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Oscillation Criteria for Certain Nonlinear Differential Equations*

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We present new oscillation criteria for certain classes of second-order nonlinear differential equations and delay differential equations obtained by using an integral averaging technique. Our theorems complement a number of existing results and handle the cases which are not covered by known criteria. © 1999 Academic Press

INTRODUCTION

It is the purpose of this paper to study the oscillatory behavior of solutions of the nonlinear differential equation,

$$x''(t) + p(t)f(x(t))g(x'(t)) = 0, \tag{1}$$

and nonlinear delay differential equations,

$$x''(t) + p(t)f(x(\tau(t)))g(x'(t)) = 0, \tag{2}$$

and

$$x''(t) + p(t)f(x(t), x(\tau(t)))g(x'(t)) = 0, \tag{3}$$

where $t \geq t_0$, and the functions p, f, g, τ are to be specified in the following text.

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By a solution of Eq. (1), we mean a continuously differentiable function $x(t)$: $[t_0, \infty) \to \mathbf{R} = (-\infty, +\infty)$ such that $x(t)$ satisfies Eq. (1) for all $t \geq t_0$. Let ϕ: $[\tau(t_0), t_0] \to \mathbf{R}$ be a continuous function. By a solution of Eq. (2) (resp., Eq. (3)), we mean a continuously differentiable function $x(t)$: $[\tau(t_0), \infty) \to \mathbf{R}$ such that $x(t) = \phi(t)$ for $\tau(t_0) \leq t_0$, and $x(t)$ satisfies Eq. (2) (resp., Eq. (3)) for all $t \geq t_0$. We restrict our attention to proper solutions of Eqs. (1)–(3), i.e., to those nonconstant solutions which exist on some ray $[T, \infty)$, where $T \geq t_0$, and satisfy condition $\sup_{t \geq T}\{|x(t)|\} > 0$. A proper solution $x(t)$ of Eq. (1) (resp., Eqs. (2) or (3)) is called oscillatory if it has arbitrarily large zeros, otherwise it is called nonoscillatory. Finally, Eq. (1) (resp., Eqs. (2) or (3)) is called oscillatory if all its proper solutions are oscillatory.

Both Eqs. (1) and (2) have been studied by Grace and Lalli [4]. They mentioned that though stability, boundedness, and convergence to zero of all solutions of Eq. (1) have been investigated in the papers of Burton and Grimmer [1], Graef and Spikes [6, 7], Lalli [10], and Wong and Burton [18], nothing much has been known regarding the oscillatory behavior of Eq. (1) except for the result by Wong and Burton [18, Theorem 4] regarding oscillatory behavior of Eq. (1) in connection with that of the corresponding linear equation,

$$x''(t) + p(t)x(t) = 0,$$

where $p(t)$ is an oscillatory coefficient in the sense of [17]. Grace and Lalli proved the following oscillation criterion for Eq. (1).

THEOREM A [4, Theorem 1]. *Suppose that the following conditions hold*:

(i) p: $I \to \mathbf{R}_0$ *is continuous, and* $p(t) \not\equiv 0$ *on any ray* $[t_*, \infty)$ *for some* $t_* \geq t_0$, *where* $I = [t_0, \infty)$ *and* $\mathbf{R}_0 = [0, +\infty)$;

(ii) f: $\mathbf{R} \to \mathbf{R}$ *is continuous, and* $xf(x) > 0$ *for* $x \neq 0$;

(iii) g: $\mathbf{R} \to \mathbf{R}$ *is continuous, and* $g(y) > 0$ *for* $y \neq 0$;

(iv) $f'(x) \geq 0$ *for* $x \neq 0$;

(v) $f(xy) \geq Kf(x)f(y)$ *for* $x, y \in \mathbf{R}_+$, *and* $-f(-xy) \geq Kf(x)f(y)$, *for* $x, y \in \mathbf{R}_-$, *where* K *is a positive constant*, $\mathbf{R}_+ = (0, +\infty)$, *and* $\mathbf{R}_- = (-\infty, 0)$;

(vi) fg *is strongly sublinear, i.e.*,

$$\int_{-0} \frac{du}{f(u)g(u)} < \infty \quad \text{and} \quad \int_{+0} \frac{du}{f(u)g(u)} < \infty;$$

(vii) $\int^{\infty} p(s)f(s)\, ds = \infty$.

Then Eq. (1) *is oscillatory.*

The proof of this result is based essentially on Ohriska's lemma [12, *Lemma* 2] *which has been also proved in* [4].

Further, using the similar technique and Erbe's lemma [2, *Lemma* 2.1], *Grace and Lalli proved also the following oscillation criterion for Eq.* (2).

