Cauchy Problem for One-Dimensional Semilinear Hyperbolic Systems: Global Existence, Blow Up

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We study the blow up or global existence of the solutions of the Cauchy problem
for 2×2 one-dimensional first order semilinear strictly hyperbolic systems with
compact support.

I. Introduction

In this paper we are concerned with the Cauchy problem for first order semilinear hyperbolic systems. This classical problem can be approached from various points of view and many papers have been devoted to the subject. Examples of such systems are Boltzmann equations of the discrete kinetic theory, N-waves type equations, and certain forms of semilinear wave equations.

Here, our aim is to classify for the Cauchy problem a family of 2×2 one-dimensional semilinear strictly hyperbolic systems with homogeneous quadratic interaction. Such an interaction remains homogeneous quadratic through a linear change of function, therefore we have to consider only diagonal systems

\[ \partial_t u_1 = q_1(u) \]
\[ \partial_t u_2 = q_2(u) \] (1.1)
\[ u_i(0, x) = u^0_i(x) \quad i = 1, 2, \] (1.2)

where

\[ \partial_i = \partial_i + c_i \partial_x \quad i = 1, 2. \]
\( c_1 \neq c_2 \) are fixed characteristic speeds and \( q = (q_1, q_2) \) is defined by

\[
q_1(u) = A_{11}^1 u_1^2 + A_{12}^1 u_1 u_2 + A_{22}^1 u_2^2 \\
q_2(u) = A_{11}^2 u_1^2 + A_{12}^2 u_1 u_2 + A_{22}^2 u_2^2
\]

\[ u = (u_1, u_2), \quad A_{jk} \in \mathbb{R}, \quad i, j, k = 1, 2. \]  \hspace{1cm} (1.3)

Note that semilinear wave equations as

\[
\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = Q(t, x, u, \frac{\partial u}{\partial x}),
\]

where \( Q \) is a quadratic form, are particular cases of such systems under a standard change of function. It is the case for the Carleman system:

\[
(t + x) u_1 = u_2 \\
(t - x) u_2 = u_1.
\]

Uniqueness and local existence of solutions of (1.1), (1.2) are well known. For example we have the following result:

Let \( p \geq 1 \) and \( \mathbf{u}_0 \in L^p(\mathbb{R})^2 \cap L^\infty(\mathbb{R})^2 \). There exists \( T^* > 0, T^* = O(\|u_0\|^{-1}) \), and a unique solution \( u \in C^0([ - T^*, T^* ], L^p(\mathbb{R})^2) \cap L^\infty([ - T^*, T^* ], L^\infty(\mathbb{R})^2) \) of (1.1), (1.2).

\[ \| u \|_\infty \text{ is the standard norm in } L^\infty(\mathbb{R}). \] More generally we note \( \| u \|_q \text{ the standard norm in } L^q(\mathbb{R}), 1 \leq q \leq \infty. \)

Global existence in time is not ensured without conditions on the interaction \( q \) or on the initial data. If the solution does not exist globally, it blows up in finite time: there exists \( T^* > 0 \) such that

\[
\lim_{t \to -(T^*)^-} \|u(t)\|_\infty = + \infty \quad \text{or} \quad \lim_{t \to -(T^*)^+} \|u(t)\|_\infty = + \infty.
\]

This property has been known for a long time but it is not often explicated. A version for quasilinear systems can be found in [6].

In this paper, we first characterize the interactions leading to global existence for any data \( u_0 \in L^\infty_{loc}(\mathbb{R})^2 \). We prove:

**Theorem 1 (Large Data).** 1. Let the interaction be:

\[
q_1(u) = \beta_2 u_2(a_1 u_1 + a_2 u_2), \\
q_2(u) = -\beta_1 u_1(a_1 u_1 + a_2 u_2) \quad \beta_1 \geq 0, \quad \beta_2 \geq 0. \]  \hspace{1cm} (1.4)

Then for any \( u_0 \in L^\infty_{loc}(\mathbb{R})^2 \), the Cauchy problem (1.1), (1.2) has a unique global solution. If in addition \( u_0 \in L^\infty(\mathbb{R}^2) \cap L^1(\mathbb{R})^2 \) then \( u \in L^\infty(\mathbb{R} \times \mathbb{R})^2 \).
2. For all other interactions (1.3), there exists \( u^0 \in \mathcal{S}(\mathbb{R})^2 \) such that the solution of (1.1), (1.2) blows up in finite time.

Now if \( g \) is not of the form (1.4), one may ask for more restrictive conditions on \( u^0 \). Is \( u = 0 \) a stable solution of (1.1)? In other words we consider initial conditions as:

\[
\mathcal{D}(\mathbb{R})^2
\]

For any \( \varphi \) in some functional space, can we find \( \varepsilon(\varphi) > 0 \) such that for \( 0 < \varepsilon < \varepsilon(\varphi) \) the Cauchy problem (1.1), (1.2) has a global solution? Here we characterize such interactions for \( \varphi \in L^\infty(\mathbb{R})^2 \) with compact support.

Definition 1. There is global existence with small data for the system (1.1) when for any \( \varphi \in L^\infty(\mathbb{R})^2 \) with compact support, one can find \( \varepsilon(\varphi) > 0 \) such that for \( 0 < \varepsilon < \varepsilon(\varphi) \) the Cauchy problem (1.1), (1.2) has a global solution. All other interactions are said to be explosive.

Let us recall known results concerning this problem.

The first is due to L. Tartar [8], 1981: if

\[
A_{ij}^\varepsilon = 0 \quad \text{for} \quad i, j = 1, 2,
\]

then for any \( \varphi \in L^1(\mathbb{R})^2 \) one can find \( \varepsilon(\varphi) > 0 \) such that for \( 0 < \varepsilon < \varepsilon(\varphi) \) the Cauchy problem (1.1), (1.2) has a global solution.

In 1985, B. Hanouzet and J.-L. Joly showed that interactions satisfying the condition

\[
\partial u, q_1(u) + q_2(u) \geq \pi(u_1 + u_2)^2
\]

are explosive and that there exists \( \varphi \in \mathcal{S}(\mathbb{R})^2 \) such that for all \( \varepsilon > 0 \) the solution of (1.1), (1.2) blows up in finite positive time [5]. We give the details of the proof here.

In 1986, J. Rauch [7], proved that interactions satisfying the condition

\[
(A_{11}^1)^2 + (A_{22}^2)^2 \neq 0
\]

are explosive: there exists a piecewise smooth function \( \varphi \) with compact support such that for all \( \varepsilon > 0 \) the solution of (1.1), (1.2) blows up in finite positive time. For example it is the case for Carleman’s equations.

Taking these results into account, the remaining unknown cases satisfy

\[
(A_{11}^1)^2 + (A_{22}^2)^2 \neq 0 \quad \text{and} \quad A_{11}^1 = A_{22}^2 = 0
\]
and then we have to consider (1.1) with \((A_{12})^2 + (A_{11})^2 \neq 0\) and
\[
\begin{align*}
q_1(u) &= u_2(A_{12}u_1 + A_{11}u_2) \\
q_2(u) &= u_1(A_{11}u_1 + A_{12}u_2).
\end{align*}
\]
(1.6)

By a diagonal linear change of functions, system (1.1) remains a strictly hyperbolic diagonal system with quadratic homogeneous interaction. We obtain two model cases for (1.6):
\[
\begin{align*}
q_1(u) &= u_2(a_1u_1 + u_2) & a_1, a_2 \in \mathbb{R} \\
q_2(u) &= u_1(u_1 + a_2u_2) & a_1 \in \mathbb{R}, a_2 \leq 0.
\end{align*}
\]
(1.7)

The following theorem ends the characterization of interactions leading to global existence with small data.

