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Note

On the ratio of maximum and minimum degree in maximal intersecting families

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ABSTRACT

To study how balanced or unbalanced a maximal intersecting family $\mathcal{F} \subseteq \binom{[n]}{r}$ is we consider the ratio $\mathcal{R}(\mathcal{F}) = \frac{\Delta(\mathcal{F})}{\delta(\mathcal{F})}$ of its maximum and minimum degree. We determine the order of magnitude of the function $m(n, r)$, the minimum possible value of $\mathcal{R}(\mathcal{F})$, and establish some lower and upper bounds on the function $M(n, r)$, the maximum possible value of $\mathcal{R}(\mathcal{F})$. To obtain constructions that show the bounds on $m(n, r)$ we use a theorem of Blokhuis on the minimum size of a non-trivial blocking set in projective planes.

1. Introduction

A family $\mathcal{F}$ of sets is said to be intersecting if $F_1 \cap F_2 \neq \emptyset$ holds for any $F_1, F_2 \in \mathcal{F}$. In their seminal paper, Erdős, Ko and Rado showed [3] that if $\mathcal{F}$ is an intersecting family of r-subsets of an n-element set X (we denote this by $\mathcal{F} \subseteq \binom{X}{r}$), then $|\mathcal{F}| \leq \binom{n-1}{r-1}$ provided that $2r \leq n$. Many generalizations of the above theorem have been considered ever since and many researchers have been interested in describing what intersecting families may look like. One of the quantities concerning intersecting families that has been studied [2,4] is the unbalance $\mathcal{U}(\mathcal{F}) = |\mathcal{F}| - \Delta(\mathcal{F})$ where $\Delta(\mathcal{F})$ denotes the maximum degree in $\mathcal{F}$.

In this paper we define another notion for measuring how balanced or unbalanced $\mathcal{F}$ is. $\mathcal{U}(\mathcal{F})$ is sensible when comparing the largest degree to the size of $\mathcal{F}$, whereas our new notion will measure how close all degrees are to each other. Denoting the minimum degree in $\mathcal{F}$ by $\delta(\mathcal{F})$, our aim is to prove lower and upper bounds on $\mathcal{R}(\mathcal{F}) = \frac{\Delta(\mathcal{F})}{\delta(\mathcal{F})}$. To avoid $\delta(\mathcal{F}) = 0$ we will always assume that $\cup_{F \in \mathcal{F}} F = X$, i.e. the degree $d(x)$ of any element x of the underlying set is at least 1. One can easily define intersecting families satisfying this condition with large $\mathcal{R}$-values: let $x, y \in X$ and let $\mathcal{F}^* = \{F \subseteq X : x \in F, y \notin F, |F| = r\} \cup \{F'\}$ where F' is any r-subset of X with $x, y \in F'$. Clearly, $d(y) = 1$ holds and also $d(x) = \mathcal{R}(\mathcal{F}) = \binom{|X|-2}{r-1} + 1$.

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We will restrict our attention to *maximal* intersecting families, i.e. families with the property $G \in \binom{X}{r} \setminus \mathcal{F} \Rightarrow \exists F \in \mathcal{F}\, F \cap G = \emptyset$, and show that for these families, at least for some range of r, the $\mathcal{R}$-value is much smaller than that of $\mathcal{F}^*$. For the sake of simplicity we will also assume that the underlying set X of our families is $[n] = \{1, 2, \ldots, n\}$.

