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A note on Peano's Theorem on time scales

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ABSTRACT

In this paper we investigate the dynamic Cauchy problem in Banach spaces. We check how dense a time scale must be in such a way that Peano's Theorem holds and we present a counterexample to Peano's Theorem on a time scale with only one right dense point.

1. Introduction

A time scale (or a measure chain) was introduced by Hilger in his Ph.D. thesis in 1988 in order to unify discrete and continuous analyses [1]. Since Hilger formed the definitions of derivative and integral on a time scale, several authors have extended various aspects of the theory [2–5]. A time scale has been shown that is applicable to any field which can be described with discrete or continuous models.

But the theory of time scales and equations involving derivatives (dynamic equations) on time scales is not only a unification of the mentioned models. One of the advantages of this theory lies in the possibilities of a unification of all models "between" continuous and discrete cases. In particular, q-difference (quantum) models and some difference schemes based on variable step sizes are covered.

Since difference [6,7] and differential equations [8,9] are also considered in (infinite dimensional) Banach spaces, it seems to be useful to extend the dynamic equations on this subject. In such a case, we need to check the necessity of compactness-type assumptions (cf. [10,11]). However, the dynamic equations in Banach spaces on arbitrary time scales are still a new research area. These kinds of dynamic equations have the same advantages as in a real-valued case and a growing number of possible applications (likewise differential and difference equations in Banach spaces).

First, for the explicit difference equations ($\mathbb{T} = \mathbb{Z}$) the existence of (forward) solutions is trivially given without imposing further assumptions on the right-hand side of $\Delta x(t) = f(t, x(t))$. There is no question about continuity hypothesis on f. Let us note that there is no necessity of any additional compactness hypothesis on the right-hand side.

The opposite situation is when $\mathbb{T} = \mathbb{R}$. Peano's Theorem (1890) is not true when the space E is infinite dimensional and so the continuity assumption is not sufficient for the existence of solutions for the Cauchy problem

$$\begin{cases} x'(t) = f(t, x(t)) \\ x(0) = x_0, \end{cases} \quad t \in [0, T]. \tag{1}$$

It can be deduced from an example of Dieudonné [12] and the problem was solved in several papers of Yorke [13], Cellina [14] or Godunov [15,16]. There is no problem to extend such a counterexample to the case of time scales of the form $[0, b_1] \cup [a_2, b_2]$ ($b_1 < a_2$). The case of a general time scale is more difficult.

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The question is how to describe the problem in a general time scale. It is known, that the terminal cases $\mathbb{R}$ and $\mathbb{Z}$ are completely different. It is really interesting to consider the time scales between these two cases. In this paper we solve Peano's problem by presenting a counterexample to Peano's Theorem in a time scale for which the initial point x_0 is right dense. It is interesting, that for time scales containing only one right dense point, Peano's Theorem fails for the problem

$$\begin{cases} x^{\Delta}(t) = f(t, x(t)) \\ x(0) = x_0, \end{cases} \quad t \in \mathbb{T}. \tag{2}$$

Let us note, that for Peano's theorem in time scales we understand the result with *continuous* right-hand side. Peano's Theorem formulated in [5] is not correctly stated (rd-continuity is not a sufficient condition and an additional continuity hypothesis is necessary). Even in such a strengthened situation we are able to show a counterexample in infinite dimensional Banach spaces.

For simplicity, we consider delta derivatives and consequently rd-continuous solutions of the problem. It will cause no problem to extend these results for nabla derivatives and ld-continuous functions.

2. Preliminaries

To understand so-called dynamic equation and easily follow this paper, we present some preliminary definitions and notations of a time scale which are very common in the literature (see [2,3,1,4] and references therein).

A time scale $\mathbb{T}$ is a nonempty closed subset of real numbers $\mathbb{R}$, with the subspace topology inherited from the standard topology of $\mathbb{R}$. Three of the most popular examples of calculus on time scales are differential calculus, difference calculus and quantum (q-difference) calculus i.e. when $\mathbb{T} = \mathbb{R}, \mathbb{T} = \mathbb{N}$ and $\mathbb{T} = q^{\mathbb{Z}} \cup \{0\} = \{q^t : t \in \mathbb{Z}\} \cup \{0\}$, where $q > 1$. More interesting time scales are also the union of non-overlapping compact intervals or in the form $\mathbb{T} = \{t_k : 0 < t_{k+1} < t_k, k \in \mathbb{N}, \lim_{k\to\infty} t_k = 0\} \cup \{0\}$. The last time scale will be really useful in our consideration.

