On the Existence of Exceptional Minimal Sets in Foliations of Codimension One

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Received March 28, 1973

INTRODUCTION

Let $M$ be a compact $C^\infty$ manifold with a codimension one foliation $\mathcal{F}$. We will assume throughout that $\mathcal{F}$ is transversely oriented (a situation which can always be obtained by passing to a two-fold covering space of $M$) and that $\mathcal{F}$ is of class $C^2$ in the sense of [4]. We will also be assuming that the individual leaves of $\mathcal{F}$ are immersed submanifolds of class $C^3$ (which is the case for example if the subbundle of the tangent bundle of $M$ which is tangent to $\mathcal{F}$ is of class $C^2$). When this extra condition obtains we say that $\mathcal{F}$ is of class $C^{2+}$. By a minimal set of $\mathcal{F}$ we shall mean a nonempty set $\mathcal{M} \subset M$ which is minimal with respect to the following two conditions:

(i) $\mathcal{M}$ is a union of leaves of $\mathcal{F}$, and
(ii) $\mathcal{M}$ is a closed subset of $M$.

It is well-known that a minimal set can be of three types:

(1) all of $M$,
(2) a single compact leaf of $\mathcal{F}$,
(3) an exceptional minimal set, i.e., a nowhere dense set which is not a compact leaf.

Denjoy (see, e.g., [5]) showed that exceptional minimal sets could not exist for $C^2$ codimension one foliations of the 2-torus. In [17], Reeb proved analogs of Denjoy's theorem for foliations of $T^2 \times [0, 1]$ transverse to the $[0, 1]$ factor and conjectured that exceptional minimal sets could not exist in $C^2$ codimension one foliations but Sacksteder [20] gave a counterexample to this conjecture. The example was a foliation of $M_2 \times S^1$ ($M_2 = \text{compact oriented surface of genus } 2$) transverse to the $S^1$ factor. In this paper we will try to put these earlier results and examples in some perspective. In particular,
for the examples mentioned above, the decisive difference turns out to be the
following: \( \pi_1(T^2) \) is abelian and thus has polynomial growth, whereas \( \pi_1(M_2) \)
has exponential growth \([9]\) (see next section for definitions). The main
results are stated in the next section and the proofs are given later on.
Applications are given to the study of locally free Lie group actions, foliations
transverse to bundles with circle fibers, and actions of finitely generated
discrete groups on \( S^1 \).

The author is grateful to R. Moussu for suggesting (2.6) and its use in
the proof of the main result (1.2).

1. GROWTH FUNCTIONS AND THE MAIN RESULTS

From now on we will assume that the manifold \( M \) has a \( \mathscr{C}^\infty \) Riemannian
metric and we define growth functions for the foliation \( \mathcal{F} \) as follows. For
\( x \in M \), let \( L_x \) denote the leaf of \( \mathcal{F} \) which contains \( x \) and \( d_\ast \) denote the distance
function derived from the metric on \( L_x \) (which is induced from the original
metric on \( M \)). The disk of radius \( R \) about \( x \) is defined by

\[
D_R(x) = \{ y \in L_x \mid d_\ast(x, y) \leq R \}.
\]

Corresponding to the Riemannian metric induced on \( L_x \) there is a volume
element which we denote by \( \Omega_x \).

**DEFINITION.** The growth function of \( \mathcal{F} \) at \( x \) is the continuous increasing
function \( g_\ast : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) defined by

\[
g_\ast(R) = \int_{D_R(x)} \Omega_x.
\]

If \( g_\ast(R) \leq \varrho(R) \) where \( \varrho(R) \) is some polynomial we say that \( \mathcal{F} \) has polynomial
growth at \( x \in M \). If \( g_\ast(R) \geq A \exp(\alpha R) \) for some \( A > 0, \alpha > 0 \), we say
that \( \mathcal{F} \) has exponential growth at \( x \in M \).

**Remarks.** 1. In general the growth function depends on the original
Riemannian metric and the point \( x \) in the leaf. However, because of the
compactness of \( M \), the type of growth (polynomial or exponential) does not
depend on either of these data. Thus, it makes sense to say that a leaf of \( \mathcal{F} \)
has polynomial (of degree \( n \)) or exponential growth.

2. Growth functions are defined for foliations of arbitrary codimension
and, in general, for any Riemannian manifold.

3. The growth function is sometimes related to the topological type of
the leaf. For example, a leaf is compact if, and only if, its growth function
is bounded. In general, however, it seems that the growth function is not related in any obvious way with the homotopy type of the leaf.