THEOREM B [4, Theorem 2]. *Let conditions* (i)–(vi) *of Theorem A hold, and suppose also that*

(viii) τ: $I \to \mathbf{R}$ *is continuous*, $\tau(t) \le t$ *for* $t \ge T_0$, *and* $\lim_{t\to\infty} \tau(t) = \infty$;

(ix) $\int^{\infty} p(s) f(\sigma(s))\, ds = \infty$, *where* $\sigma(t) = \min\{t, \tau(t)\}$.

Then Eq. (2) *is oscillatory.*

The paper by Grace and Lalli [4] *has motivated the study of the more general equation* (3) *which has been done by Hamedani and Krenz* [8]. *They proved the following oscillation criterion for Eq.* (3) *which complements in certain respect Theorem B.*

THEOREM C [8, Theorem 1]. *Let conditions* (i), (ii), *and* (viii) *of Theorem B hold, and assume also that*

(x) f: $\mathbf{R} \times \mathbf{R} \to \mathbf{R}$ *is continuous and* $f(x, y)$ *has the sign of* x *and* y *if they have the same sign*;

(xi) *there exists* $M > 0$ *such that, uniformly for* $x \ge M$,

$$\liminf_{|y|\to\infty} \left| \frac{f(x, y)}{y} \right| \ge c > 0;$$

(xii) $g(y)$ *is even and nonincreasing for* $y > 0$;

(xiii) $\limsup_{t\to\infty} t \int_t^{\infty} \tau(s) s^{-1} p(s) g(s)\, ds > c^{-1}$, *where* c *is as in* (xi).

Then Eq. (3) *is oscillatory.*

In this paper, we present new sufficient conditions which ensure oscillatory character of Eqs. (1)–(3). *They are different from those of Grace and Lalli* [4] *and Hamedani and Krenz* [8], *and are applicable to other classes of equations which are not covered by Theorems A–C. The averaging technique exploited in this paper is similar to that used by Grace* [3], *Grace and Lalli* [5], *Li* [11] (*see also corrections to this paper in Rogovchenko* [14]), *Philos* [13], *and Rogovchenko* [15, 16] *both for linear and nonlinear equations. The results obtained are presented in the form of a high degree of generality which admits rather wide possibilities for deriving various oscillation criteria with the appropriate choice of functions H and h.*

The paper is organized as follows. In Section 1, *our main results are stated and proved, and some relevant examples are indicated. In Section* 2 *we make some specific comparisons to known results and we also discuss related problems.*

1. OSCILLATION CRITERIA

Following Philos [13], we introduce a class of functions $\mathscr{P}$ which is used in the sequel.

Let $D_0 = \{(t, s): t > s \geq t_0\}$ and $D = \{(t, s): t \geq s \geq t_0\}$. The function $H \in C(D; \mathbf{R})$ is said to belong to the class $\mathscr{P}$ if

(H_1) $H(t, t) = 0$ for $t \geq t_0$, $H(t, s) > 0$ on D_0;

(H_2) H has a continuous and nonpositive partial derivative on D_0 with respect to the second variable.

Hereinafter, we assume that

(A1) the function $p: I \to \mathbf{R}_0$ is continuous and $p(t) \not\equiv 0$ on any ray $[T, \infty)$ for some $T \geq t_0$;

(A2) the function $f: \mathbf{R} \to \mathbf{R}$ is continuous and $xf(x) > 0$ for $x \neq 0$;

(A3) the function $g: \mathbf{R} \to \mathbf{R}$ is continuous and $g(y) \geq C > 0$ for $y \neq 0$.

In order to simplify presentation of examples and to avoid tedious repetition of similar formulas, in all the examples in the following text we will refer to the following

PROPOSITION 1. *For any constant* $A > 0$,

$$\limsup_{t\to\infty} \sigma_1(t, A) = \limsup_{t\to\infty} \frac{1}{t^2} \int_1^t \left[A(t-s)^2 s^2 - (2s-t)^2 \right] ds = +\infty,$$

and

$$\limsup_{t\to\infty} \sigma_2(t, A) = \limsup_{t\to\infty} \frac{1}{t^2} \int_1^t \left[A(t-s)^2 s(s-\pi) - (2s-t)^2 \right] ds = +\infty.$$

Proof. It is a matter of straightforward computation to see that

$$\limsup_{t\to\infty}\sigma_1(t,A)=\limsup_{t\to\infty}\frac{1}{t^2}\left[\frac{A}{30}t^5-\frac{1}{3}t^3+\left(1-\frac{A}{3}\right)t^2-\left(2-\frac{A}{2}\right)t+\frac{4}{3}-\frac{A}{5}\right]=+\infty,$$

$$\limsup_{t\to\infty}\sigma_2(t,A)$$
$$=\limsup_{t\to\infty}\frac{1}{t^2}\left[\frac{A}{30}t^5-\frac{A\pi}{12}t^4-\frac{1}{3}t^3+\left(1+\frac{A\pi}{2}-\frac{A}{3}\right)t^2-\left(2-\frac{A}{2}+\frac{2A\pi}{3}\right)t+\frac{4}{3}+\frac{A\pi}{4}-\frac{A}{5}\right]=+\infty,$$

for any constant $A>0$. ∎

THEOREM 1. *Let assumptions* (A1)–(A3) *hold and suppose also that*

($A4$) *there exists* $f'(x)$ *for* $x\in\mathbf{R}$ *and* $f'(x)\geq K>0$ *for* $x\neq 0$.

