \\ &\qquad\qquad G(0, x, v, \xi, \gamma) = \delta(x - \xi, v - \gamma), \end{aligned} \tag{10}$$

which is written

$$G(t, x, v, \xi, \gamma) = G_0(t, x - \xi - \gamma(1 - e^{-\beta t})/\beta, v - e^{-\beta t}\gamma), \tag{11}$$

$$G_0(t, x, v) = \frac{1}{(4\pi\sigma)^N D(t)^{N/2}} \exp -\frac{1}{4\sigma} \Phi_0(t, x, v), \tag{12}$$

$$\Phi_0(t, x, v) = \frac{1}{D(t)} \int_0^t \left| \frac{1 - e^{-\beta s}}{\beta} v - e^{-\beta s} x \right|^2 ds, \tag{13}$$

$$D(t) = \frac{1}{\beta^2} \left[\frac{1 - e^{-2\beta t}}{2\beta} t - \left(\frac{1 - e^{-\beta t}}{\beta} \right)^2 \right]. \tag{14}$$

Here we use some new formulae giving the function G, particularly suitable for convolution. They are obtained after a Fourier transformation in (x, v), and an integration of (10) along characteristics. For more details, one can refer to L. Hörmander [8], in which the following formula for $\hat{G}$ is proved

$$\hat{G}(t, \delta, \eta, \xi, \gamma) = \frac{1}{(2\pi)^N} e^{-i\delta(\xi + \gamma(1 - e^{-\beta t})/\beta) - i\eta \cdot \gamma e^{-\beta t}} e^{-\sigma\psi(t, \delta, \eta)} \tag{15}$$

$$\psi(t, \delta, \eta) = \int_0^t \left| \frac{1 - e^{-\beta s}}{\beta} \delta + e^{-\beta s} \eta \right|^2 ds. \tag{16}$$

The equations (1) and (2) are reformulated as follows, in accordance with the idea of P. L. Lions and B. Perthame

$$\begin{aligned} f(t, x, v) &= \iint_{\xi, \gamma \in \mathbb{R}^N} G(t, x, v, \xi, \gamma) f_0(\xi, \gamma)\, d\xi\, d\gamma \\ &\quad + \iiint_{\substack{0 < s < t \\ \xi, \gamma \in \mathbb{R}^N}} \nabla_\gamma G(s, x, v, \xi, \gamma)\, E(t-s, \xi)\, f(t-s, \xi, \gamma)\, d\xi\, d\gamma\, ds \\ &= f^1 + f^2. \end{aligned} \tag{17}$$

By a fixed point argument, it is easy to prove that (17) admits a unique solution $f \in C([0, \infty[, L^1(\mathbb{R}^{2N}))$, and that with this regularity, (17) is equivalent to (1) and (2). Equation (17) given the natural idea to separate f in two parts, the first one exclusively depending upon the initial datum f_0, whereas the second one essentially depends upon the field E. This second part is indeed more regular than the first one.

The index i taking the values 1 or 2, the parameter λ staying non-negative, we define the quantities

$$\rho^i_\lambda(t, x) = \int_{\mathbb{R}^N} f^i(t, x - \lambda v, v)\, dv, \tag{18}$$

$$\tilde{E}^i(t, x) = \frac{1}{|S^{N-1}|} \frac{x}{|x|^N} \underset{x}{*} \rho^i, \tag{19}$$

$$M_\lambda(t, x) = \int_{\mathbb{R}^N} E(t, x - \lambda v)\, f(t, x - \lambda v, v)\, dv, \tag{20}$$

$$\mu_\lambda(t) = \frac{1 - e^{-\beta t}}{\beta} + \lambda e^{-\beta t}, \tag{21}$$

$$d_\lambda(t) = \int_0^t \mu_\lambda(\tau)^2\, d\tau. \tag{22}$$

By convention, the missing index λ means that $\lambda = 0$. Of course we have

$$\rho_\lambda = \rho^1_\lambda + \rho^2_\lambda, \tag{23}$$

$$\tilde{E} = \tilde{E}^1 + \tilde{E}^2. \tag{24}$$

A rather tedious computation gives the formulae

$$\int_{\mathbb{R}^N} G(t, x - \lambda v, v, \xi, \gamma)\, dv = \frac{1}{(2\sigma\, d_\lambda(t))^{N/2}} \mathcal{N}\left(\frac{x - \xi - \mu_\lambda(t)\gamma}{\sqrt{2\sigma\, d_\lambda(t)}}\right)$$

with

$$\mathcal{N}(x)=\frac{1}{(2\pi)^{N/2}}\,e^{-|x|^2/2}, \tag{25}$$

$$\int_{\mathbb{R}^N}\nabla_\gamma G(t,x-\lambda v,v,\xi,\gamma)\,dv=\frac{-\mu_\lambda(t)}{(2\sigma\,d_\lambda(t))^{(N+1)/2}}\,\nabla\mathcal{N}\left(\frac{x-\xi-\mu_\lambda(t)\gamma}{\sqrt{2\sigma\,d_\lambda(t)}}\right). \tag{26}$$