Definition 1. The forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ and the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ are defined by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$ and $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$, respectively. We put $\inf \emptyset = \sup \mathbb{T}$ (i.e. $\sigma(M) = M$ if $\mathbb{T}$ has a maximum M) and $\sup \emptyset = \inf \mathbb{T}$ (i.e. $\rho(m) = m$ if $\mathbb{T}$ has a minimum m).

The jump operators σ and ρ allow the classification of points in a time scale in the following way: t is called right dense, right scattered, left dense, left scattered, dense and isolated if $\sigma(t) = t$, $\sigma(t) > t$, $\rho(t) = t$, $\rho(t) < t$, $\rho(t) = t = \sigma(t)$ and $\rho(t) < t < \sigma(t)$, respectively.

Then we define the so-called delta derivative for Banach-valued functions similar as Δ-derivative for real functions on time scales [3,5].

Definition 2. Fix $t \in \mathbb{T}$. Let E be a Banach space and $u : \mathbb{T} \to E$. Then we define $u^{\Delta}(t)$ by

$$u^{\Delta}(t) = \lim_{s\to t} \frac{u(\sigma(t)) - u(s)}{\sigma(t) - s}.$$

It turns out that

(i) $u^{\Delta} = u'$ is the usual derivative if $\mathbb{T} = \mathbb{R}$ and
(ii) $u^{\Delta} = \Delta u$ is the usual forward difference operator if $\mathbb{T} = \mathbb{Z}$,
(iii) $u^{\Delta} = D_q u$ is the q-differential operator i.e $D_q u(x) = \frac{u(qx)-u(x)}{(q-1)x}$ if $\mathbb{T} = q^{\mathbb{Z}} \cup \{0\} = \{q^t : t \in \mathbb{Z}\} \cup \{0\}$.

Hence a notion of time scales allows us to unify differential, quantum and difference equations.

Definition 3. We say that $u : \mathbb{T} \to E$ is right dense continuous (rd-continuous) if u is continuous at every right dense point $t \in I_a$ and $\lim_{s\to t^-} u(s)$ exists and is finite at every left dense point $t \in \mathbb{T}$.

By a classical solution of (2) we understand a function in $C_{rd}(\mathbb{T}, E)$ such that $x(0) = x_0$ and $x(\cdot)$ satisfies (2) for all $t \in \mathbb{T}$.

Let $(E, \|\cdot\|)$ be an arbitrary Banach space. We will consider the Cauchy problem on a time scale $\mathbb{T}$ such that $0 \in \mathbb{T}$:

$$\begin{cases} x^{\Delta}(t) = f(t, x(t)) \\ x(0) = x_0, \end{cases} \quad t \in \mathbb{T},$$

where f is a function with values in a Banach space E.

3. An example

To show, that Peano's Theorem fails in an infinite dimensional Banach space E in a time scale $\mathbb{T}$, we will modify an example of Dieudonné. The situation is a little bit more elaborate, but by utilizing the original example and due to some properties of time scales and delta derivatives we obtain the desired thesis.

Example 4. Let E be the space of real sequences $x = (x_n)_{n\in\mathbb{N}}$ convergent to zero. Consider the norm $\|x\| = \sup_{n\in\mathbb{N}} |x_n|$ on E and define a function $f : E \to E$ as:

$$f(x) = \left(\sqrt{|x_n|} + \frac{1}{n}\right)_{n\in\mathbb{N}}, \quad x \in E.$$

It is obvious that f is a continuous function.

We will show that a dynamic Cauchy problem

$$\begin{cases} x^{\Delta}(t) = f(x(t)) \\ x(0) = 0, \end{cases} \quad t \in \mathbb{T} \tag{3}$$

has no solution in E. Let us consider the time scale

$$\mathbb{T} = \{0\} \cup \left\{ t_k : 0 < t_{k+1} < t_k, k \in \mathbb{N}, \lim_{k\to\infty} t_k = 0 \right\}.$$

Observe, that 0 is a (unique) right dense point of this time scale. Without loss of generality we can assume, that $t_1 < 1$.

In fact, if $\tilde{x}(t) = \{\tilde{x}_n\}_{n\in\mathbb{N}}$ were a solution of (3), its coordinates $\tilde{x}_n$ would satisfy the dynamic Cauchy problem:

$$\begin{cases} \tilde{x}_n^{\Delta}(t) = \sqrt{|x_n(t)|} + \dfrac{1}{n} \\ \tilde{x}_n(0) = 0, \end{cases} \quad t \in \mathbb{T}. \tag{4}$$

Moreover, such a solution must be rd-continuous and for each $t_k \in \mathbb{T}$ the sequence $\tilde{x}_n(t_k)$ must converge to zero.

Fix an arbitrary $n \in \mathbb{N}$.