Further, let $h, H\colon D\to\mathbf{R}$ *be continuous functions such that* H *belongs to the class* $\mathscr{P}$ *and*

$$-\frac{\partial H}{\partial s}(t,s)=h(t,s)\sqrt{H(t,s)}\,,\quad \text{for all } (t,s)\in D_0. \tag{4}$$

If there exists a continuously differentiable function $\rho\colon I\to\mathbf{R}_+$ *such that*

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left[H(t,s)C\rho(s)p(s)-\frac{\rho(s)}{4K}\left(h(t,s)-\frac{\rho'(s)}{\rho(s)}\sqrt{H(t,s)}\right)^2\right]ds=\infty, \tag{5}$$

then Eq. (1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of Eq. (1) and let $T_0\geq t_0$ be such that $x(t)\neq 0$ for all $t\geq T_0$. Without loss of generality, we may assume that $x(t)>0$ for all $t\geq T_0$ because the similar argument holds also for the case when $x(t)$ is eventually negative. Define

$$w(t) = \rho(t)\frac{x'(t)}{f(x(t))}. \tag{6}$$

Next, differentiating (6) and making use of (1), we get, for $t \geq T_0$,

$$w'(t) = \frac{\rho'(t)}{\rho(t)}w(t) - \rho(t)p(t)g(x'(t)) - \frac{f'(x(t))}{\rho(t)}w^2(t). \tag{7}$$

Because $f'(x) \geq K > 0$ and $g(x') \geq C > 0$, we obtain by (7),

$$w'(t) + C\rho(t)p(t) + \frac{K}{\rho(t)}w^2(t) - \frac{\rho'(t)}{\rho(t)}w(t) \leq 0. \tag{8}$$

Hence, by (4) and (8), we get, for all $t \geq T \geq T_0$,

$$\begin{aligned}
&\int_T^t H(t,s)C\rho(s)p(s)\,ds \\
&\quad \leq -\int_T^t H(t,s)w'(s)\,ds - \int_T^t H(t,s)\frac{K}{\rho(s)}w^2(s)\,ds \\
&\qquad + \int_T^t H(t,s)\frac{\rho'(s)}{\rho(s)}w(s)\,ds \\
&\quad = -H(t,s)w(s)\big|_T^t - \int_T^t \left[-\frac{\partial H}{\partial s}(t,s)w(s) - H(t,s)\frac{\rho'(s)}{\rho(s)}w(s) \right. \\
&\qquad\qquad \left. + H(t,s)\frac{K}{\rho(s)}w^2(s)\right] ds \\
&\quad = H(t,T)w(T) - \int_T^t \left[\sqrt{\frac{KH(t,s)}{\rho(s)}}\,w(s) \right. \\
&\qquad\qquad \left. + \frac{1}{2}\sqrt{\frac{\rho(s)}{K}}\left(h(t,s) - \frac{\rho'(s)}{\rho(s)}\sqrt{H(t,s)}\right)\right]^2 ds \\
&\qquad + \int_T^t \frac{\rho(s)}{4K}\left(h(t,s) - \frac{\rho'(s)}{\rho(s)}\sqrt{H(t,s)}\right)^2 ds.
\end{aligned}$$

Thereby, for all $t \geq T \geq T_0$, we have

$$\int_T^t \left[H(t,s) C\rho(s) p(s) - \frac{\rho(s)}{4K} \left(h(t,s) - \frac{\rho'(s)}{\rho(s)} \sqrt{H(t,s)} \right)^2 \right] ds$$

$$\leq H(t,T) w(T) - \int_T^t \left[\frac{KH(t,s)}{\rho(s)} w(s) + \frac{1}{2} \sqrt{\frac{\rho(s)}{K}} \right.$$

$$\left. \times \left(h(t,s) - \frac{\rho'(s)}{\rho(s)} \sqrt{H(t,s)} \right) \right]^2 ds. \quad (9)$$