**Theorem 2 (Small Data).**

1. Let the interaction be one of the following:

   - interaction (1.7) with \(a_1a_2 \geq 1\) and \(a_1 < 0\) (1.9)
   - interaction (1.8) with \(a_1 \leq 0\). (1.10)

   Then there is global existence with small data for (1.1).

2. Let the interaction be one of the following:

   - interaction (1.7) with \(a_1a_2 < 1\) or \((a_1 \geq 0\) and \(a_2 \geq 0\)) (1.11)
   - interaction (1.8) with \(a_1 > 0\). (1.12)

   Then there exists \(\varphi \in \mathcal{D}(\mathbb{R})^2\) such that for any \(\epsilon > 0\) the solution of (1.1), (1.2) blows up in finite time, and one has an estimate of the blow up time \(T^*\):

   - In case (1.11): \(T^* \leq e^{\#x}\)
   - In case (1.12) or in case (1.11) with \(a_1 \geq 0\), \(a_2 \geq 0\), \(a_1 + a_2 \neq 0\): \(T^* \leq C/e^x\).

**Remark 1.** Of course global existence for any \(u^0 \in L^\infty_{\text{loc}}(\mathbb{R})^2\) should imply global existence with small data:

(1.7) satisfies (1.4) if and only if
\[
a_1a_2 = 1 \quad \text{and} \quad a_1 < 0,
\]
(1.13)
(1.8) satisfies (1.4) if and only if

\[ (a_1 < 0 \text{ and } a_2 = 0) \quad \text{or} \quad a_1 = 0, \]  

(1.14)

and both are cases where Theorem 2 ensures global existence with small data.

The remaining of the paper is devoted to the proof of Theorems 1 and 2. The first part of each of them is proved in Section II and the second part is proved in Section III.

In Section II, we first give the proof of the first part of Theorem 1. The main tool for global existence is the conservation of energy ensured by (1.4) when \( \beta_1, \beta_2 \neq 0 \): we call such systems conservative systems. The asymptotic behaviour is obtained with an additional differential inequality. In the second part of this paragraph, under a sign condition on the data, we establish a global existence theorem in positive time. For such data the energy decays and the proof uses the same method as in first part. The proof of the first part of Theorem 2 is then achieved using a local form of the previous result.

In Section III, for the second part of Theorem 1, the proof is based on a solitary wave method [3]. For Theorem 2, different cases are considered and two distinct methods are used. The first consists in making a functional blow up. We detail here the proof of this result by [5]. The second is a comparison method.

We give in the Appendix the results for nonstrictly hyperbolic \( 2 \times 2 \) systems with homogeneous quadratic interaction. In this case, one has to deal with ordinary differential systems, and the classification is found to be very different from the strictly hyperbolic case.

II. Global Solutions

Let us first detail the proof of Theorem 1, Part 1.

1. Proof of the First Part of Theorem 1

a. Global Existence. We may consider only positive times because \( q \) is quadratic homogeneous: for negative times change \( u(t, x) \) in \( -u(-t, -x) \). By finite propagation speed and continuous dependence on data, \( u^0 \) may be supposed to be in \( \mathcal{D}(\mathbb{R}) \).

Clearly, the result is true if \( \beta_1 = 0 \) or \( \beta_2 = 0 \), so let us suppose \( \beta_1 > 0 \) and \( \beta_2 > 0 \). Then we obtain:

\[ \forall u \in \mathbb{R}^2, \quad \beta_1 u_1 q_1(u) + \beta_2 u_2 q_2(u) = 0. \]  

(2.1)
A system satisfying such a relation is said to be conservative because the energy of the local solution of (1.1), (1.2) is conserved: if \( u \) exists on \([0, t]\):

\[
\int (\beta_1 u_1^2(t, x) + \beta_2 u_2^2(t, x)) \, dx = \int (\beta_1(u_1^0(x))^2 + \beta_2(u_2^0(x))^2) \, dx.
\]

One can show that all conservative interactions are of type (1.4) with \( \beta_1 > 0 \) and \( \beta_2 > 0 \).

By homogeneity, (2.1) is equivalent to the dissipative condition

\[
\forall u \in \mathbb{R}^2 \quad \beta_1 u_1 q_1(u) + \beta_2 u_2 q_2(u) \leq 0 \tag{2.1'}
\]

and also to

\[
\forall u \in \mathbb{R}^2 \quad \beta_1 u_1 q_1(u) + \beta_2 u_2 q_2(u) \geq 0 \tag{2.1''}
\]

but even if the system (1.1) is not conservative, on its domain of existence a particular solution can satisfy a dissipative condition

\[
\alpha_1 u_1 q_1(u) + \alpha_2 u_2 q_2(u) \leq 0, \quad \alpha_1 > 0, \quad \alpha_2 > 0 \tag{2.2}
\]

leading to the decay of the energy. Of course, all the solutions of conservative systems satisfy such a condition.

**Lemma 1.** Let \( T > 0 \) be a time of existence for the local solution \( u \) of (1.1), (1.2) with interaction (1.6). If there exists \( \alpha_1 > 0 \) and \( \alpha_2 > 0 \) such that \( u \) satisfies the dissipative condition (2.2), then:

\[
3C > 0 \text{ independent of } T, \forall (t, x) \in [0, T] \times \mathbb{R},
\]

\[
\int_0^t u_1^2(s, x + c_2 s) \, ds + \int_0^t u_2^2(s, x + c_1 s) \, ds \leq C \cdot \|u^0\|^2. \tag{2.3}
\]

**Proof.** Let us suppose \( c_1 > c_2 \). We define:

\[
\omega = -(\alpha_1 u_1^2 + \alpha_2 u_2^2) \, dx + (c_1 \alpha_1 u_1^2 + c_2 \alpha_2 u_2^2) \, dt.
\]

By (2.2) \( \omega \) is nonpositive. By integration on the domain

\[
D(t, x) = \{(s, y) \in [0, t] \times \mathbb{R}, x - c_1(t - s) < y < x - c_2(t - s)\}
\]

we obtain

\[
(c_1 - c_2) \left(\int_0^t \alpha_1 u_1^2(s, x - c_2(t - s)) \, ds + \int_0^t \alpha_2 u_2^2(s, x - c_1(t - s)) \, ds\right)
\]

\[
\leq \int_{x - c_2 t}^{x - c_1 t} (\alpha_1(u_1^0(y))^2 + \alpha_2(u_2^0(y))^2) \, dy
\]

and then (2.3) holds, which proves Lemma 1.
Interaction (1.4) is a particular case of (1.6). If the local solution of (1.1), (1.2) with (1.6) exists on $]-T^*, T^*[,$ $T^* > 0$, then for $0 < t < T^*$, denoting
\begin{align}
H_1(t, x) &= \int_0^t A_{11}^1 u_2(s, x + c_1 s) \, ds, \\
H_2(t, x) &= \int_0^t A_{22}^2 u_1(s, x + c_2 s) \, ds
\end{align}
we obtain
\begin{align}
u_1(t, x + c_1 t) &= e^{H_1(t, x)} \left[ u_0^1(x) + \int_0^t A_{12}^1 u_2(s, x + c_1 s) e^{-H_1(s, x)} \, ds \right] \quad \text{(a)} \\
u_2(t, x + c_2 t) &= e^{H_2(t, x)} \left[ u_0^2(x) + \int_0^t A_{21}^2 u_1(s, x + c_2 s) e^{-H_2(s, x)} \, ds \right] \quad \text{(b)}.
\end{align}

The following lemma provides a sufficient condition for continuing a local solution $u$ after a given time $T$.

**Lemma 2.** If the local solution of (1.1), (1.2) with interaction (1.6) exists on $[0, T]$ and if there exists $i \neq j$ such that
\begin{equation}
\forall t < T, \quad \int_0^t u_j^i(s, x + c_i s) \, ds \leq M(T) \quad \text{(2.6)}
\end{equation}
then $u$ can be continued after $T$.