4. The growth function of a Riemannian manifold is related to its curvature [9, 23].

Another type of growth function which we will have occasion to consider is the following. Suppose \( \Gamma \) is a finitely generated (discrete) group and let \( \gamma_1, \ldots, \gamma_k \) be a finite set of generators. Each element of \( \Gamma \) can be written as a word in the \( \gamma_i \)'s. For \( \gamma \in \Gamma \) let \( m(\gamma) \) denote the minimum length of such a word (with respect to the generating set \( \gamma_1, \ldots, \gamma_k \)).

**Definition.** The growth function of \( \Gamma \), \( g: \mathbb{Z}^+ \to \mathbb{Z}^+ \) is defined as follows: If \( n \in \mathbb{Z}^+ \), \( g(n) \) is the number of distinct elements \( \gamma \in \Gamma \) such that \( m(\gamma) \leq n \).

**Remarks.** (1) As above we define polynomial growth and exponential growth. The growth function depends on the generating set which is chosen but its type (polynomial or exponential) does not. Thus, these growth types are invariants of the group \( \Gamma \).

(2) (See [23].) Finitely generated nilpotent groups have polynomial growth. There exist solvable groups having either type of growth. Free groups, which are at the other extreme, clearly have exponential growth whenever the generating set contains more than one element.

We recall the following result which relates the two concepts of growth indicated above.

**Proposition 1.1.** Let \( M \) be a compact Riemannian manifold (with a fixed base point) and \( \tilde{M} \) its universal covering space (with induced metric). Then \( \pi_1(M) \) has the same type of growth (polynomial of degree \( n \) or exponential) as \( \tilde{M} \).

**Proof.** The proof of this result is not difficult and may be found in [15]. (Note, however, that the terminology used in [15] differs somewhat from that of the present paper.)

We are now in a position to state the main results.

**Theorem 1.2.** Let \( \mathcal{F} \) be a codimension one foliation of class \( C^{2+} \) of a compact manifold \( M \). If \( \mathcal{M} \subset M \) is an exceptional minimal set for \( \mathcal{F} \) then every leaf contained in \( \mathcal{M} \) has exponential growth.

**Corollary 1.3.** If \( \mathcal{F} \) is as in (1.2) and every leaf of \( \mathcal{F} \) has less than exponential (e.g., polynomial) growth then \( \mathcal{F} \) does not have any exceptional minimal sets.
Theorem 1.4. Let $\mathcal{F}$ be a codimension one foliation of class $\mathcal{C}^{\omega+}$ of a compact manifold $M$ and assume

1. $\pi_1(M)$ has polynomial growth,
2. $\mathcal{F}$ does not have any null-homotopic closed transversals.

Then $\mathcal{F}$ does not have any exceptional minimal sets.

Remarks. It is sufficient to prove (1.2) since (1.3) follows immediately from (1.2) and (1.4) follows from (1.2) and (2.1) of [15]. By well-known results of Novikov [12], condition (2) in (1.4) can be replaced by either of the following:

(2a) $\mathcal{F}$ does not have any one sided limit cycles;
(2b) $\mathcal{F}$ does not have any vanishing cycles, (in the sense of Novikov).

Thus, (1.4) may be thought of as a generalization of results of Moussu and Roussarie [11], which in turn generalize the results of Denjoy since fundamental groups which are abelian in particular have polynomial growth.

Using the above results and arguments used in [14] and in the proof of Theorem 9 of [21] we also conclude the following.

Theorem 1.5. If $\mathcal{F}, M$ satisfy the hypotheses of either (1.3) or (1.4) and if $\mathcal{F}$ also does not have any compact leaves then $\mathcal{F}$ has a "bundle like" metric and $H^1(M; \mathbb{R}) \neq 0$. In particular, if $\mathcal{F}$ is also transversely oriented then $\mathcal{F}$ is determined by a continuous nowhere vanishing closed one form and $M$ itself fibers over $S^1$.

The conclusion of (1.5) implies, in particular, that all of the leaves of an oriented $\mathcal{F}$ are diffeomorphic. Thus, we have the following somewhat curious result.

Corollary 1.6. Let $\mathcal{F}$ be an oriented codimension one foliation of class $\mathcal{C}^{\omega+}$ of a compact manifold and assume that $\mathcal{F}$ has no compact leaves. If there exist two leaves of $\mathcal{F}$ which are not diffeomorphic then some leaf of $\mathcal{F}$ must have exponential growth.