By virtue of (9) and by (H_2), we obtain, for every $t \geq T_0$,

$$\int_{T_0}^t \left[H(t,s) C\rho(s) p(s) - \frac{\rho(s)}{4K} \left(h(t,s) - \frac{\rho'(s)}{\rho(s)} \sqrt{H(t,s)} \right)^2 \right] ds$$

$$\leq H(t,T_0) |w(T_0)| \leq H(t,t_0) |w(T_0)|. \quad (10)$$

Thus, by (10) and by (H_2), we have

$$\int_{t_0}^t \left[H(t,s) C\rho(s) p(s) - \frac{\rho(s)}{4K} \left(h(t,s) - \frac{\rho'(s)}{\rho(s)} \sqrt{H(t,s)} \right)^2 \right] ds$$

$$\leq H(t,t_0) C \int_{t_0}^{T_0} \rho(s) p(s)\, ds + H(t,t_0) |w(T_0)|$$

$$= H(t,t_0) \left[C \int_{t_0}^{T_0} \rho(s) p(s)\, ds + |w(T_0)| \right]. \quad (11)$$

It follows from (11) that

$$\limsup_{t \to \infty} \frac{1}{H(t,t_0)}$$

$$\times \int_{t_0}^t \left[H(t,s) C\rho(s) p(s) - \frac{\rho(s)}{4K} \left(h(t,s) - \frac{\rho'(s)}{\rho(s)} \sqrt{H(t,s)} \right)^2 \right] ds$$

$$\leq C \int_{t_0}^{T_0} \rho(s) p(s)\, ds + |w(T_0)|,$$

and the latter inequality contradicts assumption (5) of the theorem. Hence, Eq. (1) is oscillatory. ∎

As an immediate consequence of Theorem 1, we get the following

COROLLARY 1. *Let assumption* (5) *in Theorem* 1 *be replaced by*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,s)\rho(s)p(s)\,ds = \infty, \tag{12}$$

and

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \rho(s)\left(h(t,s) - \frac{\rho'(s)}{\rho(s)}\sqrt{H(t,s)}\right)^2 ds < \infty, \tag{13}$$

then Eq. (1) *is oscillatory.*

With the appropriate choice of the functions H and h, it is possible to derive from Theorem 1 a number of oscillation criteria for Eq. (1). Consider, for example, the function $H(t,s) = (t-s)^{n-1}$, $(t,s) \in D$ with integer $n > 2$. Evidently, H belongs to the class $\mathscr{P}$. Further, the function $h(t,s) = (n-1)(t-s)^{(n-3)/2}$, $(t,s) \in D$ is continuous on $I \times I$ and satisfies condition (4). Therefore, by Theorem 1, we get the following oscillation criterion.

COROLLARY 2. *Let assumptions* (A1)–(A4) *hold. If there exists a continuously differentiable function* $\rho\colon I \to \mathbf{R}_+$ *such that for some integer* $n > 2$,

$$\limsup_{t\to\infty} t^{1-n} \int_{t_0}^{t} \left[(t-s)^{n-1} C\rho(s)p(s) - \frac{\rho(s)}{4K}(t-s)^{n-3}\left(n-1-\frac{\rho'(s)}{\rho(s)}(t-s)\right)^2 \right] ds = \infty,$$

then Eq. (1) *is oscillatory.*

EXAMPLE 1. Consider the nonlinear differential equation,

$$x''(t) + \frac{1}{(1+\sin^2 t)(1+\cos^2 t)} x(t)\left(1 + x^2(t)\right)\left(1 + (x'(t))^2\right) = 0, \tag{14}$$

where $t \geq 1$. It is easy to see that condition (vi) fails to hold, so Theorem A cannot be applied to Eq. (14). Nevertheless, oscillatory character of Eq.

(14) can be established by using Corollary 2. To this end, one can choose $\rho(s) = s^2$ and $n = 3$. Because for all $x, y \in \mathbf{R}$, $f'(x) = 1 + 3x^2 \geq 1$, and $g(y) = 1 + y^2 \geq 1$, one has $K = C = 1$. By Proposition 1,

$$\limsup_{t\to\infty} \frac{1}{t^2} \int_1^t \left[(t-s)^2 s^2 \frac{1}{(1+\sin^2 s)(1+\cos^2 s)} - (2s-t)^2 \right] ds$$
$$\geq \limsup_{t\to\infty} \sigma_1\left(t, \frac{4}{9}\right) = +\infty.$$

Thereby, Eq. (14) is oscillatory by Corollary 2. Observe that both $x_1(t) = \sin t$ and $x_2(t) = \cos t$ are oscillatory solutions of Eq. (14).

For the case when the function $f(x)$ is not monotonous or has no continuous derivative, the following result holds.

THEOREM 2. *Let assumption* (A4) *in Theorem* 1 *be replaced by*

(A4′) $f(x)/x \geq K > 0$ *for* $x \neq 0$, *where* K *is a constant*.