The following estimate will also be useful for the uniqueness result in Section IV:

$$\int_{\mathbb{R}^N}|\nabla_\gamma G(t,x-\lambda v,v,\xi,\gamma)|\,dv\leqslant\frac{C_N}{\sqrt{\sigma t}}\,\frac{1}{(2\sigma\,d_\lambda(t))^{N/2}}\,\mathcal{M}\left(\frac{x-\xi-\mu_\lambda(t)\gamma}{\sqrt{2\sigma\,d_\lambda(t)}}\right)$$

with

$$\mathcal{M}(x)=\sqrt{1+|x|^2}\,\frac{e^{-|x|^2/2}}{(2\pi)^{N/2}}. \tag{27}$$

Thanks to (25), we obtain for the first term f^1 implicitly defined in (17)

$$\rho_\lambda^1(t,x)=\frac{1}{(4\pi\sigma\,d_\lambda(t))^{N/2}}\,e^{-|x|^2/4\sigma\,d_\lambda(t)}\underset{x}{*}\int_{\mathbb{R}^N}f_0(x-\mu_\lambda(t)\,v,v)\,dv, \tag{28}$$

$$\tilde{E}^1(t,x)=\frac{1}{(4\pi\sigma\,d(t))^{N/2}}\,e^{-|x|^2/4\sigma\,d(t)}\underset{x}{*}E_0(\mu(t),x). \tag{29}$$

The field E_0 has been defined in (5). For the second term, we obtain from (17)

$$\rho_\lambda^2(t,x)=\int_0^t\frac{-\mu_\lambda(s)}{(2\sigma\,d_\lambda(s))^{(N+1)/2}}\,\nabla\mathcal{N}\left(\frac{x}{\sqrt{2\sigma\,d_\lambda(s)}}\right)\underset{x}{*}M_{\mu_\lambda(s)}(t-s,x)\,ds, \tag{30}$$

$$\tilde{E}^2(t,x)=\int_0^t\frac{\mu(s)}{(2\sigma\,d(s))^{N/2}}\,A\left(\frac{x}{\sqrt{2\sigma\,d(s)}}\right)\underset{x}{*}M_{\mu(s)}(t-s,x)\,ds \tag{31}$$

with

$$A_{jk}(x)=\frac{\partial^2}{\partial x_j\,\partial x_k}(-\Delta)^{-1}\mathcal{N}(x),\qquad 1\leqslant j,k\leqslant N.$$

Formula (29) is a good illustration of the two points of view for our study. The field $\tilde{E}^1$ behaves like E_0, but the convolution by a gaussian which spreads with time regularizes this reference field E_0.

II.2. *Estimates in Integral Formulae*

Let us now come to a concrete use of (30) and (31). In order to bound the L^p norm of these expressions, we are going to use the Minkowsky inequality, which enables us to take the norm under the integral, and an estimate for the convolution product.

First, let us adopt the following convention. For $0 < \alpha \leqslant 1$, the space of α-Hölderian functions is the space of all real functions u, such that

$$\|u\|_{C^{0,\alpha}} \equiv \operatorname*{ess\,sup}_{x \neq y} \frac{|x(x) - u(y)|}{|x-y|^\alpha} < \infty.$$

We set $\|u\|_p = \|u\|_{C^{0,\alpha}}$ the associated semi-norm, with $1/p = -\alpha/N$, and this space will also be denoted $C^{0,\alpha}(\mathbb{R}^N) = L^p(\mathbb{R}^N)$.

We recall two standard estimates for convolution:

(1) If $p, l \in [1, \infty]$ and $1/q \equiv 1/p - 1/l \geqslant 0$ we have

$$\|u * a\|_q \leqslant \|u\|_p \, \|a\|_{l'}. \tag{32}$$

(2) If $1 \leqslant p \leqslant \infty$, $1 \leqslant l < \infty$, $1/q \equiv 1/p - 1/l \in \,]-1/N, 0[$,

$$a \in C^1(\mathbb{R}^N \setminus \{0\}), \qquad |a(x)| \leqslant \frac{C_0}{|x|^{N/l'}}, \qquad |\nabla a(x)| \leqslant \frac{C_1}{|x|^{1+N/l'}},$$

then for every $u \in L^p(\mathbb{R}^N)$, $u * a \in C^{0,\alpha}$ with $-\alpha/N = 1/q$ and

$$\|u * a\|_{C^{0,\alpha}} \leqslant C(N, p, l)(C_0 + C_1) \, \|u\|_p. \tag{33}$$

Here, $u * a$ is defined up to a constant by the formula

$$u * a(x) = \int_{\mathbb{R}^N} (a(x-y) - a(-y)) \, u(y) \, dy.$$

We refer to L. Hörmander [9] for this inequality.

We are now going to give the simplest bound on $\|\tilde{E}\|_q$, which will be useful in proving Proposition 4. In this bound we use the quantity $\sup_{\lambda \geqslant 0} \|\rho_\lambda(t, \cdot)\|_p$. We point out that it is dominated by some v moment on f. More precisely, if $\iint |v|^m f_0(x, v) \, dx \, dv < \infty$ for some $m \geqslant 0$, and $f_0 \in L^\infty(\mathbb{R}^{2N})$, then

$$\int_{\mathbb{R}^N} f_0(x, v) \, dv \in L^{(m+N)/N}(\mathbb{R}^N).$$

It is a consequence of the inequality

$$\begin{aligned} \int f_0(x, v) \, dv &\leqslant \|f_0\|_\infty \int_{|v| \leqslant R} dv + \frac{1}{R^m} \int_{|v| > R} |v|^m f_0(x, v) \, dv \\ &\leqslant C_N \, \|f_0\|_\infty \, R^N + \frac{1}{R^m} \int |v|^m f_0(x, v) \, dv. \end{aligned}$$