**Proof.** Let us suppose $i = 1, j = 2$. Then we have:
\begin{equation}
|H_j(t, x)| \leq |A_{12}^1| \int_0^t |u_2(s, x + c_1 s)| \, ds \leq |A_{12}^1| \sqrt{M(T)}.
\end{equation}
(2.5a) becomes
\begin{equation}
|u_1(t, x + c_1 t)| \leq e^{C \sqrt{MT}} \left[ \|u_0^1\|_{\infty} + C \cdot e^{C \sqrt{MT}} M(T) \right]
\end{equation}
where $C$ is a positive constant, and so:
\begin{equation}
\|u_1(t, \cdot)\|_{\infty} \leq K(T).
\end{equation}
By (2.5b), we obtain:
\begin{equation}
\|u_2(t, \cdot)\|_{\infty} \leq C_1(T) \quad \forall t < T.
\end{equation}
A standard continuation argument allows us to conclude.

The global existence of $u$ is a direct consequence of Lemmas 1 and 2.
b. Asymptotic Behaviour. Here we suppose \( u^0 \in L^\infty(\mathbb{R})^2 \cap L^1(\mathbb{R})^2 \). As for existence, we may consider only positive times.

We are going to prove:

\[
\|u(t)\|_\infty \leq C(\|u^0\|_\infty, \|u^0\|_1, \|u^0\|_2).
\]  

(2.7)

By finite propagation speed and continuous dependence on data, it is sufficient to suppose \( u^0 \in D(\mathbb{R})^2 \).

If \( \beta_1, \beta_2 = 0 \), say \( \beta_2 = 0 \), then

\[
u_1(t, x) = u_0^1(x - c_1 t)
\]

so that

\[
H_2(t, x) = -\beta_1 a_2 \int_0^t u_0^0(x + (c_2 - c_1) s) \, ds
\]

and then by (2.5b)

\[
|u_2(t, x + c_2 t)| \leq K e^{\int_0^t \|u^0\|_\infty} \cdot [\|u^0\|_\infty + \|u^0\|_2^2]
\]

which gives (2.7).

Suppose now \( \beta_1, \beta_2 \neq 0 \) and for example \( c_1 > c_2 \). We have the differential inequality:

\[
\partial_t(a_2 \beta_1 u_1 - a_1 \beta_2 u_2) + \partial_x(a_2 c_1 \beta_1 u_1 - a_1 c_2 \beta_2 u_2) \geq 0.
\]

As in proof of Lemma 1 we integrate on the domain

\[
D(t, x) = \{(s, y) \in [0, t] \times \mathbb{R}, x - c_1(t-s) < y < x - c_2(t-s) \}
\]

and obtain

\[
\int_0^t a_2 \beta_1 u_1(s, x - c_2(t-s)) \, ds - \int_0^t a_1 \beta_2 u_2(s, x - c_1(t-s)) \, ds
\]

\[
\geq \frac{1}{c_1 - c_2} \int_{x-c_1 t}^{x-c_2 t} (a_2 \beta_1 u_0^1(y) - a_1 \beta_2 u_0^2(y)) \, dy
\]

or

\[
H_2(t, x - c_2 t) + H_1(t, x - c_1 t)
\]

\[
\leq \frac{1}{c_1 - c_2} \int_{x-c_1 t}^{x-c_2 t} (-a_2 \beta_1 u_{0}^1(y) + a_1 \beta_2 u_{0}^2(y)) \, dy
\]

\[
\leq M_1(\|u^0\|_1).
\]
Let us denote $M = e^{M_i}$:

$$
\exp(H_2(t, x - c_2 t)) \leq M \exp(-H_1(t, x - c_1 t)).
$$

(2.8)

(2.3) is satisfied for all $T > 0$. If we prove

$$
\exists A > 0, \quad \forall (t, x) \in \mathbb{R}^+ \times \mathbb{R} \quad \forall s \in [0, t],
\quad H_i(t, x) - H_i(s, x) \leq A, \quad i = 1, 2,
$$

(2.9)

where $A$ is an increasing function of $\|u^0\|_1$, then (2.5) will give (2.7).

Let us first show that $\exp(H_1)$ is bounded. (2.5.b) with $A_{i1} = -\beta_i a_1$, $A_{i2} = -\beta_i a_2$ yields:

$$
\beta_i a_1 u_2(t, x) \leq |\beta_i a_1| \exp(H_2(t, x - c_2 t)) \cdot |u^0_2(x - c_2 t)|.
$$

By (2.8)

$$
\beta_i a_1 u_2(t, x + c_1 t) \exp(H_1(t, x)) \leq M \cdot |\beta_i a_1| \cdot |u^0_2(x - (c_2 - c_1) t)|
$$

and then

$$
\exp(H_1(t, x)) - \exp(H_1(0, x)) \leq C \cdot \|u^0\|_1
$$

which proves that $\exp(H_1)$ is bounded by a constant $K$.

Multiplying (2.5.a) by $-\beta_i a_2$ and integrating between $s$ and $t$ on characteristic 2 we can then estimate

$$
H_2(t, x) - H_2(s, x) = -\int_s^t \beta_i a_2 u_1(\sigma, x + c_2 \sigma) d\sigma,
$$

$$
\|H_2(t, x) - H_2(s, x)\|_1 \leq |\beta_i a_2| K \int_s^t |u^0_2(x - (c_2 - c_1) \sigma)| d\sigma \leq C_1 \|u^0\|_1 = K,'
$$

Multiplying (2.5.b) by $\beta_i a_1$ and integrating between $s$ and $t$ on characteristic 1 we can then estimate

$$
H_1(t, x) - H_1(s, x) = \int_s^t \beta_i a_1 u_2(\sigma, x + c_1 \sigma) d\sigma,
$$

$$
\|H_1(t, x) - H_1(s, x)\|_1 \leq C_2 \|u^0\|_1
$$

which proves (2.9) and ends the proof.

Let us recall the two model cases:

$$
\begin{align*}
q_1(u) &= u_2(a_1 u_1 + u_2) & a_1, a_2 &\in \mathbb{R} \\
q_2(u) &= u_1(a_1 u_1 + a_2 u_2) & & \\
q_1(u) &= a_1 u_1 u_2 & a_1 &\in \mathbb{R}, \quad a_2 &< 0.
\end{align*}
$$

(1.7)
We have to prove global existence with small data for (1.1) under one of the following conditions:

interaction (1.7) with $a_1a_2 \geq 1$ and $a_1 < 0$  \hspace{1cm} (1.9)

interaction (1.8) with $a_1 \leq 0$.  \hspace{1cm} (1.10)

According to Remark 1, Theorem 1 proves a part of Theorem 2: cases (1.13) and (1.14). Hence we do no longer consider these cases. (1.9) and (1.10) become:

interaction (1.7) with $a_1a_2 > 1$ and $a_1 < 0$  \hspace{1cm} (2.10)

interaction (1.8) with $a_1 < 0$ and $a_2 < 0$.  \hspace{1cm} (2.11)

For these interactions, we show in section III that there exists $u^0 \in (\mathcal{S}(\mathbb{R}))^2$ so that the solution of (1.1), (1.2) blows up in finite time (Theorem 1, Part 2). In order to prove global existence for small data we show now an intermediate result.

2. A Global Existence Result with Positive Data

By homogeneity of the interaction $q$, changing the data $u^0(x)$ in $-u^0(-x)$ is equivalent to change the time direction, so that without sign condition, global existence in positive time is equivalent to global existence in time. We already used this property in the last part. But here, we impose a sign condition on data and our global existence result is unilateral in time.