Further, let the functions h, H *be the same as in Theorem* 1. *If there exists a continuously differentiable function* ρ: $I \to \mathbf{R}_+$ *such that*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \times \int_{t_0}^t \left[H(t,s)CK\rho(s)p(s) - \frac{\rho(s)}{4}\left(h(t,s) - \frac{\rho'(s)}{\rho(s)}\sqrt{H(t,s)} \right)^2 \right] ds = \infty, \tag{15}$$

then Eq. (1) *is oscillatory*.

Proof. Let $x(t)$ be a nonoscillatory solution of Eq. (1). As in Theorem 1, without loss of generality, we assume that $x(t) > 0$ for all $t \geq T_0 \geq t_0$. Define

$$w(t) = \rho(t)\frac{x'(t)}{x(t)}. \tag{16}$$

Next, differentiating (16) and making use of (1), we obtain

$$w'(t) = \frac{\rho'(t)}{\rho(t)} w(t) - \rho(t)p(t)\frac{f(x(t))}{x(t)} g(x'(t)) - \frac{1}{\rho(t)} w^2(t). \tag{17}$$

In view of (A3) and (A4′), we get by (17),

$$w'(t) \leq \frac{\rho'(t)}{\rho(t)} w(t) - \frac{1}{\rho(t)} w^2(t) - p(t)\rho(t)CK.$$

The rest of the proof runs as before. ∎

We note that if the assumption (15) of Theorem 2 is replaced by (12) and (13), the conclusion of the theorem remains valid.

COROLLARY 3. *Let assumptions* (A1)–(A3) *and* (A4′) *hold. If there exists a continuously differentiable function* $\rho: I \to \mathbf{R}_+$ *such that for some integer* $n > 2$,

$$\limsup_{t\to\infty} t^{1-n} \int_{t_0}^{t} \left[(t-s)^{n-1} CK\rho(s)p(s) - \frac{\rho(s)}{4}(t-s)^{n-3}\left(n-1-\frac{\rho'(s)}{\rho(s)}(t-s)\right)^2 \right] ds = \infty,$$

then Eq. (1) *is oscillatory.*

EXAMPLE 2. Consider the nonlinear differential equation,

$$x''(t) + \frac{32(1+\sin^2 t)}{(35+3\sin^2 t)(1+\cos^2 t)} \times x(t)\left(\frac{3}{32} + \frac{1}{1+x^2(t)}\right)\left(1+(x'(t))^2\right) = 0, \tag{18}$$

where $t \geq 1$. Observe that the derivative $f'(x)$ exists for all $x \in \mathbf{R}$ and

$$f'(x) = \frac{(x^2-7)(3x^2-5)}{32(x^2+1)^2}.$$

Nevertheless, because both (iv) and (vi) fail to hold, we cannot apply Theorem A to Eq. (18). In spite of this, with $\rho(s) = s^2$ and $n = 3$, one can prove the oscillatory character of Eq. (18) by using Corollary 3. Because for all $x, y \in \mathbf{R}$,

$$\frac{f(x)}{x} = \frac{3}{32} + \frac{1}{1+x^2(t)} \geq \frac{3}{32},$$

and $g(y) = 1 + y^2 \geq 1$, one has $K = \frac{3}{32}$ and $C = 1$. By Proposition 1, we obtain

$$\limsup_{t\to\infty} \frac{1}{t^2} \int_1^t \left[(t-s)^2 s^2 \frac{32(1+\sin^2 s)}{(35 + 3\sin^2 s)(1+\cos^2 s)} \frac{3}{32} - (2s-t)^2 \right] ds$$

$$\geq \limsup_{t\to\infty} \sigma_1\left(t, \frac{3}{70}\right) = +\infty.$$

Thereby, Eq. (18) is oscillatory by Corollary 3. Observe that $x(t) = \sin t$ is an oscillatory solution of Eq. (14).

The following result is concerned with the oscillatory behavior of solutions of Eq. (2). To prove it, in addition to the aforementioned conditions we assume that

(A5) $\tau: I \to \mathbf{R}$ is continuous, $\tau(t) \leq t$ for $t \geq t_0$, and $\lim_{t\to\infty} \tau(t) = \infty$.

THEOREM 3. *Let assumptions* (A1)–(A3), (A4′) *and* (A5) *hold, and let the functions* h, H *be the same as in Theorem* 1. *If there exists a continuously differentiable function* $\rho: I \to \mathbf{R}_+$ *such that*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^t \left[H(t,s) KCk\rho(s)p(s)\frac{\tau(s)}{s} - \frac{\rho(s)}{4}\left(h(t,s) - \frac{\rho'(s)}{\rho(s)}\sqrt{H(t,s)} \right)^2 \right] ds = \infty,$$