With the above notation and motivation we define our two main functions as follows:

$$M(n,r) = \max\left\{\mathcal{R}(\mathcal{F}) : \mathcal{F} \subseteq \binom{[n]}{r} \text{ is maximal intersecting with } \cup_{F\in\mathcal{F}} F = [n]\right\},$$

$$m(n,r) = \min\left\{\mathcal{R}(\mathcal{F}) : \mathcal{F} \subseteq \binom{[n]}{r} \text{ is maximal intersecting with } \cup_{F\in\mathcal{F}} F = [n]\right\}.$$

We will use standard notation to compare the orders of magnitude of two positive functions. We will write $f(n) = o(g(n))$ to denote the fact that $f(n)/g(n)$ tends to 0, and $f(n) = \omega(g(n))$ to denote that $g(n)/f(n)$ tends to 0. We will write $f(n) = O(g(n))$ if there exists a positive constant C such that $f(n) \le Cg(n)$ holds for all n and $f(n) = \Omega(g(n))$ if there exists a positive constant C such that $Cg(n) \le f(n)$ holds for all n. If both $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$ hold, then we will write $f(n) = \Theta(g(n))$. Finally, $f(n) \sim g(n)$ denotes the fact that $f(n)/g(n)$ tends to 1.

The family giving the extremal size in the theorem of Erdős, Ko and Rado seems to be a natural candidate for achieving the value of $M(n,r)$. In fact, most families $\mathcal{F}$ that occur in the literature have $\mathcal{R}(\mathcal{F}) = \Theta\left(\frac{n}{r}\right)$. In Section 2 we will prove the following theorems showing that $M(n,r)$ and $m(n,r)$ have different orders of magnitude.

Theorem 1.1. (i) *For all $r \le n$ we have $M(n,r) \le n + r^r$. In particular, if $r < \frac{\log n}{\log\log n}$, then $M(n,r) \le (1+o(1))n$ holds.*
(ii) *If $2r - 2 < n$, then*

$$M(n,r) \ge n - 2r + 3 - \frac{n-2r+2}{\binom{2r-3}{r-2}}$$

holds. In particular, if $r < \frac{\log n}{\log\log n}$, then we obtain $M(n,r) \sim n$.

At first sight, the upper bound $n + r^r$ seems to be very weak, but we will show in Section 3 that it cannot be strengthened too much in general.

Certainly $\mathcal{R}(\mathcal{F}) \ge 1$ is true for all families $\mathcal{F} \subseteq \binom{[n]}{r}$ with $\cup_{F\in\mathcal{F}} F = [n]$, so a trivial lower bound on $m(n,r)$ is 1. The next theorem states that for intersecting families n/r^2 is also a lower bound and we construct maximal intersecting families showing that this is the order of magnitude of $m(n,r)$ as long as $r \le n^{1/2}$. For larger values of r we obtain regular maximal families showing the tightness of the trivial lower bound.

Theorem 1.2. (i) *$m(n,r) \ge \frac{n}{r^2}$ holds for all $r \le n$.*
(ii) *$m(n,r) = \Theta\left(\frac{n}{r^2}\right)$ holds for all $r \le n^{1/2}$.*
(iii) *If $\omega(n^{1/2}) = r = o(n)$ and $r(n)/n$ is monotone, then there exist infinitely many n' and $r' = r'(n')$ with $m(n', r'(n')) = 1$ and $r \sim r'$.*

2. Proofs

In this section we prove Theorems 1.1 and 1.2.

Proof of Theorem 1.1. To prove (i), let us consider a maximal intersecting family $\mathcal{F} \subseteq \binom{[n]}{r}$. Let us partition $\mathcal{F}$ into two subfamilies $\mathcal{F}_1$ and $\mathcal{F}_2$ where $\mathcal{F}_1 := \{F \in \mathcal{F} : \exists x \in F \text{ with } (F \setminus \{x\}) \cap F' \ne \emptyset \text{ for all } F' \in \mathcal{F}\}$ and $\mathcal{F}_2 = \mathcal{F} \setminus \mathcal{F}_1$.