**Theorem 3.** Let $u^0$ be in $L^\infty(\mathbb{R})^2$ with $u^0_1 \geq 0$ or $u^0_2 \geq 0$. If one of the two conditions (2.10), (2.11) holds, then the Cauchy problem (1.1), (1.2) has a unique global solution in positive time $u \in L^\infty_{\text{loc}}([0, +\infty[; L^2(\mathbb{R})^2)$. For $u^0_1 \geq 0$ and $u^0_2 \geq 0$, $u^0 \in L^\infty(\mathbb{R})^2 \cap L^2(\mathbb{R})^2$, the global solution in positive time satisfies:

$$0 \leq u_i \leq C\left(\|u^0_1\|_\infty, \|u^0_2\|_2\right), \quad i = 1, 2.$$  

**Proof.** We can suppose $u^0$ in $\mathcal{S}(\mathbb{R})^2$.

1. **Global Existence.** First note that for interactions (1.7) and (1.8), we have $A_{12}^{\pm} \geq 0$ (1 or 0), $A_{11}^{\pm} = 1$. Thus by (2.5) for all $(t, x)$ in $\mathbb{R}^+ \times \mathbb{R}$ and $i = 1, 2$:

if $u^0_i(x) \geq 0$ then $u_i(t, x + c_i t) \geq 0$. 


Hence for $i = 1, 2$:

if $u_0^i \geq 0$ then $u_i(t, \cdot) \geq 0$ for all $t > 0$.

1.a. Interaction (2.11) with $u_0^a \geq 0$. Then $u_0^a(t, x) \geq 0$ for $t \geq 0$. Let us suppose that $u$ exists on $[0, T] \times \mathbb{R}$, $T > 0$. $a_1 < 0$, so by (2.5.a) we have

\[
\forall x \in \mathbb{R} \quad \forall t \in [0, T] \quad |u_i(t, x + c_1 t)| \leq |u_0^i|_\infty
\]

and by (2.5.b)

\[
\forall x \in \mathbb{R} \quad \forall t \in [0, T] \quad |u_2(t, x + c_2 t)| \leq C e^{K |u_1|_\infty T} |u_0^a|_\infty + T |u_0^a|_\infty^2
\]

where $K$ and $C$ are positive constants. Hence for $0 \leq t < T$

\[
|u(t, \cdot)|_\infty \leq M(\|u^0\|_\infty, T)
\]

which ensures global existence.

1.b. Interaction (2.10) with $u_0^1 \geq 0$ or $u_0^2 \geq 0$, and interaction (2.11) with $u_0^a \geq 0$. In both cases, we show that the local solution satisfies (2.2):

(i) In the case of (2.10), the local solution satisfies

\[
\begin{align*}
\partial_x u_1^1(x) - a_1 u_2 q_2(u) &= u_1^a u_2 q_2(1 - a_1 a_2) \\
\partial_x u_1^a(x) + u_2 q_2(u) &= u_1^a u_2(1 - a_1 a_2).
\end{align*}
\]

(ii) In the case of (2.11)

\[
\begin{align*}
\partial_x u_1^1(x) - a_1 u_2 q_2(u) &= -u_1^a u_2^a a_1 a_2.
\end{align*}
\]

Hence, if the solution exists on $[0, T] \times \mathbb{R}$, (2.2) holds on this domain with:

$(\sigma_1, \sigma_2) = (1, -a_1) \text{ or } (-a_2, 1)$.

We conclude by Lemma 1 and Lemma 2.

2. Asymptotic behaviour of $u$ for $u^0 \geq 0$ (i.e., $u_0^a \geq 0$ and $u_0^a \geq 0$). Consider (2.10). The solution is nonnegative. By (2.5.a)

\[
u_1(t, x + c_1 t) = e^{H_1(t, x)} \left[ u_0^1(x) + \int_0^t u_2^a(s, x + c_1 s) e^{-H_1(s, x)} \, ds \right]
\]

with

\[
H_1(t, x) = \int_0^t a_1 u_2^1(s, x + c_1 s) \, ds
\]
thus
\[ H_1(t, x) - H_1(s, x) \leq 0 \quad \text{for} \quad 0 \leq s \leq t \]
and then
\[ 0 \leq u_1(t, x + c_1 t) \leq u_0^1(x) + \int_0^t u_2^1(s, x + c_1 s) \, ds \quad \forall x \in \mathbb{R}. \]

Furthermore, \( u^0 \in (L^2(\mathbb{R}))^2 \), and from (2.3) we get
\[ \int_0^t u_2^1(s, x + c_1 s) \, ds \leq C \| u^0 \|_2 \]
and then, for \( t \geq 0, x \in \mathbb{R} \)
\[ 0 \leq u_1(t, x + c_1 t) \leq \| u_0^1 \|_\infty + C \| u^0 \|_2 \]
and similarly
\[ 0 \leq u_2(t, x + c_2 t) \leq \| u_0^2 \|_\infty + C \| u^0 \|_2. \]

The case of interaction (2.11) is similar.

In the following, we use a local form of Theorem 3 (finite speed propagation):

**Corollary 1.** Let \( c_M = \max(c_1, c_2), \ c_m = \min(c_1, c_2), \ -\infty \leq a < b \leq +\infty, \) and \( u^0 \in L^\infty(\mathbb{R})^2 \) with \( u_0^1 \geq 0 \) or \( u_0^2 \geq 0 \) on \((a, b)\).

If the interaction is of type (2.10) or (2.11), then (1.1), (1.2) has a unique solution \( u \) which exists in the causal domain:
\[ \{ (t, x); 0 \leq t \leq \frac{b-a}{c_M - c_m} \text{ and } a + c_M t < x < b + c_m t \}. \]

We are now in position to achieve the proof of the first part of Theorem 2.

3. **Proof of the First Part of Theorem 2**

We have to prove that in cases (2.10) and (2.11):

For any \( \varphi \in L^\infty(\mathbb{R})^2 \) with compact support, one can find \( \varepsilon > 0 \) such that for \( 0 < \varepsilon < \varepsilon(\varphi) \) the Cauchy problem (1.1), (1.2) has a global solution.

First, let us suppose (2.10) or (2.11) and \( c_1 > c_2 \). Let \( \varphi \in (L^\infty(\mathbb{R}))^2, \supp \varphi \subset (a, b) \).
Let $T^*$ be such that the local solution exists in $[0, T^*] \times \mathbb{R}$. It is known that there exists $K > 0$ (depending on the interaction) such that:

$$T^* \geq K \varepsilon^{-1} \| \varphi \|^{-1}_\infty. \quad (2.12)$$

**Lemma 3.** Let us suppose that the interaction is of type (2.10) or (2.11). Let

$$\varepsilon_0 = (c_1 - c_2) K (b - a)^{-1} \| \varphi \|^{-1}_\infty.$$

For $0 \leq \varepsilon < \varepsilon_0$, the Cauchy problem (1.1), (1.2') has a global solution $u \in L^\infty_\text{loc}(\mathbb{R}; L^\infty(\mathbb{R})^2)$.

**Proof.** We can suppose $\varphi$ in $\mathcal{D}(\mathbb{R})^2$ and $t \geq 0$. Let $\tilde{T}$ be the vertex of the causal domain $C(a, b) = \{(t, x), t \geq 0, a + c_1 t \leq x \leq b + c_2 t\}$

$$\tilde{T} = \frac{b - a}{c_1 - c_2},$$

so that $T^* \geq \tilde{T}$.