2. Proof of the Main Result

Essentially, the proof of (1.2) involves combining the Poincaré–Bendixson theorem as proved in [16] together with results of Moussu and Sacksteder. We begin with a definition and a technical result which shows where our smoothness assumption originates.
DEFINITION. Let \( D_R(x) \) be a disk of radius \( R \) about \( x \) in some Riemannian manifold. A point \( z_0 \) contained in the topological boundary of \( D_R(x) \) is called a regular point if there exists a coordinate neighborhood \( U \) containing \( z_0 \) and a chart map \( \varphi: U \to \mathbb{R}^m \) such that

\[
\varphi(U \cap D_R(x)) \cap \{(x_1, \ldots, x_m) \in \mathbb{R}^m \mid x_m \leq 0\}
\]

and for any point \( z \in U \) which is in the topological boundary of \( D_R(x) \) we have

\[
\varphi(z) \in \{(x_1, \ldots, x_m) \in \mathbb{R}^m \mid x_m = 0\}.
\]

**Lemma 2.1.** Let \( L \) be a Riemannian manifold, \( x \in L \) and assume that the exponential map \( \exp_x : T_xL \to L \) is of class \( C^1 \). Also let

\[
\partial D_R(x) \subset \{y \in L \mid d(x, y) = R\}
\]

denote the set of regular boundary points of \( D_R(x) \). Then:

1. for almost all values of \( R \), the pair \( D_R(x), \partial D_R(x) \) satisfies Stokes' theorem, and
2. the derivative of the growth function is given by \( \text{vol} \partial D_R(x) \) (for almost all \( R \)).

**Proof.** Let \( \mathcal{L} \) denote the set of points \( X \) in \( T_xL \) such that \( \exp_x tX \) is a minimal geodesic from \( x \) to \( \exp_x tX \) for \( t < 1 \) but not for any \( t > 1 \). \( \mathcal{L} \) is the image of the continuous map from an open subset of the unit hypersphere in \( T_xL \) to \( T_xL \) which sends \( X/\|X\| \) to \( X \) ([3]) and, hence, we see that \( \mathcal{L} \) is a set of measure zero in \( T_xL \). Let \( \Delta_R \subset T_xL \) be the set of points contained in and interior to both \( \mathcal{L} \) and \( S_R \) where \( S_R = \{X \in T_xL \mid \|X\| = R\} \). Clearly \( \exp_x(\Delta_R) = D_R(x) \) and since \( \exp_x \) is \( C^1 \) and the topological boundary of \( \Delta_R \) has measure zero the integral of any form over \( D_R(x) \) is equal to the integral over \( \Delta_R \) of its pullback via \( \exp_x \). By Fubini's theorem \( \mathcal{L} \cap S_R \) has measure zero in \( S_R \) for almost all \( R \) and for such \( R \) Stokes' theorem is (14A) (see also 13b) of [22]. This proves (1) and the proof of (2) is completely straightforward since the integral of a form over \( \partial D_R(x) \) is the integral of its pullback (via \( \exp_x \)) over \( \Delta_R \cap S_R \) whenever \( \mathcal{L} \cap S_R \) has measure zero in \( S_R \).

**Lemma 2.2.** Let \( L \subset M \) be a leaf of \( \mathcal{F} \), \( x \in L \), and assume that \( \mathcal{F} \) is of class \( C^2 \). If

\[
\liminf_{R \to \infty} \frac{\text{vol} \partial D_R(x)}{\text{vol} D_R(x)} > 0
\]

then the leaf \( L \) has exponential growth.
Proof. Let \( g(t) \) be the growth function of \( L \) at \( x \). \( g(t) \) is increasing and differentiable almost everywhere. The above condition implies that there exists \( T > 0 \) such that for \( t \geq T \) we have
\[
g'(t)/g(t) \geq \tau > 0
\]
whenever \( g'(t) \) is defined. Since \( g(t) \) is increasing we have
\[
g(T + 1) - g(T) \geq \int_{T}^{T+1} g'(t) \, dt \\
\geq \tau \int_{T}^{T+1} g(t) \, dt \\
\geq \tau g(T) \\
g(T + 1) \geq (1 + \tau) g(T).
\]
Iterating this, we have for any positive integer \( N \),
\[
g(T + N) \geq (1 + \tau)^N g(T)
\]
which clearly implies that \( g(t) \) has exponential growth. This proves (2.2).