If we choose R to minimize the right hand side, we find

$$\int f_0(x, v)\, dv \leqslant C_N \|f_0\|_\infty^{m/(m+N)} \left(\int |v|^m f_0(x, v)\, dv\right)^{N/(m+N)},$$

$$\left\| \int_{\mathbb{R}^N} f_0(x, v)\, dv \right\|_{(m+N)/N} \leqslant C_N \|f_0\|_\infty^{m/(m+N)} \left(\iint_{\mathbb{R}^{2N}} |v|^m f_0(x, v)\, dx\, dv\right)^{N/(m+N)}. \tag{34}$$

Notice also that we have a bound independent of λ

$$\left\| \int_{\mathbb{R}^N} f_0(x - \lambda v, v)\, dv \right\|_{(m+N)/N} \leqslant C_N \|f_0\|_\infty^{m/(m+N)} \left(\iint_{\mathbb{R}^{2N}} |v|^m f_0(x, v)\, dx\, dv\right)^{N/(m+N)}.$$

PROPOSITION 1. *Assume $N \geqslant 2$ and let f_0 belong to $L^\infty(\mathbb{R}^{2N})$ with the conditions* (7), (8) *and*

$$K_r(T) \equiv \operatorname*{ess\,sup}_{0 \leqslant t \leqslant T} \|E(t, \cdot)\|_r < \infty \qquad \textit{for some} \quad r \in \left]\frac{N}{2}, \infty\right], \tag{35}$$

$$S_p(T) \equiv \sup_{\substack{\lambda \geqslant 0 \\ 0 \leqslant t \leqslant T}} \|\rho_\lambda(t, \cdot)\|_p < \infty \qquad \textit{for some} \quad p \in [1, \infty[. \tag{36}$$

Then for every $q \in [1, \infty]$ satisfying

$$\frac{1}{pr'} - \frac{2}{3}\left(\frac{2}{N} - \frac{1}{r}\right) < \frac{1}{q} < \frac{1}{pr'} \tag{37}$$

the following regularity holds

$$\sup_{0 \leqslant t \leqslant T} \|\tilde{E}^2(t, \cdot)\|_q \leqslant C(r, p, q, N, \beta, \sigma, T)\, \|f_0\|_\infty^{1/r} K_r(T)\, S_p(T)^{1/r'}. \tag{38}$$

Proof. First, we have the following elementary estimate on M_λ

$$\forall \lambda > 0\ \forall r \in [1, \infty]\ |M_\lambda(t, x)| \leqslant \frac{1}{\lambda^{N/r}} \|E(t, \cdot)\|_r \|f(t, \cdot)\|_\infty^{1/r} \times \left(\int_{\mathbb{R}^N} f(t, x - \lambda v, v)\, dv\right)^{1/r'}, \tag{39}$$

as well as

$$\forall t \geqslant 0\ \|f(t, \cdot)\|_\infty \leqslant e^{N\beta t} \|f_0\|_\infty. \tag{40}$$

Using these inequalities in (31) we obtain

$$|\tilde{E}^2| \leqslant K_r(\|f_0\|_\infty e^{N\beta t})^{1/r} \int_0^t \frac{\mu(s)^{1-N/r}}{(2\sigma\, d(s))^{N/2}} \times \left| A\left(\frac{x}{\sqrt{2\sigma\, d(s)}}\right)\right| \underset{x}{*} \rho_{\mu(s)}(t-s, x)^{1/r'}\, ds. \tag{41}$$

Setting $1/l = 1/pr' - 1/q \in]0, 1[$, we find using (32) and Minkowsky's inequality

$$\|\tilde{E}^2(t,\cdot)\|_q \leqslant K_r(\|f_0\|_\infty e^{N\beta t})^{1/r} \int_0^t \frac{\mu(s)^{1-N/r}}{(2\sigma\, d(s))^{N/2l}} \|A\|_{l'} S_p^{1/r'}\, ds. \tag{42}$$

Since

$$\mu(s) \underset{s\to 0}{\sim} s, \qquad d(s) \underset{s\to 0}{\sim} \frac{s^3}{3} \qquad \text{and} \qquad 1 - \frac{N}{r} - \frac{3N}{2l} > -1, \tag{43}$$

the above integral converges, and we obtain (38) (it is well known that the matrix A belongs to L^k for every $k \in]1, \infty[$). ∎

Remarks. (1) In the case $r = \infty$, the assumption $f_0 \in L^\infty$ is not really needed.

(2) Using the convolution inequality (33) instead of (32), we could obtain some estimate similar to (38), but with $1/q < 0$.

(3) With the help of formula (29), it is easy to obtain bounds on $\|\tilde{E}^1\|_q$ with suitable conditions on f_0.

(4) To understand the meaning of Proposition 1, let us consider the case $r = \infty$. If "$\rho \in L^p$" (which is more or less like (36)), we obtain $\tilde{E}^2 \in L^q$ for q such that $1/q$ is close to $1/p - 4/3N$. This represents in terms of Sobolev imbeddings a gain of 4/3 of derivative over ρ, although by its definition, $\tilde{E}$ has a priori just one more derivative than ρ! When r is small, this gain is unfortunately little, or even vanishes. Nevertheless, Proposition 1 can be useful to conclude by a boot-strap argument, as we will see later. Indeed when r is small, the factor $S_p^{1/r'}$ in (38) is less important.

Let us now give the second main estimate. Now, we bound ρ instead of $\tilde{E}$.