Let $\tilde{T} < T < T^*$. By (2.5) we have

$$u_i(t, x + c_1 t) \geq 0 \quad \text{for} \quad x \leq a$$

$$u_i(t, x + c_2 t) \geq 0 \quad \text{for} \quad x \geq b$$

and then at $T$

$$u_i(T, y) \geq 0 \quad \text{for} \quad y \leq a + c_1 T$$

$$u_i(T, y) \geq 0 \quad \text{for} \quad y \geq b + c_2 T.$$

We consider the Cauchy problem:

$$\partial_t v_i = q_i(v)$$

$$v_i(T, x) = u_i(T, x) \quad i = 1, 2.$$ 

Clearly: $v = u$ for $T \leq t < T^*$. By Corollary 1, $v$ exists in

$$\{(t, x); t \geq T, x < a + c_1 T + c_2 (t - T) \text{ or } x > b + c_2 T + c_1 (t - T)\}$$

and consequently for $x \in \mathbb{R}$, $T \leq t \leq 2 T - \tilde{T} = T^*$. 

Then, we choose $T \geq (\tilde{T} + T^*)/2$: $u$ can be continued after $T^*$. Hence, the solution $u$ exists for all positive times.

If $c_1 < c_2$, $u_1$ and $u_2$ can be exchanged since only $c_M$ and $c_m$ appear in Corollary 1.
III. Blow Up

In this section we prove the second part of Theorem 1 and Theorem 2.

1. Proof of Second Part of Theorem 1

The proof is based on a solitary wave method, following an idea by M. Balabane \[3\].

We note $c_m = \min(c_1, c_2)$ and $c_M = \max(c_1, c_2)$. Suppose there exists $\sigma_1 < \sigma_2$, $c \in \mathbb{R}$, $\phi \in C^\infty(\sigma_1, \sigma_2)$ such that:

$$u^*(t, x) = \phi(x - ct)$$

is a solution of (1.1) in

$$B = \{(t, x) \in \mathbb{R}^2; x - ct \in [\sigma_1, \sigma_2]\} \quad (3.1)$$

$$\lim_{\sigma \to \sigma_1} |\phi(\sigma)| = +\infty \quad \text{or} \quad \lim_{\sigma \to \sigma_2} |\phi(\sigma)| = +\infty. \quad (3.2)$$

Then $u^*$ blows up along the line $L = \{(t, x) \in \mathbb{R}^2; x - ct = \sigma^*\}$ where $\sigma^* = \sigma_1$ in case (3.2.a) and $\sigma^* = \sigma_2$ in case (3.2.b). We choose $\sigma_1 < a < b < \sigma_2$ and construct $u^0 \in \mathcal{D}(\mathbb{R}^2)$, $u^0 = \phi$ on $[a, b]$, and supp $u^0 \subset \sigma_1, \sigma_2$.

Let $C(a, b)$ be the causal domain $\{(t, x), t \geq 0, a + c_M t \leq x \leq b + c_M t\}$. In $B \cap C(a, b)$, the solution $u$ of the Cauchy problem (1.1), (1.2) is $u^*$ and it blows up in $t > 0$ if

$$L \cap C(a, b) \neq \emptyset \quad (3.3)$$

$a$ and $b$ can be chosen as close to $\sigma_1$ and $\sigma_2$ as we wish, thus (3.3) holds if

(a) $\begin{cases} c < c_m \\ \lim_{\sigma \to \sigma_1} |\phi(\sigma)| = +\infty \end{cases}$ or (b) $\begin{cases} c > c_M \\ \lim_{\sigma \to \sigma_2} |\phi(\sigma)| = +\infty \end{cases}. \quad (3.4)$

Actually in our case we will look for polarized solitary waves:

$$\phi = \lambda \psi, \quad \lambda \in \mathbb{R}^2, \quad \lambda \neq 0, \quad \psi \in C^\infty(\sigma_1, \sigma_2).$$

Then (3.1) is satisfied if and only if:

$$\lambda_i (c_i - c) \psi' = \psi^2 \psi_i(\lambda), \quad i = 1, 2.$$

The solution of

$$\psi' = \psi^2$$

$$\psi(0) = \psi_0 > 0$$

blows up at $\sigma = 1/\psi_0$ and one can easily check the
**Lemma 4.** Suppose there exists \( \lambda \in \mathbb{R}^2, \lambda \neq 0, c \in ] - \infty, c_m[ \cup ]c_M, + \infty[ \) satisfying:

\[
\hat{\lambda}_i(c_i - c) = q_i(\lambda) \quad i = 1, 2. \tag{3.5}
\]

Then there exists \( u^0 \in \mathcal{D}(\mathbb{R})^2 \) and \( T^* > 0 \) such that the solution \( u \) of (1.1), (1.2) satisfies:

\[
\lim_{t \to T^*} \|u(t)\|_{\infty} = + \infty.
\]

We are now in position to prove the second part of Theorem 1. In order to apply Lemma 4, we look for \( \lambda \in \mathbb{R}^2, \lambda \neq 0, c \in ] - \infty, c_m[ \cup ]c_M, + \infty[ \) satisfying (3.5).

First we examine the possibility for \( \hat{\lambda}_1 \) or \( \hat{\lambda}_2 \) to be equal to zero. It leads to:

\[
(a) \begin{cases} \hat{A}_{11} \neq 0 \\ \hat{A}_{11} = 0 \end{cases} \quad \text{or} \quad \begin{cases} \hat{A}_{12} \neq 0 \\ \hat{A}_{12} = 0 \end{cases} \tag{3.6}
\]

If (3.6) does not hold, we may look for \( \lambda \in (\mathbb{R}^*)^2, c \in ] - \infty, c_m[ \cup ]c_M, + \infty[ \) such that

\[
\hat{\lambda}_i(c_i - c) = q_i(\lambda) \quad i = 1, 2 \tag{3.7}
\]

or, taking into account the homogeneity of \( q \)

\[
\frac{c_1 - c}{c_2 - c} = \frac{\hat{\lambda}_2 q_1(\hat{\lambda})}{\hat{\lambda}_1 q_2(\hat{\lambda})} \tag{3.8}
\]

We denote \( \varphi(\lambda) = \frac{\hat{\lambda}_2 q_1(\hat{\lambda})}{\hat{\lambda}_1 q_2(\hat{\lambda})} \).

As \( (c_1 - c)/(c_2 - c) \) takes all values in \( ]0, 1[ \cup ]1, + \infty[ \) one has to find \( \lambda \in (\mathbb{R}^*)^2 \) such that \( \varphi(\lambda) > 0 \) and \( \varphi(\lambda) \neq 1 \). \( \varphi \equiv 1 \) being a trivial case, we have to study the sign of the homogeneous polynomial

\[
Q(\lambda) = \hat{\lambda}_1 \hat{\lambda}_2 q_1(\hat{\lambda}) q_2(\hat{\lambda})
\]

One can show that it always takes positive values except in the case of interaction (1.4).

**Corollary 2.** Suppose \( p > 1 \). If the interaction is not of form (1.4) then for any \( \epsilon > 0 \) there exists \( \epsilon^0 \in \mathcal{D}(\mathbb{R})^2 \) such that:

(i) \( \|\epsilon^0\|_p < \epsilon \)

(ii) the solution of (1.1), (1.2) blows up in finite time.
Proof. If \( u \) is solution of (1.1), (1.2), then for \( \gamma > 0 \), \( \gamma u(\gamma t, \gamma x) \) is solution of (1.1), (1.2) for the initial data \( u^0 = \gamma u^0(\gamma x) \) and
\[
\|u^0\|_p = \gamma^{1 - \frac{1}{p}} \|u^0\|_p
\]
which tends to zero as \( \gamma \) tends to zero.

Remark 2. This corollary characterizes the blow up for a certain type of small data and it is here the same as for large data. Definition 1 is much more restrictive: it requires smallness in all \( L^q \) norms including \( q = 1 \) and the cases of blow up and global existence are not the same. For example, consider the interaction:

\[
q_1(u) = a_1 u_1 u_2, \quad a_1 < 0, \quad a_2 < 0. \tag{2.11}
\]

It is not of form (1.4), so that one can apply Corollary 2. But according to Theorem 2, first part, there is global existence for small data: one cannot find \( \varphi \in L^+([\mathbb{R}]^2) \) with compact support such that for any \( \varepsilon > 0 \), the solution of (1.1), (1.2') blows up in finite time.