Lemma 2.3. Let \( \rho > 0 \), \( R > 0 \) and denote by \( N_{\rho}(x, R) \) the maximum number of disjoint disks of radius \( \rho \) which are contained in \( D_R(x) \). Then
\[
\liminf_{R \to \infty} \frac{N_{\rho}(x, R)}{\text{vol } D_R(x)} > 0.
\]
Proof. Let \( \mu > 0 \) be an upper bound for the volume of a disk of radius \( 2\rho \) in a leaf of \( \mathcal{F} \). (That such an upper bound exists follows easily from the compactness of \( M \).) The claim is that any disk \( D_R(x) \) contains at least \( \text{vol } D_R(x)/\mu \) disjoint disks of radius \( \rho \). If \( D_{\rho}(x_1), \ldots, D_{\rho}(x_k) \) are disjoint disks in \( D_R(x) \) with \( k < \text{vol } D_{R-\rho}(x)/\mu \), then we have
\[
\text{vol } \left( D_{R-\rho}(x) - \bigcup_{i=1}^{k} D_{2\rho}(x_i) \right) \geq \text{vol } D_{R-\rho}(x) - \sum_{i=1}^{k} \text{vol } D_{2\rho}(x_i) \\
\geq \text{vol } D_{R-\rho}(x) - k\mu > 0.
\]
Hence, there exists \( x_{k+1} \in (D_{R-\rho}(x) - \bigcup_{i=1}^{k} D_{2\rho}(x_i)) \) and \( D_{\rho}(x_1), \ldots, D_{\rho}(x_{k+1}) \) are disjoint so the above claim is proved. The lemma now follows since \( \liminf_{R \to \infty} (\text{vol } D_{R-\rho}(x)/\text{vol } D_R(x)) > 0 \). (The verification of this last fact is tedious but straightforward.)

Lemma 2.4. Let \( \mathcal{M} \) be a minimal set for \( \mathcal{F} \). Then given \( \epsilon > 0 \) there exists
$R > 0$ such that for each $x \in \mathcal{M}$, $D_R(x)$ is $\varepsilon$-dense in $\mathcal{M}$ (i.e., $D_R(x)$ intersects the $\varepsilon$-neighborhood in $M$ of every point of $\mathcal{M}$).

Proof. Suppose the lemma is false. Then there exist sequences (which may be assumed convergent by taking appropriate subsequences) $x_n \to x$, $R_n \to \infty$, $z_n \to z$ ($x_n, z_n \in \mathcal{M}$) such that $D_{R_n}(x_n)$ does not intersect the $\varepsilon$-neighborhood of $z_n$. Thus, for sufficiently large $n$, $D_{R_n}(x_n)$ does not intersect the $\varepsilon/2$-neighborhood of $z$. This is impossible, however, since the leaf $L_x$ is dense in $\mathcal{M}$ and any point in $L_x$ is the limit of a sequence $y_n \in D_{R_n}(x_n)$. This proves the lemma.

**Lemma 2.5.** Let $\gamma$ be a smoothly embedded closed curve in an orientable manifold and $U$ a neighborhood of $\gamma$. Then there exists a smooth volume preserving flow having $\gamma$ as a closed orbit and whose tangent vector field has support contained in $U$.

Proof. First consider the case $\gamma = S^1 \times \{0\} \subset S^1 \times D^n$. Let $\theta, x_1, \ldots, x_n$ denote the coordinates. If $f: S^1 \times D^n \to \mathbb{R}$ is independent of $\theta$ then the vector field $f(\partial/\partial \theta)$ preserves the volume element $d\theta \wedge dx_1 \wedge \cdots \wedge dx_n$. Choose $f$ so that $f \geq 0$, $f|_{\gamma} > 0$, and $f = 0$ in a neighborhood of the boundary of $S^1 \times D^n$. The general case is now accomplished by taking appropriate coordinates for a tubular neighborhood of $\gamma$ and modifying the volume element so that it agrees with the pull-back of $d\theta \wedge dx_1 \wedge \cdots \wedge dx_n$ via the coordinate chart map.

**Definition.** For an oriented codimension one foliation an element of holonomy is said to be contracting if it is conjugate to the restriction to a neighborhood of zero of a map $h: \mathbb{R} \to \mathbb{R}$ such that $h(0) = 0$ and such that for sufficiently small $|t| > 0$, we have $|h(t)| < |t|$.

The following result is due to R. Moussu and is essentially proved in [10].

**Theorem 2.6.** Let $L$ be a non proper leaf in an oriented $C^1$ codimension one foliation such that $L$ contains an element of contracting holonomy. Then there is a null homologous closed transversal which intersects $L$.