where a constant $k \in (0,1)$, *then Eq.* (2) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of Eq. (2). As in Theorem 1, without loss of generality, we assume that $x(t) > 0$ for all $t \geq T_0 \geq t_0$. Further, by (A5), if $x(t)$ is a nonoscillatory solution of Eq. (2) such that $x(t) > 0$ for $t \geq T_0$, then there exists $T_1 \geq T_0$ such that $x(\tau(t)) > 0$, for $t \geq T_1$, and it follows from (A1) and Eq. (2) that $x''(t) \leq 0$ for $t \geq T_1$. It is not difficult to see that $x'(t) > 0$ for $t \geq T_1$. Hence, by [2, Lemma 2.1], for any $k \in (0,1)$ there exists a $T_2 \geq T_1$ such that

$$x(\tau(t)) \geq k\frac{\tau(t)}{t}x(t), \quad \text{for all } t \geq T_2. \tag{19}$$

Let us define the function $w(t)$ again by relation (16). Then, differentiating (16) and making use of Eq. (2), we obtain

$$w'(t) = \frac{\rho'(t)}{\rho(t)} w(t) - \rho(t) p(t) \frac{f(x(\tau(t)))}{x(\tau(t))} \frac{x(\tau(t))}{x(t)} g(x'(t)) - \frac{1}{\rho(t)} w^2(t). \tag{20}$$

In view of (A3), (A4′), and (A5), we get by (20),

$$w'(t) \leq \frac{\rho'(t)}{\rho(t)} w(t) - p(t)\rho(t) Kk \frac{\tau(t)}{t} C - \frac{1}{\rho(t)} w^2(t),$$

for $t \geq T_2$. The rest of the proof is analogous to that of Theorem 1 and hence is omitted. ▮

With the same choice of the functions H and h as previously, by Theorem 3, we get the following oscillation criterion.

COROLLARY 4. *Let assumptions* (A1)–(A3), (A4′), *and* (A5) *hold. If there exists a continuously differentiable function* $\rho\colon I \to \mathbf{R}_+$ *such that for some integer* $n > 2$,

$$\limsup_{t\to\infty} t^{1-n} \int_{t_0}^{t} \left[(t-s)^{n-1} KkC\rho(s)p(s)\frac{\tau(s)}{s} - \frac{\rho(s)}{4}(t-s)^{n-3}\left(n-1-\frac{\rho'(s)}{\rho(s)}(t-s)\right)^2 \right] ds = \infty,$$

where a constant $k \in (0,1)$, *then Eq.* (2) *is oscillatory*.

The following examples are illustrative.

EXAMPLE 3. Consider the nonlinear delay differential equation,

$$x''(t) + \frac{4}{1+4\cos^2 2t} x(t-\pi)\left(1+(x'(t))^2\right) = 0, \qquad t \geq 1. \tag{21}$$

Evidently, condition (vi) fails to hold, so Theorem B cannot be applied to Eq. (21). Nevertheless, oscillatory character of Eq. (21) can be easily proved by using Corollary 4 with $\rho(s) = s^2$ and $n = 3$. Because for all $x, y \in \mathbf{R}$ $f(x)/x = 1$ and $g(y) = 1 + y^2 \geq 1$, one has $K = C = 1$. Let $k \in (0,1)$. By Proposition 1, we obtain

$$\limsup_{t\to\infty} \frac{1}{t^2}\int_1^t \left[(t-s)^2 s^2 k \frac{s-\pi}{s} \frac{4}{1+4\cos^2 2s} - (2s-t)^2 \right] ds$$
$$\geq \limsup_{t\to\infty} \sigma_2\left(t, \frac{4k}{5}\right) = +\infty,$$

for any $k \in (0,1)$. Hence, Eq. (21) is oscillatory by Corollary 4. In fact, $x(t) = \sin 2t$ is an oscillatory solution of Eq. (21).

EXAMPLE 4. Consider the nonlinear delay differential equation,

$$x''(t) + \frac{4(1+\cos^2 2t)}{1+(1+4\cos^2 2t)^2} x(t-\pi)$$
$$\times \left(1 + (x'(t))^2 + \frac{1}{1+(x'(t))^2}\right) = 0, \tag{22}$$

where $t \geq 1$. One can easily observe that condition (xii) fails to hold, so Theorem C cannot be applied to Eq. (22), though one can prove oscillatory character of Eq. (22) by using Corollary 4. To do it, let us choose, as earlier, $\rho(s) = s^2$ and $n = 3$. Further, for all $x, y \in \mathbf{R}$,

$$\frac{f(x)}{x} = 1 \quad \text{and} \quad g(y) = 1 + y^2 + \frac{1}{1+y^2} \geq 2,$$

so $K = 1$ and $C = 2$. Let $k \in (0,1)$. By Proposition 1, we have

$$\limsup_{t\to\infty} \frac{1}{t^2}\int_1^t \left[(t-s)^2 s^2 2k \frac{s-\pi}{s} \frac{4(1+4\cos^2 2s)}{1+(1+4\cos^2 2s)^2} - (2s-t)^2 \right] ds$$
$$\geq \limsup_{t\to\infty} \sigma_2\left(t, \frac{20k}{13}\right) = +\infty,$$

for any $k \in (0,1)$. Hence, Eq. (22) is oscillatory by Corollary 4. We note that, as in Example 3, $x(t) = \sin 2t$ is also an oscillatory solution of Eq. (22).