PROPOSITION 2. *Assume* (7), (8) *and let* f_0 *belong to* $L^\infty(\mathbb{R}^{2N})$. *If*

$$K_r(T) \equiv \operatorname*{ess\,sup}_{0\leqslant t\leqslant T} \|E(t,\cdot)\|_r < \infty \qquad \textit{for some} \quad r \in]2N, \infty], \tag{44}$$

$$Q_p \equiv \sup_{\lambda \geqslant 0} \left\| \int_{\mathbb{R}^N} f_0(x - \lambda v, v)\, dv \right\|_p < \infty \qquad \textit{for some} \quad p \in [1, \infty] \tag{45}$$

then

$$S_p(T) \equiv \sup_{\substack{\lambda \geq 0 \\ 0 \leq t \leq T}} \|\rho_\lambda(t, \cdot)\|_p < \infty, \tag{46}$$

and when the parameters $p_0, l \in [1, \infty]$ *satisfy*

$$0 \leq \frac{1}{l} < \frac{1}{3N} - \frac{2}{3}\frac{1}{r}, \qquad \frac{1}{p} \leq \frac{1}{p_0}, \qquad \frac{1}{p_0 r} + \frac{1}{l} > 0, \tag{47}$$

we have the estimate

$$S_p(T) \leq C(r, p_0, p, l)\, Q_p + C(r, p_0, p, l, N, \beta, \sigma, T)\, S_{p_0}(T)^\gamma (K_r(T)\, \|f_0\|_\infty^{1/r})^\delta, \tag{48}$$

with

$$\gamma = \frac{1/pr + 1/l}{1/p_0 r + 1/l} \qquad \textit{and} \qquad \delta = \frac{1/p_0 - 1/p}{1/p_0 r + 1/l}.$$

Similarly, when $1/p \in \,](2/3)(1/r) - 1/3N, 0[$ *(with the conventions stated before on* $C^{0,\alpha}$ *spaces), this remains true, under the additional condition*

$$\frac{1}{p} + \frac{1}{l} \geq 0. \tag{49}$$

Proof. We denote

$$S_k^{(i)} = \sup_{\substack{\lambda \geq 0 \\ 0 \leq t \leq T}} \|\rho_\lambda^i(t, \cdot)\|_k \qquad (\leq \infty). \tag{50}$$

We clearly have

$$S_k \leq S_k^{(1)} + S_k^{(2)}, \qquad S_k^{(1)} \leq S_k \tag{51}$$

(thanks to (28), $\|\rho_\lambda^1(t, \cdot)\|_k \leq \|\rho_{\mu_\lambda(t)}(0, \cdot)\|_k$),

$$S_k^{(2)} \leq 2S_k, \qquad S_1 \leq \|f_0\|_1 < \infty. \tag{52}$$

We want to prove that $S_p < \infty$, and since $S_p^{(1)} \leq Q_p$, we must bound $S_p^{(2)}$. Let us estimate (30) with (39) and (40). We find

$$|\rho_\lambda^2| \leq K_r (e^{N\beta T} \|f_0\|_\infty)^{1/r} \int_0^t \frac{\mu_{\lambda(s)} 1 - N/r}{(2\sigma\, d_\lambda(s))^{(N+1)/2}}$$

$$\times \left| \nabla \mathcal{N}\left(\frac{x}{\sqrt{2\sigma\, d_\lambda(s)}}\right)\right| \underset{x}{*} \rho_{\mu_\lambda(s)}(t-s, x)^{1/r'}\, ds. \tag{53}$$

First Step. We prove an inequality with

$$0 \leqslant \frac{1}{l} < \frac{1}{3N} - \frac{2}{3}\frac{1}{r}, \qquad r' \leqslant k \leqslant \infty, \qquad \frac{1}{q} = \frac{1}{k} - \frac{1}{l}, \tag{54}$$

if $S_{k/r'} < \infty$ then $S_q^{(2)} < \infty$ and

$$S_q^{(2)} \leqslant C(N, r, l, k, \beta, \sigma, T)\, K_r\, \|f_0\|_\infty^{1/r}\, S_{k/r'}^{1/r'}. \tag{55}$$

To prove it, we use the convolution inequality (32) in (53). We obtain in the case $1/q \geqslant 0$

$$\|\rho_\lambda^2(t,\cdot)\|_q \leqslant K_r (e^{N\beta t}\, \|f_0\|_\infty)^{1/r}\, S_{k/r'}^{1/r'}\, \|\nabla A\|_{l'} \times \int_0^t \frac{\mu_\lambda(s)^{1-N/r}}{(2\sigma\, d_\lambda(s))^{1/2+(N/2)(1-1/l')}}\, ds. \tag{56}$$

With the condition over l, the integral converges. We deduce (55) with the following lemma (whose proof is left to the reader), which ensures that the above integral is bounded independently of λ.

LEMMA 1. *When $\alpha \geqslant 0$, $\theta \leqslant 2\alpha$, $s > 0$, $\mu_\lambda(s)^\theta / d_\lambda(s)^\alpha$ is a non-increasing function of $\lambda \in [0, \infty[$.*

The inequality (55) is now proved in the case $1/q \geqslant 0$. In the case $1/q < 0$ we write an x-translation in (30) and use the convolution inequality (33). The remainder is identical to the case $1/q \geqslant 0$.

Second Step. Obtaining bounds.

Let us start from $S_1 < \infty$. By iteration and using (55), we obtain more and more regularity until $S_p < \infty$. Now assume that l and p_0 are given and satisfy the conditions (47), (49). We are going to prove (48). The case $p_0 = p$ is trivial.