Proof. Let $x_0 \in L$ be a point and $\gamma$ a smooth loop in $L$ which passes through $x_0$ and represents an element of contracting holonomy $h$. Let $\tau$ be a transverse segment through $x_0$ and $x_1 \in \tau \cap L$ sufficiently close to $x_0$ such that $h(x_1)$ is strictly between $x_0$ and $x_1$. By a well-known argument a closed transversal $\alpha$ may be constructed which passes through $x_1$ and if $x_0$ and $x_1$ are sufficiently close we may assume that $\alpha$ is homotopic to the loop $\tau \gamma \tau^{-1}$ where $\tau$ now merely denotes the segment from $x_1$ to $x_0$. Now let $\beta$ be a path in $L$ from $x_1$ to $x_0$. The loop (based at $x_1$) given by $\alpha \beta \gamma^{-1} \beta^{-1}$ is homotopic
to a commutator in \( \pi_1(M, x) \) and, hence, is null homologous. However, a standard deformation argument shows that \( \alpha \beta \gamma^{-1} \beta^{-1} \) can be freely homotoped into a closed transversal.

The following result of Sacksteder is proved in [21].

**Theorem 2.7.** Let \( \mathcal{M} \) be an exceptional minimal set in a \( \mathcal{C}^2 \) codimension one foliation. Then some leaf in \( \mathcal{M} \) has an element of nontrivial linear holonomy.

**Remark.** This means, in particular that there is an element of contracting holonomy.

We now define an asymptotic homology class for a leaf which does not have exponential growth in a foliation of class \( \mathcal{C}^{2+} \). Since we assume \( M \) compact of dimension \( m \), let \( \eta_1, ..., \eta_r \) be closed \((m - 1)\) forms on \( M \) which determine a basis of \( H^{m-1}(M; \mathbb{R}) \). Now let \( R_n \to \infty \) be a sequence of positive real numbers such that

\[
\begin{align*}
(a) & \quad \lim_{n \to \infty} \left( \frac{\text{vol } D_{R_n}(x)}{\text{vol } D_{R_n}(x)} \right) = 0, \\
(b) & \quad \text{the pair } D_{R_n}(x), D_{R_n}(x) \text{ satisfies Stokes' theorem,} \\
(c) & \quad \lim_{n \to \infty} \frac{1}{\text{vol } D_{R_n}(x)} \int_{D_{R_n}(x)} j_x \ast \eta_k \text{ exists for } k = 1, ..., r
\end{align*}
\]

where \( j_x : L_x \to M \) is the inclusion map. That (a) and (b) can be obtained follows from (2.1) and (2.2) and (c) follows by taking successive subsequences since \( M \) is compact. We define an element \( A_x \in H_{m-1}(M; \mathbb{R}) \) (which we think of as being the dual of \( H^{m-1}(M; \mathbb{R}) \)) by

\[
A_x(\eta) = \lim_{n \to \infty} \frac{1}{\text{vol } D_{R_n}(x)} \int_{D_{R_n}(x)} j_x \ast \eta
\]

where \( \eta \) is a closed \((m - 1)\) form. Stokes' theorem (2.1) implies that \( A_x \) is a well-defined linear functional on \( H^{m-1}(M; \mathbb{R}) \). (The definition may, however, depend on the sequence \( R_n \to \infty \) which is chosen.)

We are now in a position to prove the main result.

**Proof of Theorem 1.2.** The proof is by contradiction. Let \( \mathcal{M} \) be an exceptional minimal set of the foliation and assume that some leaf \( L \) of \( \mathcal{M} \) does not have exponential growth. By (2.6) and (2.7), \( L \) intersects a null-homologous closed transversal \( \gamma \) (in fact, every leaf in \( \mathcal{M} \) will intersect \( \gamma \)). Let \( X \) be the divergence free vector field constructed in the proof of (2.5) and let \( \Omega \) be the volume element which is preserved by the \( X \)-flow. The form \( i_x \Omega \) is closed (since \( di_x \Omega = L_x \Omega = 0 \)) and its cohomology class is a multiple of the Poincaré-dual of the homology class of \( \gamma \) ([18]) and, hence, must be zero. Let \( x \in \gamma \cap L \) and \( A_x \in H_{m-1}(M; \mathbb{R}) \) be the asymptotic homology
class defined above. We claim that \( A_2(i_x\Omega) \neq 0 \) which contradicts the fact that \( i_x\Omega \) is cohomologous to zero. Let \( U \) be a small tubular neighborhood of \( \gamma \). By (2.4) any disk of a fixed sufficiently large radius in a leaf of \( \mathcal{M} \) must intersect \( U \) and, hence, there exists a \( \rho > 0 \) such that any disk of radius \( \rho \) in \( \mathcal{M} \) cuts through \( U \). This means that the integral of \( i_x\Omega \) over a disk of radius \( \rho \) in \( \mathcal{M} \) always has the same sign and has absolute value greater than some fixed positive number. (2.3) now implies that \( A_2(i_x\Omega) \neq 0 \) and this completes the proof of (1.2).