Consider now Eq. (3) and assume that the function f is such that

(B0) $f(x, y)$ has the sign of x and y if they have the same sign and $f(x(t), x(\tau(t))) \geq f_1(x(t)) f_2(x(\tau(t)))$, where the functions f_1 and f_2 satisfy the following assumptions:

(B1) $f_1(x) \geq K_1 > 0$, where K_1 is a constant;

(B2) $f_2(x)/x \geq K_2 > 0$, where K_2 is a constant.

THEOREM 4. *Let assumptions* (A1), (A3), (A5), *and* (B0)–(B2) *hold, and let the functions* h, H *be the same as in Theorem* 1. *If there exists a continuously differentiable function* $\rho\colon I \to \mathbf{R}_+$ *such that*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[H(t,s) K_1 K_2 C k \rho(s) p(s) \frac{\tau(s)}{s} - \frac{\rho(s)}{4} \left(h(t,s) - \frac{\rho'(s)}{\rho(s)} \sqrt{H(t,s)} \right)^2 \right] ds = \infty,$$

where a constant $k \in (0,1)$, *then Eq.* (3) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of Eq. (3). As in the foregoing text, without loss of generality, we assume that $x(t) > 0$ for all $t \geq T_0 \geq t_0$. In the same way as in Theorem 3, we can prove that there exists $T_1 \geq T_0$ such that $x(\tau(t)) > 0$, $x'(t) > 0$, and $x''(t) \leq 0$ for $t \geq T_1$. Hence, by [2, Lemma 2.1], for any $k \in (0,1)$ there exists a $T_2 \geq T_1$ such that (19) holds.

Let us define the function $w(t)$ again by relation (16). Then, differentiating (16) and making use of Eq. (3), we obtain

$$\begin{aligned} w'(t) &= \frac{\rho'(t)}{\rho(t)} w(t) - \rho(t) \frac{p(t) f(x(t), x(\tau(t))) g(x'(t))}{x(t)} - \frac{1}{\rho(t)} w^2(t) \\ &\leq \frac{\rho'(t)}{\rho(t)} w(t) - \rho(t) p(t) f_1(x(t)) \frac{f_2(x(\tau(t)))}{x(\tau(t))} \frac{x(\tau(t))}{x(t)} g(x'(t)) \\ &\quad - \frac{1}{\rho(t)} w^2(t) \\ &\leq \frac{\rho'(t)}{\rho(t)} w(t) - \rho(t) p(t) K_1 K_2 k \frac{\tau(t)}{t} C - \frac{1}{\rho(t)} w^2(t). \end{aligned} \tag{23}$$

The rest of the proof runs as that of Theorem 3 and hence is omitted. ∎

With the same choice of the functions H and h as previously, by Theorem 4, we get the following oscillation criterion.

COROLLARY 5. *Let assumptions* (A1), (A3), (A5), *and* (B0)–(B2) *hold. If there exists a continuously differentiable function* $\rho: I \to \mathbf{R}_+$ *such that for some integer* $n > 2$,

$$\limsup_{t\to\infty} t^{1-n} \int_{t_0}^{t} \Bigg[(t-s)^{n-1} K_1 K_2 k C \rho(s) p(s) \frac{\tau(s)}{s} - \frac{\rho(s)}{4}(t-s)^{n-3}\left(n-1-\frac{\rho'(s)}{\rho(s)}(t-s)\right)^2 \Bigg] ds = \infty,$$

where a constant $k \in (0,1)$, *then Eq.* (3) *is oscillatory.*

EXAMPLE 5. Consider the nonlinear delay differential equation,

$$x''(t) + \frac{4}{(1+\sin^2 2t)(1+4\cos^2 2t)} \times (1+x^2(t))x(t-\pi)\left(1+(x'(t))^2\right) = 0, \tag{24}$$

where $t \geq 1$. Evidently, condition (xii) fails to hold, so we cannot apply Theorem C to Eq. (24), though oscillatory character of Eq. (24) is implied by Corollary 5 with $\rho(s) = s^2$ and $n = 3$. Because for all $x, y \in \mathbf{R}$, $f_1(x) = 1 + x^2 \geq 1$, $f_2(x)/x = 1$, and $g(y) = 1 + y^2 \geq 1$, one has $K_1 = K_2 = C = 1$. Let $k \in (0,1)$. By Proposition 1, we obtain

$$\limsup_{t\to\infty} \frac{1}{t^2} \int_1^t \Bigg[(t-s)^2 s^2 k \frac{s-\pi}{s} \frac{4}{(1+\sin^2 2s)(1+4\cos^2 2s)} - (2s-t)^2 \Bigg] ds \geq \limsup_{t\to\infty} \sigma_2\left(t, \frac{64k}{81}\right) = +\infty,$$

for any $k \in (0,1)$. Hence, Eq. (24) is oscillatory by Corollary 5. In fact, $x(t) = \cos 2t$ is an oscillatory solution of Eq. (24).