2. Proof of Second Part of Theorem 2

Let us first recall and prove a result from [5].

**Theorem 4.** Suppose \( u \) solution of (1.1), (1.2') and suppose that the interaction satisfies
\[
\exists \alpha > 0 \quad \forall u \in \mathbb{R}^2 \quad q_1(u) + q_2(u) \geq \alpha(u_1 + u_2)^2 \tag{3.9}
\]
and that \( \varphi \in \mathcal{D}(\mathbb{R})^2 \) is such that
\[
\int_{\mathbb{R}} (\varphi_1(x) + \varphi_2(x)) \, dx = A > 0. \tag{3.10}
\]

Then for \( \varepsilon > 0 \) there exists \( T^*_\varepsilon > 0 \) such that
\[
\lim_{t \to (T^*_\varepsilon)^-} \|u(t)\|_\infty = +\infty
\]
and we have the estimate
\[
T^*_\varepsilon \leq e^{B\varepsilon}, \tag{3.11}
\]
where \( B > 0 \) depends on \( \varepsilon_1, \varepsilon_2, \alpha, \) and \( \varphi. \)
Proof. Suppose \( \text{supp} \varphi \subseteq [-R, R], R > 0 \). As long as the local solution remains bounded, we know that \( x \rightarrow u(t, x) \) is \( C^\infty \) with compact support and for \( t > 0 \)

\[
(i) \quad \text{supp} \, u(t, \cdot) \subseteq \left[ -(R + ct), R + ct \right]
\]

where \( c = \text{supp}([|c_1|, |c_2|]) \).

For \( t \geq 0 \), we define:

\[
J(t) = \int_{-R}^R (q_1(u(s, x) + q_2(u(s, x))) \, ds \, dx + \varepsilon \int_{-R}^R (\varphi_1(x) + \varphi_2(x)) \, dx.
\]

We have

\[
0 \leq J(t) = \int_{-R}^R (u_1(t, x) + u_2(t, x)) \, dx
\]

and

\[
(J(t))^2 \leq 2(R + ct) \int_{-R}^R (u_1(t, x) + u_2(t, x))^2 \, dx
\]

\[
\leq \frac{2}{\alpha} (R + ct) \int_{-R}^R \left[ q_1(u(t, x)) + q_2(u(t, x)) \right] \, dx.
\]

In addition,

\[
\frac{d}{dt}(J(t)) = \int_{-R}^R \left[ q_1(u(t, x)) + q_2(u(t, x)) \right] \, dx.
\]

Hence

\[
J'(t) \geq \frac{\alpha}{2(R + ct)} J^2(t)
\]

\[
J(0) = \varepsilon A
\]

and \( J \geq Y \) where \( Y \), solution of

\[
Y'(t) = \frac{\alpha}{2(R + ct)} Y^2(t)
\]

\[
Y(0) = \varepsilon A
\]
satisfies
\[
\frac{1}{Y(t)} = \frac{1}{eA} - \frac{x}{2c} \ln \frac{R + ct}{R}.
\]

Hence, \( Y \) blows up in finite time
\[
T^* = \frac{R}{c} (e^{2c/eA} - 1)
\]
and we obtain (3.11).

We now apply Theorem 4 in order to obtain blow up results for the model case:
\[
q_1(u) = u_2(a_1 u_1 + u_2) \quad a_1, a_2 \in \mathbb{R} \tag{1.7}
\]
with condition

interaction (1.7) with \( a_1,a_2 < 1 \) or \( a_1 \geq 0 \) and \( a_2 \geq 0 \). \tag{1.11}

**Lemma 5.** If the interaction is one of the following:

interaction (1.7) with \( a_1,a_2 < 1 \) \hspace{1cm} (1.11.a)

interaction (1.7) with \( a_1a_2 \geq 1 \) and \( a_1 > 0 \) \hspace{1cm} (1.11.b)

there exists a function \( \varphi \in \mathcal{D}(\mathbb{R}) \) such that the solution of (1.1), (1.2) blows up in finite time. In addition one has the estimate (3.11) of the blow up time.

**Proof.** By the transformation \((u_1, u_2) \rightarrow (\lambda_1 u_1, \lambda_2 u_2), \lambda_1 \neq 0, \lambda_2 \neq 0\), Theorem 4 can be applied if there exists \( \lambda_1 \neq 0, \lambda_2 \neq 0, x > 0 \) such that for any \( u \in \mathbb{R}^2 \):
\[
\frac{1}{\lambda_1} q_1(\lambda_1 u_1, \lambda_2 u_2) + \frac{1}{\lambda_2} q_2(\lambda_1 u_1, \lambda_2 u_2) \geq x(u_1 + u_2)^2. \tag{3.12}
\]

Consider interaction (1.7) with \( a_1a_2 < 1 \). (3.12) is satisfied if and only if:
\[
\lambda_1^2 \lambda_2^{-1} u_1^2 + (a_1 \lambda_2 + a_2 \lambda_1) u_1 u_2 + \lambda_2^2 \lambda_1^{-1} u_2^2 \geq x(u_1 + u_2)^2.
\]

We suppose \( x = 1 \), since otherwise \( \lambda \) can be replaced by \( \lambda x^{-1} \). So we want \( \lambda_1, \lambda_2 \) such that:
\[
(\lambda_1^2 \lambda_2^{-1} - 1) u_1^2 + (a_1 \lambda_2 + a_2 \lambda_1 - 2) u_1 u_2 + (\lambda_2^2 \lambda_1^{-1} - 1) u_2^2 \geq 0.
\]
Necessarily
\[ \lambda_1^2 \lambda_2^{-1} - 1 > 0 \]
so that \( \lambda_2 > 0 \) and similarly \( \lambda_1 > 0 \).

It is sufficient to find \((\lambda_1, \lambda_2) = (\lambda, p\lambda), \lambda > 0, p > 0\) such that
\[ \lambda^2 - p\lambda > 0 \] (a)
\[ p^2\lambda^2 - \lambda > 0 \] (b) \(3.13\)
\[ \lambda(\lambda(a_1p + a_2)^2 - 4p) + 4(p^2 + p^{-1}) - 4(a_1p + a_2) < 0 \] (c)

\(a_1a_2 < 1\), so there exists \(p > 0\) such that
\[ (a_1p + a_2)^2 - 4p < 0 \] (3.14)
and then there exists \(\lambda_0\) such that for \(\lambda > \lambda_0\), (3.13) holds. Hence there exists \(\lambda_1 > 0, \lambda_2 > 0\), \(\alpha > 0\) such that (3.12) is realised for every \(u\) in \(\mathbb{R}^2\).

Thus, Theorem 4 can be applied in the case (1.11.a).

Actually, to apply Theorem 4, it is sufficient to prove that the solution of (1.1), (1.2) satisfies:
\[ \exists \alpha > 0 \quad q_1(u) + q_2(u) \geq \alpha(u_1 + u_2)^2. \] (3.12‘)

If \(q_1 \geq 0, q_2 \geq 0, a_1 \geq 0, a_2 \geq 0\) then in the case of interaction (1.7) the solution is positive and
\[ q_1(u) + q_2(u) \geq u_1^2 + u_2^2 \]
and consequently (3.12‘) is satisfied. Hence Theorem 4 allows us to conclude in the case of (1.11.b).