Remark. The above proof actually shows that if a nonproper leaf in a minimal set has an element of contracting holonomy then the leaf has exponential growth (assuming the leaf is an immersed submanifold of class \( C^0 \)). This is the case, for example, if the minimal set is all of \( M \) and contains a nontrivial element of holonomy. The contraction obtained in this case may only be one sided but the same argument will work (see proof of Theorem 9 of [21]).

3. Locally Free Lie Group Actions

In this section we consider group actions \( \Phi: G \times M \to M \) where \( G \) is a Lie group, \( M \) is a compact manifold and the isotropy group at each point of \( M \) is a discrete subgroup of \( G \). Further, we make the special assumptions that the map \( \Phi \) is of class \( C^2 \) and that \( \dim M = (\dim G) + 1 \). We assume also that \( G \) has a fixed right invariant Riemannian metric. \( G \) is said to have polynomial growth if the volume of the disk of radius \( R \) about the identity in \( G \) is dominated by a polynomial in \( R \). As before this concept (and that of exponential growth) is independent of the (invariant) metric chosen but the specific polynomial does depend on the metric. We also assume that \( M \) has a Riemannian metric which agrees on orbits of \( \Phi \) with the metric induced from \( G \).

Theorem 3.1. If \( G \) has polynomial growth then the group action \( \Phi: G \times M \to M \) does not have any exceptional minimal sets.

Proof. If we assume that the group action is of class \( C^3 \) the result would follow directly from (1.2). To prove the \( C^2 \) case we merely repeat the proof of (1.2) with some small changes. The asymptotic homology class is in this case defined by

\[
A_2(\eta) = \lim_{n \to \infty} \left( \frac{1}{\text{vol } D_{p_n}} \right) \int_{p_n} j_{p_n}^* \eta
\]
where $\eta$ is a closed $m-1$ form on $M$ ($m = \dim M$), $D_R$ is the disk of radius $R$ about the identity of $G$, $j_z : G \to M$ is defined by $j_z(g) = \Phi(g, x)$, and $R_n \to \infty$ is a sequence such that Stokes' theorem is valid on $D_{R_n}$ and the limit in the above formula exists for a collection of closed $m-1$ forms which determine a basis of $H^{m-1}(M; \mathbb{R})$. (Note that the argument for the validity of Stokes' theorem for almost all values of $R$ is no problem here since $G$ itself is a $C^\infty$ manifold.) The other necessary modifications of the proof of (1.2) are completely straightforward and are therefore omitted.

The question of which Lie groups have polynomial growth is answered by the following result of Jenkins [7].

**Theorem 3.2.** Let $G$ be a simply connected Lie group and $\mathfrak{G}$ its Lie algebra. Then:

(i) $G$ has either polynomial growth or exponential growth;

(ii) $G$ has polynomial growth iff all the eigenvalues of the adjoint representation $\text{ad} : \mathfrak{G} \to \text{gl}(\mathfrak{G})$ are imaginary.

**Remarks.**

1. Jenkins considers the growth of the measure of the sets $U^n$ ($n = 1, 2, 3, ...$) where $U$ is a compact neighborhood of the identity in $G$. It is easily shown, however, that this growth type is the same as that obtained when $G$ is considered as a Riemannian manifold.

2. Note that nilpotent groups have polynomial growth and nonunimodular groups have exponential growth. Solvable groups can have either type of growth. Semisimple groups are either compact or have exponential growth.

(3.1) and (3.2) combine to give the following.

**Corollary 3.3.** If $\Phi : G \times M \to M$ is a $C^2$ codimension one Lie group action with $M$ compact and $\text{ad} : \mathfrak{G} \to \text{gl}(\mathfrak{G})$ has only imaginary eigenvalues then $\Phi$ does not have any exceptional minimal sets.

**Remark.** This result generalizes Sacksteder's result [21, Theorem 8] that a $C^2$ action $\mathbb{R}^{m-1} \times M \to M$ cannot have an exceptional minimal set.

The following result is parallel to (1.5).