2. DISCUSSION AND REMARKS

1. Though Eq. (1) is a particular case of the more general equation,

$$(r(t)x'(t))' + p(t)x'(t) + Q(t, x(t)) = P(t, x(t), x'(t)), \quad (25)$$

and the technique exploited in this paper is similar in the main to that used in [16] for the study of Eq. (25), Theorems 1 and 2 cannot be obtained as a simple consequence of more general results [16, Theorems 1 and 4] because of assumptions on functions P and Q which make impossible the direct reduction of Eq. (25) to Eq. (1).

2. As it has been mentioned in [8], in all the examples considered in [4], $g(y)$ is nondecreasing in $|y|$, while criteria given in [8] handle cases where $g(y)$ is nonincreasing. As it can be easily seen, the function $g(y)$ in Example 3 is strictly increasing for $y > 0$, while in Example 4 it is not monotonous on $\mathbf{R}$ at all. Further, in contrast to [4], our Theorems 2 and 3 do not require $f(x)$ to be nondecreasing for $x \neq 0$, and it is not difficult to see that Example 2 presents the oscillatory equation of type (1), where f is not monotonous on $\mathbf{R}$ at all, because $f'(x)$ exists for all $x \in \mathbf{R}$ and changes sign on $\mathbf{R}$ four times. Hence, it can be said that our criteria complement those given in [4, 8] in the sense that our results do not require any monotonicity properties of $f(x)$ (except for Theorem 1) and $g(y)$.

3. In the formulation of the original result [4, Theorem 2] assumption $\tau(t) \geq t$ for $t \geq T_0$ has been omitted and the authors considered also the advanced equation [4, Example 4] and equation with deviating argument of the form $\tau(t) = t + \sin t$ [4, Example 6] to illustrate the relevance of Theorem B. Nevertheless, because Erbe's lemma [2, Lemma 2.1] which has been used for the proof of this result is applicable only to delay equations, it is necessary to give another proof of Theorem B for the case of advanced equations as well as for other classes of equations apart from delay differential equations, if at all.

4. The results of the paper are presented in the form of a high degree of generality and give rather wide possibilities of deriving the different oscillation criteria similar to Corollaries 2–5 with the appropriate choice of the functions H and h. Though throughout the paper we have always chosen $H(t, s) = (t - s)^{n-1}$, where $n > 2$ is an integer, there are interesting perspectives to apply our results, for example, with

$$H(t, s) = \left[\int_s^t \frac{dz}{\Theta(z)} \right]^{n-1}, \qquad (t, s) \in D,$$

where $n > 2$ is a constant and Θ is a positive continuous function on $[t_0, \infty)$ such that

$$\int_{t_0}^{\infty} \frac{dz}{\Theta(z)} = \infty,$$

(one of the important cases to be considered is $\Theta(z) = z^{\alpha}$ with α real).

5. The oscillation criteria presented in this paper require the assumption (A4) or (A4′) on $f(x)$ and thus the results obtained here are not applicable to equations of type (1) or (2) where, for example, $f(x) = |x|^{\lambda} \operatorname{sgn} x$, $\lambda > 0$, $\lambda \neq 1$. This problem is related to the question posed by Kamenev [9] whether condition (A4) can be relaxed to

$$\text{(A4'')} \quad f'(x) \geq 0 \text{ for } x \neq 0,$$

or whether it can be omitted. For the case of equation,

$$(a(t)x'(t))' + p(t)x'(t) + q(t)|x(t)|^{\lambda} \operatorname{sgn} x(t) = 0, \qquad \lambda > 0,$$

an affirmative answer to the preceding question has been given by Grace and Lalli [5, Theorem 8]. The study of this problem for Eqs. (1)–(3) will form the subject of one of our forthcoming papers.

6. It is possible to extend the present results to equations of the form,

$$x''(t) + F(t, x(t), x'(t), x(\tau(t)), x'(\tau(t))) = 0,$$

under suitable assumptions on the function F. For instance, F could be considered as bounded from below by a function of the form

$$p(t) f_1(x(t)) f_2(x(\tau(t))) g(x'(t)),$$

where the functions p, f_1, f_2, g satisfy assumptions of Theorem 4.

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