First Case. Assume $1/r'p_0 - 1/l \leqslant 1/p$ with $1/p < 1/p_0$.

Let $k = r'p_0$. Inequality (55) gives $1/q = 1/r'p_0 - 1/l \leqslant 1/p$, and

$$S_q^{(2)} \leqslant C K_r\, \|f_0\|_\infty^{1/r}\, S_{p_0}^{1/r'}.$$

By interpolation we have also

$$S_p^{(2)} \leqslant C_N S_{p_0}^{(2)(1-\Theta)} S_q^{(2)\Theta},$$

with

$$\Theta = \frac{1/p_0 - 1/p}{1/p_0 - 1/q},$$

and thus

$$S_p \leqslant Q_p + 2C_N (CK_r \|f_0\|_\infty^{1/r})^\Theta S_{p_0}^{1-\Theta+\Theta/r'}.$$

Second Case. Assume $1/r'p_0 - 1/l > 1/p$ with $1/p < 1/p_0$. Let $1/k = 1/l + 1/p < 1/r'p_0$. We obtain by (55)

$$S_p^{(2)} \leqslant CK_r \|f_0\|_\infty^{1/r} S_{k/r'}^{1/r'}.$$

But by interpolation

$$S_{k/r'} \leqslant C_N S_{p_0}^{1-\Theta} S_p^\Theta,$$

with

$$\Theta = \frac{1/p_0 - r'/k}{1/p_0 - 1/p},$$

thus

$$S_p \leqslant Q_p + CK_r \|f_0\|_\infty^{1/r} S_{p_0}^{(1-\Theta)/r'} S_p^{\Theta/r'},$$

and then

$$S_p \leqslant \frac{1}{1-\Theta/r'} Q_p + (CK_r \|f_0\|_\infty^{1/r} S_{p_0}^{(1-\Theta)/r'})^{1/(1-\Theta/r')}. \quad \blacksquare$$

Remarks. (1) Proposition 2 shows the conservation of the L^p regularity of ρ, as well as the $C^{0,\alpha}$ regularity if r is large enough (if $r = \infty$, we get $0 < \alpha < 1/3$).

(2) Condition (44) with $r > 2N$ is essential. Indeed, if $r \leqslant 2N$, the integral in (56) never converges, for any choice of p and l.

Let us now give a final result which supplements Proposition 1. For simplicity, we restrict it to the case $r = \infty$.

PROPOSITION 3. *With the assumptions* (7) *and* (8), *if* $S_p(T) < \infty$ *for some* $p \in [1, \infty[$ *then for every* q *such that*

$$\frac{1}{p} - \frac{1}{3N} < \frac{1}{q} < \frac{1}{p}$$

($1/q$ *is allowed to be negative*) *we have*

$$\sup_{0 \leqslant t \leqslant T} \left\| \frac{\partial \tilde{E}^2}{\partial x_j}(t, \cdot) \right\|_q \leqslant C(p, q, N, \beta, \sigma, T) \operatorname*{ess\,sup}_{0 \leqslant t \leqslant T} \|E(t, \cdot)\|_\infty S_p(T). \quad (57)$$

The proof is again based on the convolution inequalities (32) and (33), applied to the formula

$$\frac{\partial \tilde{E}^2}{\partial x_j} = \int_0^t \frac{\mu(s)}{(2\sigma\, d(s))^{(N+1)/2}} \frac{\partial A}{\partial x_j}\left(\frac{x}{\sqrt{2\sigma\, d(s)}}\right) \underset{x}{*} M_{\mu(s)}(t-s, x)\, ds, \tag{58}$$

using that $\partial A/\partial x_j \in L^l$ for every $l \in\,]1, \infty[$. Notice that this amounts to gain 1/3 derivative in terms of Sobolev imbeddings.

III. Existence Proof for Theorem 1

We are now interested in solutions of (1), (2), (3), (4). It is well known (see the Introduction) that a weak solution of (1)–(4) is obtained as a limit of an approximate problem where (4) is replaced by

$$E = \omega\zeta \underset{x}{*} \tilde{E}. \tag{59}$$

The function ζ is a mollifier in $\mathbb{R}^N$, and tends to a Dirac mass. Henceforth we will always assume

$$f_0 \in L^1 \cap L^\infty(\mathbb{R}^{2N}), \qquad f_0 \geqslant 0, \qquad \iint_{\mathbb{R}^{2N}} |v|^2 f_0(x, v)\, dx\, dv < \infty. \tag{60}$$

With the additional following conditions, we are sure that the approximation (59) admits a solution (by an easy fixed point theorem) which converges when $\zeta \to \delta_0$.

$$\omega = +1 \qquad \text{if} \quad N \geqslant 4, \tag{61}$$

$$E_0(0, \cdot) \in L^2(\mathbb{R}^N). \tag{62}$$

Notice that condition (62) is not satisfactory in dimension $N = 1$ or 2 because in these cases it cannot be fulfilled unless $f_0 = 0$ (because $\hat{\rho}_0(\xi)/|\xi|$ must belong to $L^2(\mathbb{R}^N_\xi)$).

The regularity of the weak solution obtained is not relevant here, but for the apprximate one (i.e., with (59)), we know that E satisfies (7), and so the results of Section II are applicable. Our method is the following: we prove bounds independent of ζ for the approximate system. The limit will automatically satisfy the same bounds. In this perspective, every bound we will obtain must be understood as independent of ζ.