Now the remaining cases are the interaction
\[ q_1(u) = a_1u_1u_2 \]
\[ q_2(u) = u_1(u_1 + a_2u_2) \]
with the condition
interaction (1.8) with \(a_1 > 0\) (1.12)
and the second estimate of blow up time for (1.7) with \(a_1 \geq 0, a_2 \geq 0, a_1 + a_2 \neq 0\).

**Theorem 5.** Let \(u\) be the solution of (1.1), (1.2) with:
\[ q_1(u) = a_1u_1u_2 \]
\[ q_2(u) = u_1^2 \quad a_1 > 0. \] (3.15)
Then, there exists \( \varphi \in \mathcal{D}(\mathbb{R})^2 \), \( \varphi \geq 0 \), such that for \( \varepsilon > 0 \), there exists \( T^*_\varepsilon > 0 \) with

\[
\lim_{t \to (T^*_\varepsilon)^-} \| u(t) \|_\infty = + \infty
\]

and

\[
T^*_\varepsilon \leq \frac{C}{\varepsilon^2}.
\] (3.16)

**Proof.** For \( \varphi \in \mathcal{D}(\mathbb{R})^2 \) the local solution is \( C^\infty \) and (2.5.b) can be written:

\[
u_2(t, x + c_2 t) = \varepsilon \varphi_2(x) + \int_0^t u_2^\varepsilon(s, x + c_2 s) \, ds.
\] (3.17)

Consequently, if \( \varphi_2 \geq 0 \), then \( u_2 \geq 0 \). Moreover

\[u_1 \partial_t u_1 - a_1 u_2 \partial_2 u_2 = 0.
\]

As in the proof of Lemma 1, we obtain:

\[
(c_1 - c_2) \left( \int_0^r u_2^\varepsilon(s, x - c_2(t - s)) \, ds - \int_0^r a_1 u_2^\varepsilon(s, x - c_1(t - s)) \, ds \right)
\]

\[= \varepsilon^2 \int_{x - c_1 t}^{x - c_2 t} (\varphi_2^\varepsilon(y) - a_1 \varphi_2^\varepsilon(y)) \, dy.
\] (3.18)

Let us choose \( \varphi \in \mathcal{D}(\mathbb{R})^2 \) satisfying the following conditions:

\[\varphi_1^\varepsilon - a_1 \varphi_2^\varepsilon \equiv 0\]

\[0 \leq a_1 \varphi_2 \leq 1,\]

\[\exists x_0 \in \mathbb{R}, \quad a_1 \varphi_2(x_0) = 1,\]

\[
\text{supp} \, \varphi_2 \subset [x_0 - |c_1 - c_2|, x_0 + |c_1 - c_2|].
\] (3.19)

By (3.18) and (3.19), we have:

\[
a_1 u_2(t, x + c_1 t) = a_1 \varphi_2(x + (c_1 - c_2) t) + \int_0^t a_1 \varphi_2(s, x + c_1 s) \, ds.
\] (3.20)

Let us denote

\[f(t) = a_1 u_2(t, x_0 + c_1 t), \quad \psi(t) = a_1 \varphi_2(x_0 + (c_1 - c_2) t)\]

Theorem 5 is then a consequence of:
**Lemma 6.** If \( \psi \in \mathcal{D}(\mathbb{R}) \) with \( \psi(0) = 1 \), \( 0 \leq \psi \leq 1 \), supp \( \psi \subset [-1, 1] \) and if \( f \) is solution of
\[
f(t) = z\psi(t) + \int_0^t f^2(s) \, ds \quad t \geq 0
\]
then there exists \( T^*_n > 0 \) satisfying (3.16) such that
\[
\lim_{t \to (T^*_n)^-} f(t) = +\infty.
\]

**Proof.** We compare \( f \) to the solution of
\[
y' = y^2
\]
\[
y(0) = \varepsilon.
\]
Introducing \( z = y - f \), we have:
\[
y(t) - f(t) = \varepsilon(1 - \psi(t)) + \int_0^t (y(s) + f(s)) z(s) \, ds.
\]
As long as they exist, \( f \) and \( y \) are positive for positive time and \( (1 - \psi(t)) \) is positive, thus \( z \) is positive for positive time.
As \( y(t) = \varepsilon(1 - \alpha t)^{-1} \), for \( \varepsilon \) small enough \( (\varepsilon < 1) \), \( y(t) \) and \( f(t) \) exist on \([0, 1]\) and
\[
\varepsilon^2 \int_0^1 y^2(s) \, ds \leq f(1) \leq \varepsilon^2 \frac{1}{1-\varepsilon}.
\]
For \( t \geq 1 \) we have \( f'(t) = f^2(t) \) so:
\[
f(t) = \frac{f(1)}{1 - f(1)(t - 1)}
\]
The lemma is then obtained with \( T^*_n = (1/f(1)) + 1 \), which satisfies (3.16) by (3.22).

This last theorem allows us to end the proof of Theorem 2.

**Lemma 7.** If the interaction is one of the following:
\[
\text{interaction (1.8) with } a_1 > 0 \\
\text{interaction (1.7) with } a_1 \geq 0, a_2 \geq 0, a_1 + a_2 \neq 0,
\]
there exists a function \( \varphi \in \mathcal{D}(\mathbb{R})^2 \) such that the solution of (1.1), (1.2') blows up in finite time. In addition one has the estimate (3.16) for the blow up time.
Proof. Consider (1.12). We choose \( \varphi \) as in Theorem 5 with \( \varphi_1 \leq 0 \). Then the solution \( u \) of (1.1), (1.2') satisfies \( u_1 \leq 0 \) and \( u_2 \geq 0 \).

Let \( v \) be the solution of (3.15), (1.2') with the initial condition \((-e\varphi_1, e\varphi_2)\) which also satisfies (3.19). Then \( v_1 \geq 0 \) and \( v_2 \geq 0 \).

We define \( w_1 = -u_1 - v_1, w_2 = u_2 - v_2 \). Then

\[
\begin{align*}
\partial_t w_1 &= a_1 u_2 w_1 + a_1 v_1 w_2 \\
\partial_t w_2 &= (-u_1 + v_1) w_1 + a_2 u_1 u_2
\end{align*}
\]

and recalling that \( a_2 \leq 0 \), we have \( a_2 u_1 u_2 \geq 0 \).

**Lemma 8.** Let \( T \) be a positive time and \( a, b, c \) smooth non negative functions from \([0, T] \times \mathbb{R}\) to \( \mathbb{R}^2 \). Then a solution of

\[
\begin{align*}
\partial_t w_1 &= a_1(t, x) w_1 + b_1(t, x) w_2 + c_1(t, x) \\
\partial_t w_2 &= a_2(t, x) w_1 + b_2(t, x) w_2 + c_2(t, x)
\end{align*}
\]

is nonnegative on \([0, T] \times \mathbb{R}\).

Consequently, for every \( T > 0 \) such that \( u \) and \( v \) exist on \([0, T] \times \mathbb{R} \) we have

\[
-w_1(0, x) \geq v_1(0, x) \geq 0 \quad \text{and} \quad u_2(0, x) \geq v_2(0, x) \geq 0
\]

and \( u \) blows up in finite time.

In the case of interaction (1.7) with \( a_1 > 0 \) and \( a_2 \geq 0 \), we choose \( \varphi \) as in Theorem 5 with \( \varphi_1 \geq 0 \) and define \( w = u - v \). We have

\[
\begin{align*}
\partial_t w_1 &= a_1 v_2 w_1 + a_1 u_1 w_2 + u_2^2 \\
\partial_t w_2 &= (u_1 + v_1) w_1 + a_2 u_1 u_2
\end{align*}
\]

from which we conclude. Similarly comparison with

\[
\begin{align*}
\partial_t u_1 &= u_2^2 \\
\partial_t u_2 &= a_2 u_1 u_2 \quad a_2 > 0
\end{align*}
\]

allows us to conclude for interaction (1.7) with \( a_1 \geq 0 \) and \( a_2 > 0 \).