Claim 2.1. *Let d_j denote the maximum number of sets in $\mathcal{F}_2$ that contain the same j-subset. Then $d_j \le r^{r-j}$ holds. In particular, we have $d_0 = |\mathcal{F}_2| \le r^r$.*

Proof of Claim 2.1. By the definition of $\mathcal{F}_2$, for any $j < r$ and any j-subset J that is contained in some $F \in \mathcal{F}_2$ there exists an $F' \in \mathcal{F}$ with $J \cap F' = \emptyset$. Since $\mathcal{F}$ (and so $\mathcal{F}_2$) is intersecting, any set in $\mathcal{F}_2$ containing J must intersect F'; thus summing the number of sets of $\mathcal{F}_2$ containing $J \cup \{x\}$ for all $x \in F'$ we obtain $d_j \le rd_{j+1}$. Since $d_r = 1$, the claim follows. □

Let τ denote the *covering number* of $\mathcal{F}$, i.e. the minimum size of a set meeting all sets of $\mathcal{F}$. Clearly, if $\tau = r$, then $\mathcal{F}_1 = \emptyset$ and thus by Claim 2.1, $|\mathcal{F}| \le r^r$ and $\mathcal{R}(\mathcal{F}) \le r^r$.

Assume $\tau < r$. We will show a mapping f from $\mathcal{F}_1$ to $\mathcal{F}_{\min}$, the subfamily containing one fixed vertex y of minimum degree. For any $F \in \mathcal{F}_1$ let $g(F)$ be an element of F such that $(F \setminus \{g(F)\}) \cap F' \ne \emptyset$ for all $F' \in \mathcal{F}$ (such an element exists by definition of $\mathcal{F}_1$). Let us define $f(F) = F$ if $y \in F$, and $f(F) = (F \setminus \{g(F)\}) \cup \{y\}$ if $y \notin F$. Note that $f(F) \in \mathcal{F}$ as already $F \setminus \{g(F)\}$ meets all sets in $\mathcal{F}$ and by assumption $\mathcal{F}$ is a maximal intersecting family. Observe that at most $n-r+1$ sets can be mapped to the same set G since all such sets should contain $G \setminus \{y\}$. This concludes the proof of (i) as $\mathcal{R}(\mathcal{F}) \le \mathcal{R}(\mathcal{F}_1) + |\mathcal{F}_2|$.

To prove (ii) we need a construction. Let us write $S = [2, 2r-2]$ and $S_0 = [2, r-1]$ and define

$$\mathcal{F}_1 = \left\{\{1\} \cup G : G \in \binom{S}{r-1}\right\}, \qquad \mathcal{F}_2 = \left\{\{1, i\} \cup H : 2r-1 \le i \le n, H \in \binom{S}{r-2} \setminus \{S_0\}\right\},$$

$$\mathcal{F}_3 = \binom{S}{r}, \qquad \mathcal{F}_4 = \{(S \setminus S_0) \cup \{i\} : 2r-1 \le i \le n\}, \qquad \mathcal{F} = \cup_{j=1}^{4} \mathcal{F}_j.$$

Claim 2.2. *The family $\mathcal{F}$ is maximal intersecting.*

Proof of Claim 2.2. $\mathcal{F}$ is clearly intersecting as all of its sets, except those coming from $\mathcal{F}_2$, meet S in at least $r-1$ elements. A set F_2 from $\mathcal{F}_2$ meets any other from $\mathcal{F}_1 \cup \mathcal{F}_2$ as they both contain 1, a set from $\mathcal{F}_3$ because of the pigeon-hole principle, and a set from $\mathcal{F}_4$ as by definition $F_2 \cap S \neq S_0$.

To prove the maximality of $\mathcal{F}$ let us consider a set $T \notin \mathcal{F}$. If $|T \cap S| < r-2$, then any r-subset of $S \setminus T$ is in $\mathcal{F}$ and thus T cannot be added to $\mathcal{F}$. Since all r-subsets of S are already in $\mathcal{F}$, it remains to deal with the cases $|T \cap S| = r-1$ and $|T \cap S| = r-2$. If $|T \cap S| = r-1$, then $1 \notin T$ as otherwise T is in $\mathcal{F}_1$, and $T \cap S \neq S \setminus S_0$ as otherwise T is in $\mathcal{F}_4$. But then a set F from $\mathcal{F}_2$ with $F \cap S = S \setminus T$ is disjoint from T; thus T cannot be added to $\mathcal{F}$.