We now restrict our focus to the most interesting case $N = 3$. The parameter ω can be $+1$ or -1. Notice that in three dimensions, (62) is a consequence of (60). We first use Proposition 1 to show how we can improve the known regularity which is

$$\sup_{0 \leq t \leq T} \|\rho(t, \cdot)\|_q < \infty \qquad \text{for} \quad 1 \leq q \leq \tfrac{5}{3}, \tag{63}$$

$$\sup_{0 \leq t \leq T} \|E(t, \cdot)\|_p < \infty \qquad \text{for} \quad \tfrac{3}{2} < p \leq \tfrac{15}{4}. \tag{64}$$

This comes from the energy equation stated in Section V, and from (34). As it was pointed out in Proposition 2 the key point in the next results is to gain the L^6 regularity for E.

PROPOSITION 4. *Assume that $N=3$, f_0 satisfies* (60), *and*

$$\exists p \in]\tfrac{3}{2}, 6[\sup_{t \geq 0} \|E_0(t, \cdot)\|_p < \infty. \tag{65}$$

Then there exists a weak solution of (1)–(4) *such that*

$$\sup_{0 \leq t \leq T} \|E(t, \cdot)\|_p < \infty \qquad \textit{for every} \quad T > 0. \tag{66}$$

The field E_0 was defined in (5).

Proof. We use Proposition 1, applied to the approximate solution. We know that $S_{5/3}(T)$ is bounded (see (34)). Choose q such that $1/q > 3/5p' - (2/3)(2/3 - 1/p)$, but with equality nearly satisfied. Since $p < 6$, we have $p < q < 6$. The inequality (38) gives

$$\sup_{0 \leq t \leq T} \|\tilde{E}^2(t, \cdot)\|_q \leq C(p, q, \beta, \sigma, T) \|f_0\|_\infty^{1/p} S_{5/3}(T)^{1/p'} \operatorname*{ess\,sup}_{0 \leq t \leq T} \|E(t, \cdot)\|_p$$

and we deduce that

$$\sup_{0 \leq t \leq T} \|\tilde{E}^2(t, \cdot)\|_p \leq C(\operatorname*{ess\,sup}_{0 \leq t \leq T} \|E(t, \cdot)\|_p)^\alpha \qquad \text{for some} \quad \alpha \in]0, 1[.$$

On the other hand, (29) gives

$$\sup_{t \geq 0} \|\tilde{E}^1(t, \cdot)\|_p \leq \sup_{t \geq 0} \|E_0(t, \cdot)\|_p < \infty.$$

Adding these inequalities, we find

$$\sup_{0 \leq t \leq T} \|\tilde{E}(t, \cdot)\|_p \leq C(1 + \operatorname*{ess\,sup}_{0 \leq t \leq T} \|E(t, \cdot)\|_p)^\alpha,$$

and thanks to (59) this implies the claim. ∎

Using Proposition 2, we can now pass beyond the exponent 6.

PROPOSITION 5. *Assume that* $N=3$, f_0 *satisfies* (60),

$$\exists p > \tfrac{5}{3} \sup_{t \geqslant 0} \left\| \int_{\mathbb{R}^3} f_0(x - tv, v)\, dv \right\|_p < \infty, \tag{67}$$

$$\exists k > 6 \sup_{t \geqslant 0} \|E_0(t, \cdot)\|_k < \infty. \tag{68}$$

Then there exists a weak solution to (1)–(4) *such that*

$$\exists q \in]6, k[\text{ such that } \sup_{0 \leqslant t \leqslant T} \|E(t, \cdot)\|_q < \infty \text{ for every } T > 0. \tag{69}$$

Proof. We can assume $p < \infty$. Let us fix $q \in]6, k[$. We are going to prove that if q is close enough to 6, we can bound $K_q \equiv \sup_{0 \leqslant t \leqslant T} \|E(t, \cdot)\|_q$ for the approximate solution. Proposition 2 gives (with $1/l = 0$)

$$S_p(T) \leqslant C(1 + K_q^{(5/3)q(3/5 - 1/p)}).$$

Now choose $r \in]3/2, 6[$ such that

$$\frac{1}{pr'} > \frac{1}{q} > \frac{1}{pr'} - \frac{2}{3}\left(\frac{2}{3} - \frac{1}{r}\right) \tag{70}$$

(this condition will be examined at the end). By Propositions 1 and 4 we get

$$\sup_{0 \leqslant t \leqslant T} \|\tilde{E}^2(t, \cdot)\|_q \leqslant C S_p(T)^{1/r'},$$

$$\begin{aligned} \sup_{0 \leqslant t \leqslant T} \|\tilde{E}(t, \cdot)\|_q &\leqslant C(1 + S_p(T)^{1/r'}) \\ &\leqslant C(1 + K_q^{(5/3)(q/r')(3/5 - 1/p)}). \end{aligned}$$

This will give a bound on K_q as soon as $(5/3)(q/r')(3/5 - 1/p) < 1$. If we take p small enough, it will be fulfilled. Then we take $r < 6$ close to 6 and $q > 6$ close to 6 so that (70) is fulfilled. ∎

Let us now consider the context of Theorem 1. Let us assume that $N = 3$, f_0 satisfies (60), and (67) for some $p > 2$. By "Sobolev imbeddings," and by (34), (68) is fulfilled. Thus we obtain (69) according to Proposition 5 (for the weak solution and for the approximate solution). Applying again Proposition 2, we find that

$$\sup_{\substack{\lambda \geqslant 0 \\ 0 \leqslant t \leqslant T}} \left\| \int_{\mathbb{R}^3} f(t, x - \lambda v, v)\, dv \right\|_p$$

is bounded, and also $\sup_{0 \leqslant t \leqslant T} \|\rho(t, \cdot)\|_p$.