It will be useful to consider also the (ordinary) differential Cauchy problem

$$\begin{cases} x'(t) = \sqrt{|x(t)|} \\ x(0) = 0, \end{cases} \quad t \in [0, 1]. \tag{5}$$

It is easy to check, that the problem (5) has a solution $x(t) = \frac{t^2}{4}$. This function is differentiable at each point t. Since $x'(t) = \frac{t}{2}$ and its restriction to $\mathbb{T}$ (again denoted by x) has a delta derivative $x^{\Delta}(t_k) = \frac{t_k+\sigma(t_k)}{4}$ (cf. [3]), it has $x^{\Delta}(t_k) > x'(t_k)$. Thus for the points t_k we have

$$x^{\Delta}(t_k) > x'(t_k) = \sqrt{|x(t_k)|}.$$

Let us remark, that all considered functions have non-negative derivatives (or delta derivatives) and the initial value is zero, so the functions are positive for $t > 0$ and we can omit the absolute value in the next formulas.

From the definition of delta derivative we have

$$x^{\Delta}(t_k) = \frac{x(\sigma(t_k)) - x(t_k)}{\sigma(t_k) - t_k}.$$

From the differentiability of x in t_k we have

$$x(t_k + h) - x(t_k) = x'(t_k) \cdot h + R(t_k, h),$$

where $\frac{R(t_k,h)}{h} \to 0$ as $h \to 0$. For arbitrary $\varepsilon > 0$ there exists a number $\delta > 0$ such that if $|h| < \delta$, then

$$\left| \frac{x(t_k + h) - x(t_k)}{h} - x'(t_k) \right| < \varepsilon.$$

It means that there exists N_0 such that for $k > N_0$ we are able to put $h = \sigma(t_k) - t_k$ in such a way that $h < \delta$. Fix a point t_k and a number h with the above property.

Then

$$\begin{aligned} \left| \frac{x(t_k + h) - x(t_k)}{h} - x'(t_k) \right| &= \frac{x(\sigma(t_k)) - x(t_k)}{\sigma(t_k) - t_k} - x'(t_k) \\ &= x^{\Delta}(t_k) - x'(t_k) \\ &= \frac{R(t_k, \sigma(t_k) - t_k)}{\sigma(t_k) - t_k}. \end{aligned}$$

Denote by ε_k a positive number $\frac{R(t_k,\sigma(t_k)-t_k)}{\sigma(t_k)-t_k}$ which is dependent only on the (fixed) point t_k.

As $x'(t_k) = \sqrt{x(t_k)}$, we obtain $x^{\Delta}(t_k) = \sqrt{x(t_k)} + \varepsilon_k$.

We can assume, that $\varepsilon_k < \frac{1}{n}$ (recall, that $\frac{R(t_k,\sigma(t_k)-t_k)}{\sigma(t_k)-t_k}$ is arbitrarily small when $t_k \to 0$ and 0 is a limit of the sequence (t_k)).

Suppose, that the problem (3) has a solution $\tilde{x}$. Thus, $\tilde{x}_n$ should satisfy the equation $\tilde{x}_n^{\Delta}(t_k) = \sqrt{\tilde{x}_n(t_k)} + \frac{1}{n}$. From the above consideration it follows that a function $x(t_k) = \frac{t_k^2}{4}$ satisfies the equation $x^{\Delta}(t_k) = \sqrt{x(t_k)} + \varepsilon_k$. By theorem on differential inequalities (Theorem 6.9 in [3]) we can ensure, that

$$\tilde{x}_n(t_k) > \frac{{t_k}^2}{4}$$

for sufficiently small t_k. This is a contradiction with the fact that $\tilde{x}_n(t_k) \to 0$ as n tends to ∞ and then $\tilde{x} \notin c_0 = E$.

It is worthwhile to remark, that the number of right dense points is not a crucial thing of the above counterexample. In fact, by changing the initial point in the considered problem, we obtain forward difference scheme and we get a local solution. Let us stress, that in such a situation we still have only one right dense point in $\mathbb{T}$.

Peano's Theorem is true in the following cases (for any norm topology τ on a Banach space E and delta derivatives):

- $\mathbb{T} = \mathbb{R}$, f is τ-continuous and the space is finite dimensional,
- $\mathbb{T} = \mathbb{Z}$, for arbitrary f and E,
- for arbitrary time scale $\mathbb{T}$ if an initial point is not a right dense point of $\mathbb{T}$.

It is clear, that the character of research for the dynamic equations is much closer to the differential equations rather than difference equations and it is expected to use the techniques from differential equations when the space is infinite dimensional (cf. [10,11]).

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