Lemmas 5 and 7 end the proof of Theorem 2.

We have characterized the quadratic interactions leading to global existence for the Cauchy problem in two cases: global existence for any data in \( L^\infty_{loc}(\mathbb{R})^2 \), global existence for small data \( \varphi \in L^\infty(\mathbb{R})^2 \) with compact support. Moreover, in each case where blow up holds, we have an estimate of the explosion time.
Concerning complex valued systems with $N$ equations ($N \geq 2$)
\[
\begin{align*}
\partial_t u_i + c_i \partial_x u_i &= q_i(u), \quad 1 \leq i \leq N \\
q_i(u) &= \sum_{1 \leq j < k \leq N} (A'_{jk} u_j u_k + B'_{jk} u_j^* u_k^* + C'_{jk} u_j^* u_k^*)
\end{align*}
\]

one can generalize conservative interactions of type (1.4): if there exists $\beta_1 > 0$, $\beta_2 > 0$, ..., $\beta_N > 0$ such that
\[
\sum_{1 \leq i < N} \beta_i \Re(u_i^* q_i(u)) = 0
\]
then for any $u^0 \in L_{loc}^\infty(\mathbb{R})^N$, the Cauchy problem has a unique global solution [1, 4]. For example, the well-known 3-waves system
\[
\begin{align*}
\partial_t u_1 + c_1 \partial_x u_1 &= Ku_3 u_2 \\
\partial_t u_2 + c_2 \partial_x u_2 &= -K^* u_1 u_3^* \quad \text{K complex} \\
\partial_t u_3 + c_3 \partial_x u_3 &= -K^* u_1 u_2^*
\end{align*}
\]
satisfies this relation. But for $N = 3$ there exists real non conservative interactions for which there is global existence.

The solitary wave method can also be generalized to systems with more than two equations [2]. For example, this method allows us to find data such that the solution of the Cauchy problem for the non conservative 3-waves system
\[
\begin{align*}
\partial_t u_1 + c_1 \partial_x u_1 &= Ku_3 u_2 \\
\partial_t u_2 + c_2 \partial_x u_2 &= K^* u_1 u_3^* \quad \text{K complex} \\
\partial_t u_3 + c_3 \partial_x u_3 &= K^* u_1 u_2^*
\end{align*}
\]
blows up in finite time [1].

**Appendix**

*First Order Ordinary Differential Systems with Quadratic Interaction*

We study the Cauchy problem for system (1.1) when it is not strictly hyperbolic:
\[
c_1 = c_2 = c.
\]
In this case, it can be written as an ordinary differential system along the characteristic, so that we can suppose $c = 0$ and consider the initial value problem

\begin{align}
    u'_1 &= q_1(u) \\ 
    u'_2 &= q_2(u) \\ 
    u(0) &= u^0 \in \mathbb{R}^2
\end{align}  \tag{A.1} \tag{A.2}

where $q$ satisfies (1.3).

Local theory is well known, and if the solution of (A.1), (A.2) is not global, it blows up in finite time: there exists $T^* > 0$ such that

$$
\lim_{t \to (T^*)^-} |u(t)| = +\infty \quad \text{or} \quad \lim_{t \to (-T^*)^+} |u(t)| = +\infty.
$$

Initial value problems for small and large data are the same because if $u$ is solution of (A.1), (A.2) then $v(t) = u(t)$ is also solution with the initial value:

$$
v(0) = eu^0. \tag{A.2'}
$$

Hence the problem is to characterize the interactions leading to global existence for any $u^0 \in \mathbb{R}$. For all other interactions, the system is said to be explosive.

To find out if an interaction is explosive we look for rays, as in the solitary wave method of Section III: can we find $\lambda \in \mathbb{R}^2$, $\lambda \neq 0$, and a smooth function $\psi$ blowing up in finite time such that $\lambda \psi$ is solution of (A.1)? One obtains:

$$
\lambda_1 \psi' = q_1(\lambda) \psi^2 \\
\lambda_2 \psi' = q_2(\lambda) \psi^2.
$$

Using the fact that the solution of

\begin{align}
\psi' &= \psi^2 \\
\psi(0) &= \psi_0
\end{align}

blows up in finite time, one can show that if there exists rays for (A.1), then (A.1) is explosive.

Let us note iso$(q)$ the set of elements of $\mathbb{R}^2$ which are isotropic for $q_1$ and $q_2$.

**Lemma A.1.** If there exists $\lambda \in \mathbb{R}^2$, $\lambda \neq 0$, such that

$$
\lambda \notin \text{iso}(q) \quad \text{and} \quad \lambda_1 q_1(\lambda) = \lambda_2 q_2(\lambda) \tag{A.3}
$$

there exists rays for (A.1).
Then we look for rays for each of the three possible cases: \( \text{iso}(q) \) is empty, \( \text{iso}(q) \) is a single line, \( \text{iso}(q) \) consists of two lines or a double line.

If \( \text{iso}(q) \) is empty, one can prove that there exists \( \lambda \in \mathbb{R}^2, \lambda \neq 0 \), such that (A.3) is satisfied.

If \( \text{iso}(q) \) is a single line, \( q \) is written as

\[
q_1(u) = l_1(u) \, l(u), \quad q_2(u) = l_2(u) \, l(u)
\]

where \( l_1, l_2 \) are independent linear forms. Let us note

\[
l_i(u) = \sum_{j=1}^{2} l_{ij} u_j, \quad i = 1, 2, \quad L = (l_{ij})_{1 \leq i, j \leq 2}.
\]

One can see that (A.3) is connected with the eigenvalues of \( L \). If \( L \) has two distincts real eigenvalues, there exists \( \lambda \in \mathbb{R}^2, \lambda \neq 0 \), such that (A.3) holds.

If \( L \) has a double eigenvalue \( \lambda_0 \) then one can show that if \( \lambda \in \mathbb{R}^2, \lambda \neq 0 \), satisfying (A.3), does not exist, the solution of (A.1), (A.2) satisfies

\[
l(u) = k(l(u))^2, \quad k \neq 0 \text{ real}
\]

and then blows up if \( l(u^0) \neq 0 \). If the eigenvalues of \( L \) are not real, the change of variable

\[
\tau'(t) = l(u(t)), \quad \tau(0) = 0
\]

leads to a linear system, and noting that the orbits cannot cross the line of stationary points \( l(u) = 0 \), one concludes to global existence.

If \( \text{iso}(q) \) consists of two lines or a double one, \( q \) can be written as

\[
q_1(u) = a_1 l_1(u) \, l_2(u), \quad q_2(u) = a_2 l_1(u) \, l_2(u)
\]

where \( l_1, l_2 \) are linear forms. Let us note

\[
l_j(u) = a_1 u_1 - a_2 u_j
\]

One can show that if \( l_3 \) is independent from \( l_1 \) and from \( l_2 \) there exists \( \lambda \in \mathbb{R}^2, \lambda \neq 0 \), such that (A.3) holds. If this is not the case, we have for example \( l_3 = k l_1 \), \( k \) real constant, and \( l_3(u') = 0 \), so that \( u \) satisfies a linear differential system

\[
u_i' = a_i l_1(u^0) l_2(u), \quad i = 1, 2
\]

and then exists globally.
This study shows that ordinary differential systems have very different properties from strictly hyperbolic systems with the same interaction. For example, Carleman’s system is explosive, but the same interaction for an ordinary differential system leads to global existence. Conversely for the interaction

\[ q_1(u) = u_1 u_2, \quad q_2(u) = u_1 u_2 \]

there is global existence with small data for (1.1) but there is blow up for (A.1). And for the interaction (1.4), there is global existence for (1.1) (Theorem 1) and for (A.1).

REFERENCES