If $p > 3$, $\sup_{0 \leq t \leq T} \|E(t, \cdot)\|_\infty$ is therefore bounded and the solution is a strong one. The existence is now proved for Theorem 1, since by (34), condition (6) implies (67) for some $p > 3$.

Using the regularity conditions obtained by G. Rein and J. Weckler in [14], it follows from Theorem 1 that if f_0 is rapidly decreasing as well as its first derivatives, we can find a solution with a field E satisfying in addition

$$\sup_{0 \leq t \leq T} \|\rho(t, \cdot)\|_\infty < \infty, \qquad \sup_{0 \leq t \leq T} \left\| \frac{\partial E}{\partial x_i}(t, \cdot) \right\|_\infty < \infty.$$

IV. Uniqueness for a Weak Initial Datum

We are going to prove a uniqueness result where no assumption on the derivatives of f_0 is needed. It is quite surprising, since such a result does not exist for the Vlasov–Poisson system.

Proposition 6. *Asssume $N = 3$, f_0 satisfies the energy estimate* (60). *Then there exists at most one couple (E, f) solution of* (1)–(4) *with*

$$f \in C([0, \infty[, L^1(\mathbb{R}^6)), \tag{71}$$

$$\sup_{0 \leq t \leq T} \|E(t, \cdot)\|_\infty < \infty \qquad \textit{for every} \quad T > 0. \tag{72}$$

Proof. Given two solutions (E_1, f_1) and (E_2, f_2), we write following (31)

$$\tilde{E}_2 - \tilde{E}_1 = \int_0^t \frac{\mu(s)}{(2\sigma\, d(s))^{N/2}} A\left(\frac{x}{\sqrt{2\sigma\, d(s)}}\right) \underset{x}{*} I_{\mu(s)}(t - s, x)\, ds \tag{73}$$

with

$$I_\lambda(t, x) = \int_{\mathbb{R}^N} (E_2 f_2 - E_1 f_1)(t, x - \lambda v, v)\, dv. \tag{74}$$

Our aim is to deduce from (73) a contraction property. We start from the inequality

$$|I_\lambda| \leq J_\lambda \equiv \int_{\mathbb{R}^N} |E_2 f_2 - E_1 f_1|\, (t, x - \lambda v, v)\, dv \tag{75}$$

$$\leq \int |E_2 - E_1|\, f_2(t, x - \lambda v, v)\, dv + C \int |f_2 - f_1|\, (t, x - \lambda v, v)\, dv, \tag{76}$$

since E_1 satisfies (72). The same estimate as in (39) gives

$$\forall \lambda > 0\ \forall q \in [1, \infty]\ \left\| \int_{\mathbb{R}^N} |E_2 - E_1|\, f_2(t, x - \lambda v, v)\, dv \right\|_{q'} \leqslant \frac{1}{\lambda^{N/q}} \|(E_2 - E_1)(t, \cdot)\|_q\, \|f_2(t, \cdot)\|_{q'}. \tag{77}$$

Let us choose $q > 2$. Because of (77) (and $N = 3$), if q is close enough to 2, the first term of (73), arising in the splitting relative to (76), will be bounded in L^q_x by

$$\int_0^t \frac{\mu(s)^{1 - N/q}}{(2\sigma\, d(s))^{N/2l}} \|A\|_{l'}\, \|(E_2 - E_1)(t - s, \cdot)\|_q\, \|f_2(t - s, \cdot)\|_{q'}\, ds \leqslant C t^{\alpha_1} \sup_{0 \leqslant s \leqslant t} \|(E_2 - E_1)(s, \cdot)\|_q, \tag{78}$$

with $1/q = 1/q' - 1/l$, and $\alpha_1 \simeq 1/2$.

Let us now deal with the second term. Following (17) we write

$$f_2 - f_1 = \iiint_{\substack{0 < s < t \\ \xi, \gamma \in \mathbb{R}^N}} \nabla_\gamma G(s, x, v, \xi, \gamma)(E_2 f_2 - E_1 f_1)(t - s, \xi, \gamma)\, d\xi\, d\gamma\, ds, \tag{79}$$

and we get with the help of (27)

$$\int_{\mathbb{R}^N} |f_2 - f_1|\, (t, x - \lambda v, v)\, dv \leqslant C_N \int_0^t \frac{1}{\sqrt{\sigma s}} \frac{1}{(2\sigma\, d_\lambda(s))^{N/2}} \mathcal{M}\left(\frac{x}{\sqrt{2\sigma\, d_\lambda(s)}}\right) \underset{x}{*} J_{\mu_\lambda(s)}(t - s, x)\, ds \tag{80}$$

and for $1 \leqslant p \leqslant \infty$

$$\left\| \int_{\mathbb{R}^N} |f_2 - f_1|\, (t, x - \lambda v, v)\, dv \right\|_p \leqslant C_N \int_0^t \frac{ds}{\sqrt{\sigma s}} \|J_{\mu_\lambda(s)}(t - s, \cdot)\|_p. \tag{81}$$

Denote

$$\delta_p(t) = \sup_{\substack{\lambda \geqslant 0 \\ 0 \leqslant s \leqslant t}} \left\| \int_{\mathbb{R}^N} |f_2 - f_1|\, (s, x - \lambda v, v)\, dv \right\|_p. \tag{82}$$