Finally, suppose $|T \cap S| = r-2$. If $1 \notin T$, then $\{1\} \cup (S \setminus T) \in \mathcal{F}_1$ is disjoint from T and thus T cannot be added to $\mathcal{F}$. If $1 \in T$, then $T \cap S = S_0$ as $T \notin \mathcal{F}_2$. Then we can find a set disjoint from T in $\mathcal{F}_4$. □

Note that for any $x, y \in [n]$, the ratio $d(x)/d(y)$ is a lower bound for $\mathcal{R}(\mathcal{F})$. Thus all we have to observe is that in $\mathcal{F}$ the degree of 1 is $\binom{2r-3}{r-1} + \left(\binom{2r-3}{r-2} - 1\right)(n - 2r + 2)$, and the degree of i is $\binom{2r-3}{r-2}$ for any $2r-1 \le i \le n$. Dividing $d(1)$ by $d(i)$ yields the result.

Note that the proof of Theorem 1.1(i) gives an upper bound $M(n, r) \le n + r^r$ for any value of r and n.

Conjecture 2.3. *If $r = o(n)$ holds, then the order of magnitude of $M(n, r)$ is $\Theta(n)$.*

Now we turn our attention to the function $m(n, r)$. In the proof of Theorem 1.2 we will use the following theorem of Blokhuis on blocking sets of projective planes (for a short survey on the topic see [6]). Let us briefly introduce the properties of projective planes that we will use in our proofs. A *projective plane Q of order q* is a family of subsets of $V(Q)$ (the points of the projective plane) of size $q+1$ such that any two sets intersect in exactly one point and for any $x, y \in V(Q)$ there is exactly one $F \in Q$ with $x, y \in F$. For every prime power $q = p^n$ there exists a projective plane Q of order q with the following properties:

- both the number of points and the number of lines are $q^2 + q + 1$,
- for any $x \in V(Q)$, we have $d(x) = q + 1$.

Theorem 2.4 (*Blokhuis, [1]*)**.** *Let Q be a projective plane of order q and B be a blocking set (a set that meets all lines of the projective plane) of size less than $\frac{3}{2}(q+1)$. If q is prime, then B contains a line of the projective plane.*

We will also need the following strengthening of Chebyshev's theorem.

Theorem 2.5 (*Nagura, [5]*)**.** *For every integer $n \ge 25$ there exists a prime p with $n \le p \le (1 + 1/5)n$.*

Proof of Theorem 1.2. To prove (i) we make the following two easy observations: for any intersecting family $\mathcal{F}$ we have $\Delta(\mathcal{F}) \ge |\mathcal{F}|/r$ as for any set $F \in \mathcal{F}$ the inequality $\sum_{x \in F} d(x) \ge |\mathcal{F}|$ holds. Also, the average degree in $\mathcal{F}$ equals $\frac{r|\mathcal{F}|}{n}$. As the average degree is at least as large as the minimum degree, we obtain

$$\mathcal{R}(\mathcal{F}) = \frac{\Delta(\mathcal{F})}{\delta(\mathcal{F})} \ge \frac{\frac{|\mathcal{F}|}{r}}{\frac{r|\mathcal{F}|}{n}} = \frac{n}{r^2}.$$

Note that the proof does not use the fact that $\mathcal{F}$ is maximal.

To prove (ii) and (iii) we need constructions. Suppose first that $r \le n^{1/2}$ holds. By Theorem 2.5 we can pick a prime p such that $\frac{2}{3}r \le p \le \frac{2}{3}\left(1 + \frac{1}{5}\right)r = \frac{4}{5}r$. Let P denote a projective plane of order p with vertex set $[p^2 + p + 1]$. Let us define the following maximal intersecting family:

$$\mathcal{F}_{n,r,p} = \left\{F \in \binom{[n]}{r} : l \subset F \text{ for some line } l \in P\right\}.$$