**Corollary 3.4.** If $\Phi : G \times M \to M$ is as above (3.1 and 3.3) and $\Phi$ has no compact orbits then the orbit foliation of $\Phi$ has finite holonomy groups, a bundle like metric, and $H^3(M; \mathbb{R}) \neq 0$.

**Proof.** Corollary 3.4 follows using the same arguments as in the proof of Theorem 9 of [21].

**Remark.** Orbit foliations of locally free Lie group actions have the
property that we can assume that the growth function of each leaf is independent of the point chosen in the leaf. In Section 2 we saw that a (sufficiently smooth) leaf having polynomial growth and contained in a minimal set could not have an element of contracting holonomy. The same conclusion can be obtained if we replace the assumption that the leaf is contained in a minimal set with the assumption that the growth function is independent of the point in the leaf. The idea of the proof is to consider a fixed disk and to project nearby (in $M$) disks of approximately the same radius (and in the same leaf) onto the original disk along the integral curves of a smooth vector field which is transverse to the foliation. As the disks are chosen larger and closer to the original one their projections wrap around it and we find that the growth function $g(t)$ satisfies

$$\liminf_{t \to \infty} \frac{(g(t) - g(t - \delta))/g(t/2)}{g(t/2)} \geq 1$$

where $\delta > 0$ is some constant. Such a growth function cannot be dominated by a polynomial.

One might suppose that the codimension one orbit foliation of a sufficiently smooth Lie group action could not have an exceptional minimal set. To show that this is not the case we give an example of a $C^\infty$ locally free Lie group action whose (codimension one) orbit foliation contains an exceptional minimal set.

First we recall a basic construction of foliations from discrete group actions [4]. Let $\Gamma$ be a finitely generated discrete group and let $\Gamma \times F \to F$ be a $C^r (r \geq 1)$ action where $F$ is a smooth manifold. Also let $U$ be another smooth manifold and suppose there is a $C^r$ action $\Gamma \times U \to U$ which is properly discontinuous. This means, in particular, that $U/\Gamma$ is a $C^r$ manifold. Now define an action $\Gamma \times (F \times U) \to (F \times U)$ by $(\gamma, f, u) \to (\gamma(f), \gamma(u))$. This action is also properly discontinuous and we have a $C^r$ fibration $F \to (F \times U)/\Gamma \to U/\Gamma$. Furthermore, the trivial foliation of $F \times U$ having leaves diffeomorphic to $U$ is invariant under the action of $\Gamma$ and thus induces a foliation $\mathcal{F}$ of $(F \times U)/\Gamma$ which is transverse to the fibers. Conversely, a foliation which is transverse to the fibers of a (compact) fibration and such that the leaves have the same dimension as the base may be obtained from such a construction (using an appropriate action $\pi_1(B) \times F \to F$ where $B$ is the base and $F$ is the fiber).

Now let $M_2$ denote the compact orientable 2-dimensional manifold of genus 2. Sacksteder [20] constructs a $C^\infty$ action $\pi_1(M_2) \times S^1 \to S^1$ which has an exceptional minimal set. Let $G$ denote the universal covering group of $SL(2, \mathbb{R})$. It is well-known (see, e.g., [1]) that $G$ has a uniform discrete subgroup $\Gamma$ (i.e., $G/\Gamma$ is compact) such that there is a surjective group homomorphism $\eta: \Gamma \to \pi_1(M_2)$. We define an action $\Gamma \times S^1 \to S^1$ by
(γ, θ) → η(γ)(θ), γ ∈ Γ, θ ∈ S¹. This new action has the same exceptional minimal set as the original one. Now define Γ × (G × S¹) → G × S¹ by (γ, g, θ) → (gγ⁻¹, η(γ)θ). This action preserves the trivial codimension one foliation with leaves diffeomorphic to G and this foliation induces a codimension one foliation $\mathcal{F}$ of (G × S¹)/Γ having an exceptional minimal set. Furthermore, $\mathcal{F}$ is the orbit foliation of the action $G × (G × S¹)/Γ → (G × S¹)/Γ$ given by left translation.

4. SOME ERGODIC PROPERTIES

Suppose we have a $C^2$ locally free action $\Phi: G × M → M$ where $G$ has polynomial growth and $M$ is compact and oriented. If $\Phi$ has no compact orbits then by (3.4) there is a vector field $X$ on $M$ which is transverse to the orbit foliation $\mathcal{F}$ of $\Phi$ and such that the one form $\omega$ on $M$ defined by $\omega(X) = 1$, $\omega|T\mathcal{F} = 0$ ($T\mathcal{F}$ = tangent bundle of $\mathcal{F}$) is invariant under the action of $\Phi$. It is well known [21] that the one form $\omega$ is closed (i.e., for suitable coordinates on $M$ it is locally the differential of a $C^1$ real valued map). The cohomology class of $\omega$ is essentially the Poincaré dual of the class $A = H_*(M, \mathbb{R})$. This is made precise below but first we state the individual ergodic theorem as proved in [2].