Because of (60), and of Proposition 2, we know that $\delta_p(t)$ is finite for $1 \leqslant p \leqslant 5/3$. Using (76) again, in (81), we find that if $\varepsilon > 0$ is small enough

$$\delta_p(\varepsilon) \leqslant C \sup_{\substack{\lambda \geqslant 0 \\ 0 \leqslant t \leqslant \varepsilon}} \int_0^t \frac{ds}{\sqrt{\sigma s}} \left\| \int |E_2 - E_1|\, f_2(t - s, x - \mu_\lambda(s) v, v)\, dv \right\|_p. \tag{83}$$

But

$$\int |E_2 - E_1| f_2(t-s, x-\mu_\lambda(s)v, v)\, dv$$
$$= \frac{1}{\mu_\lambda(s)^N} \int |E_2 - E_1| f_2\left(t-s, v, \frac{x-v}{\mu_\lambda(s)}\right) dv,$$

hence

$$\|\cdots\|_p \leqslant \frac{1}{\mu_\lambda(s)^{N/p'}} \|(E_2 - E_1)(t-s, \cdot)\|_q \left\| \left(\int f_2^p(t-s, x, v)\, dv \right)^{1/p} \right\|_{q'}. \tag{84}$$

As soon as $1 \leqslant q'/p \leqslant 5/3$ we obtain

$$\delta_p(\varepsilon) \leqslant C \sup_{0 \leqslant t \leqslant \varepsilon} \int_0^t \frac{ds}{\sqrt{\sigma s}} \frac{\|(E_2 - E_1)(t-s, \cdot)\|_q}{\mu(s)^{N/p'}}. \tag{85}$$

Choose $q'/p = 5.3$, for example, so that $p < 6/5$ (and p is close to $6/5$). We get

$$\delta_p(\varepsilon) \leqslant C \sup_{0 \leqslant s \leqslant \varepsilon} \|(E_2 - E_1)(s, \cdot)\|_q. \tag{86}$$

For any $t \leqslant \varepsilon$ we have consequently

$$\left\| \int_0^t \frac{\mu(s)}{(2\sigma\, d(s))^{N/2}} \left| A\left(\frac{x}{\sqrt{2\sigma\, d(s)}}\right)\right| * \left(\int |f_2 - f_1|\, (t-s, x-\mu(s)v, v)\, dv \right) ds \right\|_q$$
$$\leqslant \delta_p(\varepsilon) \int_0^t \frac{\mu(s)}{(2\sigma\, d(s))^{(N/2)(1/p - 1/q)}} C_N\, ds$$
$$\leqslant C\varepsilon^{\alpha_2}\, \delta_p(\varepsilon), \qquad \text{with} \quad \alpha_2 \simeq \frac{1}{2}. \tag{87}$$

Substituting (86) in (87), and adding to (78) gives

$$\|(\tilde{E}_2 - \tilde{E}_1)(\varepsilon, \cdot)\|_q \leqslant C\varepsilon^{\alpha} \sup_{0 \leqslant s \leqslant \varepsilon} \|(E_2 - E_1)(s, \cdot)\|_q \tag{88}$$

with $\alpha \simeq 1/2$. Because of (4), this gives $E_2 = E_1$ for small times t, and so $f_2 = f_1$. The global equality is easily deduced. ∎

V. Additional Results and Open Problems

(1) First let us give additional properties of solution (E, f) of Theorem 1. We have seen that $f \geqslant 0$, $f(t, \cdot) \in L^1 \cap L^\infty$, and

$$\|f(t, \cdot)\|_\infty \leqslant e^{3\beta t} \|f_0\|_\infty,$$

$$\iint_{\mathbb{R}^6} f(t, x, v)\, dx\, dv = \iint_{\mathbb{R}^6} f_0(x, v)\, dx\, dv,$$

which is the conservation of particles. We also have

$$f \in C(]0, \infty[, C_0(\mathbb{R}^6)).$$

Since E is bounded, the velocity moments are propagated, and the energy equation holds in the classical sense

$$\frac{d}{dt}\left(\iint_{\mathbb{R}^6} |v|^2 f(t, x, v)\, dx\, dv + \omega \iint_{\mathbb{R}^3} |E(t, x)|^2\, dx\right)$$
$$= -2\beta \iint_{\mathbb{R}^6} |v|^2 f(t, x, v)\, dx\, dv + 6\sigma \iint_{\mathbb{R}^6} f_0(x, v)\, dx\, dv \text{ in } [0, \infty[.$$

Proposition 2 applies of course, and gives the conservation of the L^p regularity of the "ahead of time" velocity averages ρ_λ.

(2) Propositions 1 and 2 do not prove existence of strong solutions in dimension $N \geqslant 4$. Nevertheless, we obtain in four dimensions a gain of L^p regularity for E, up to $p \simeq 3$.

(3) Concerning the gain of 1/3 derivative in Propositions 1, 2, 3, it seems difficult to improve it only with the condition $E \in L^\infty$. On the other hand, when $\nabla_x E \in L^\infty$, we can obtain better results (see H. D. Victory and B. P. O'Dwyer [15]).

(4) All the above results hold for the system corresponding to several different species of particles, as well as in the limiting case $\beta = 0$ (make formally $\beta \to 0$ in all expressions).

(5) Let us finally point out an interesting problem which is to find an asymptotic behaviour of the solution of Theorem 1 when $t \to \infty$, even in the simplest case $\beta = 0$.

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