Note that $\mathcal{F}_{n,r,p}$ is intersecting because any two of its sets intersect as they both contain lines of a projective plane, and $\mathcal{F}_{n,r,p}$ is maximal because if $G \in \binom{[n]}{r}$ does not contain any line of P; then by Theorem 2.4 and $r < \frac{3(p+1)}{2}$ we know that there exists a line l in P such that $l \cap G = \emptyset$ and this line can be extended to a set $l \subset F_l \in \binom{[n]}{r}$ such that $F_l \cap G = \emptyset$ holds. As every vertex is contained in $p+1$ lines of P we have that $d(x)=(p+1)\binom{n-p-1}{r-p-1}+p^2\binom{n-p-2}{r-p-2}$ if $x\in[p^2+p+1]$. Indeed, either we pick one of the $p+1$ lines of P containing x and add $r-p-1$ other points, or we pick one of the p^2 lines of P not containing x and add x and $r-p-2$ further points. Note that as $r\le 2p$, none of the sets can contain two lines and thus we did not count any set $F\in\mathcal{F}_{n,r,p}$ twice.

Also for any $y\in[p^2+p+2,n]$ we have $d(y)=(p^2+p+1)\binom{n-p-2}{r-p-2}$ as we can pick any of the p^2+p+1 lines of P and extend it by y and any $r-p-2$ other points. Therefore we obtain

$$\mathcal{R}(\mathcal{F}_{n,p,r})=\frac{p^2\binom{n-p-2}{r-p-2}+(p+1)\binom{n-p-1}{r-p-1}}{(p^2+p+1)\binom{n-p-2}{r-p-2}}\le 1+\frac{1}{p}\cdot\frac{n-p-1}{r-p-1}\le\frac{17}{2}\cdot\frac{n}{r^2}$$

where the last inequality follows from $\frac{2}{3}r\le p\le\frac{4}{5}r$ and $n\ge r^2$.

It remains to prove (iii). Consider the following general construction $\mathcal{F}'_{k,p,s}\subseteq\binom{[n]}{r}$ where $1\le k$ is an odd integer, p is a prime, $0\le s\le\frac{p}{2}$ and $n=k(p^2+p+1)$, $r=\frac{k+1}{2}(p+1)+s$. For $1\le i\le k$ let P_i be a projective plane of order p with underlying set $[(i-1)(p^2+p+1)+1,i(p^2+p+1)]$ and let us write

$$\mathcal{F}'_{k,p,s}=\left\{F\in\binom{[n]}{r}:F\text{ contains a line of }P_i\text{ if }i\in I\text{ for some }I\in\binom{[k]}{\frac{k+1}{2}}\right\}.$$

As any two lines of a projective plane intersect each other and so do any $I,I'\in\binom{[k]}{\frac{k+1}{2}}$, the family $\mathcal{F}'_{k,p,s}$ is intersecting.

To obtain the maximality of $\mathcal{F}'_{k,p,s}$ we need to show that for any r-subset $G\notin\mathcal{F}'_{k,p,s}$ there exists an $F\in\mathcal{F}'_{k,p,s}$ with $F\cap G=\emptyset$. Let $G\notin\mathcal{F}'_{k,p,s}$ and let us write $t=|\{i:\exists\ell\in P_i,\ \ell\subset G\}|$, $b=|\{i:G\cap P_i\text{ is a blocking set in }P_i\text{ and }\nexists\ell\in P_i,\ \ell\subset G\}|$ and $u=|\{i:G\cap P_i\text{ is not a blocking set in }P_i\}|$. Since $G\notin\mathcal{F}'_{k,p,s}$, we have $t\le(k-1)/2$. By Theorem 2.4, we know that whenever $G\cap P_i$ is a blocking set, then $|G\cap P_i|\ge p+1$ and if $G\cap P_i$ does not contain any line of P_i, then $|G\cap P_i|\ge\frac{3}{2}(p+1)$. Therefore we must have

$$t\cdot(p+1)+b\cdot\frac{3}{2}(p+1)\le r=\frac{k+1}{2}(p+1)+s.$$

Since $s\le\frac{p}{2}$, it follows that $t+b\le\frac{k-1}{2}$ and thus $u\ge\frac{k+1}{2}$ holds. Therefore we can pick lines $\ell_{i_1},\ell_{i_2},\ldots,\ell_{i_{(k+1)/2}}$ of different P_{i_j}'s such that $\ell_{i_j}\cap G=\emptyset$. By the definition of $\mathcal{F}'_{k,p,s}$, every r-set containing all ℓ_{i_j}'s belongs to $\mathcal{F}'_{k,p,s}$ and therefore by adding s elements not in G we can find a set $F\in\mathcal{F}'_{k,p,s}$ with $F\cap G=\emptyset$. This finishes the proof of the maximality of $\mathcal{F}'_{k,p,s}$. As the construction is symmetric, all degrees are equal and therefore we obtain $\mathcal{R}(\mathcal{F}'_{k,p,s})=1$.

Assume that we are given a sequence of integers $r=r(n)$ with $r=\omega(n^{1/2})$. Let us pick a prime p with $p\sim\frac{n}{2r}$ and an odd integer $k\sim\frac{4r^2}{n}$. Then we can consider the family $\mathcal{F}'_{k,p,s}$ with any $0\le s\le p/2$. Its vertex set has size $k(p^2+p+1)=n'\sim n$ and by the monotonicity of r/n and $r=\omega(n^{1/2})$ we obtain that the sets of $\mathcal{F}'_{k,p,s}$ have size $\frac{k+1}{2}(p+1)+s=r'\sim r$. $\square$

3. Concluding remarks

As we mentioned in the Section 1, the bound of Theorem 1.1(i) cannot be greatly improved in general, as the following example shows. If $n=2r$, then a maximal intersecting family $\mathcal{F}$ contains one set from every pair of complement sets. Thus the family $\mathcal{F}^*=\left\{F\in\binom{[n]}{r}:1\notin F,F\neq[r+1,n]\right\}\cup\{[r]\}$ is maximal intersecting and $\mathcal{R}(\mathcal{F}^*)=\Theta\left(\binom{n}{r}\right)=e^{\Theta(n)}$ holds while $r^r=e^{\Theta(n\log n)}$.

In Theorem 1.2(iii), we could show regular maximal intersecting families only for special values of n and r. There are two ways to generalize our construction. First, one need not insist that all projective planes should be of the same order, but for the maximality one still needs that they should be of the same asymptotic order (one will have to choose s a bit more carefully). This will ruin the regularity, but for families $\mathcal{F}$ obtained this way, $\mathcal{R}(\mathcal{F})=1+o(1)$ would still hold. The other possibility is to add extra vertices that do not belong to $\cup P_i$, like in the construction used for Theorem 1.2(iii). This will enable us to obtain constructions for arbitrary values of n and r (provided n is large enough) but for these families $\mathcal{F}'$ we will have $\mathcal{R}(\mathcal{F}')=\Theta\left(\frac{n}{r^2}\right)$.

It remains open whether one can construct maximal intersecting families with $\mathcal{R}$-value $1+o(1)$ for any $r(n)$.

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References

[1] A. Blokhuis, On the size of a blocking set in $PG(2, p)$, Combinatorica 14 (1994) 111–114.
[2] I. Dinur, E. Friedgut, Intersecting families are essentially contained in Juntas, Combin. Probab. Comput. 18 (2009) 107–122.
[3] P. Erdős, C. Ko, R. Rado, Intersection theorems for systems of finite sets, Quart. J. Math. Oxford 12 (1961) 313–318.
[4] N. Lemons, C. Palmer, The unbalance of set systems, Graphs Combin. 24 (2008) 361–365.
[5] J. Nagura, On the interval containing at least one prime number, P. Jpn Acad. A 28 (1952) 177–181.
[6] T. Szőnyi, Blocking sets in desarguesian affine and projective planes, Finite Fields Appl. 3 (1997) 187–202.