**Theorem 4.1.** Let $\nu$ be a left invariant Haar measure on $G$ and let $\mu$ be a finite complete measure on $M$ which is invariant under a measurable action $\Phi: G × M → M$. Let $A_n$ be a sequence of measurable subsets of $G$ such that $0 < \nu(A_k) < \infty$ for all $k$ and such that:

1. $A_k \subseteq A_{k+1}$;
2. $\lim_{k→∞} \nu(A_k \Delta xA_k)/\nu(A_k) = \lim_{k→∞} \nu(A_k \Delta A_kx)/\nu(A_k) = 0$ for all $x$ in $G$;
3. for each $k$ and $n$, $A_kA_n = \{xy \mid x ∈ A_k, y ∈ A_n\}$ is measurable and $\lim_{n→∞} \nu(A_kA_n \Delta A_n)/\nu(A_n) = 0$;
4. there exists $K > 1$ such that $\nu(A_k^{-1}A_k) ≤ K\nu(A_k)$ for all $k$.

Then if $f ∈ L_1(M)$ we have

\[
\lim_{k→∞} \frac{1}{\nu(A_k)} \int_{A_k} f(\Phi(g, x)) \, d\nu(g) = \int_M f(x) \, d\mu(x)
\]

for almost all $x ∈ M$.

**Proposition 4.2.** Let $\Phi: G × M → M$ a $C^2$ codimension one locally free action where $M$ is compact, $G$ has polynomial growth. Assume $\Phi$ has no compact
orbits and let $\omega$ be the $\Phi$ invariant one form on $M$ as described above. Then, up to a constant multiple, the cohomology class of $\omega$ is the Poincaré dual of the asymptotic homology class $A_x \in H_{m-1}(M; \mathbb{R})$.

Proof. We sketch the proof which uses the ergodic theorem. Let $R_k \to \infty$ be a sequence such that the sets (e denotes the identity of $G$)

$$A_k = \{ g \in G \mid d(e, g) \leq R_k \}$$

satisfy the hypotheses of the ergodic theorem. (This is done easily using the assumption of polynomial growth of $G$ and the fact that $d(e, xy) \leq d(e, x) + d(x, xy) = d(e, x) + d(e, y)$.)

Now let $X_0$ be a vector field on $M$ such that $\omega(X_0) = 1$ and let $X_1, \ldots, X_{m-1}$ be the vector fields tangent to the orbit foliation which correspond to a basis $Z_1, \ldots, Z_{m-1}$ of the Lie algebra of $G$. We define a measure $\mu$ on $M$ as follows. Let $\varphi_t$ be a continuous flow with tangent vector field $X_0$ and which leaves the orbit foliation invariant. Thus, locally $M$ looks like the product of an open set in $G$ with an interval and we let $\mu$ be the local product measure. Clearly $\mu$ is invariant under $\varphi$. If $\eta$ is an arbitrary closed $m - 1$ form on $M$ then by the ergodic theorem (recall $\varphi$ has no compact orbits and the orbit foliation is, therefore, transitive [14])

$$\lim_{k \to \infty} \frac{1}{\nu(A_k)} \int_{A_k} \eta(Z_1, \ldots, Z_{m-1}) \, d\nu$$

$$= \int_M \eta(X_1, \ldots, X_{m-1}) \, d\mu$$

$$- \int_M (\omega \wedge \eta)(X_0, X_1, \ldots, X_{m-1}) \, d\mu$$

which proves the proposition.

Remark. If a locally free action $\Phi: G \times M \to M$ has an invariant one form $\omega$ as above then $G$ must be unimodular for otherwise some $g \in G$ would increase the $\mu$-measure of $M$. In particular, if $G$ is not unimodular there can be no $C^2$ free codimension one action $\Phi: G \times M \to M$ where $M$ is compact. This generalizes a result in [13].

5. Discrete Group Actions on $S^1$

In this section we consider group actions of the form $\Phi: \Gamma \times S^1 \to S^1$ where $\Phi$ is a $C^2$ map and $\Gamma$ is a finitely presented discrete group.
**Theorem 5.1.** If \( \Phi: \Gamma \times S^1 \to S^1 \) is a \( C^2 \) group action where \( \Gamma \) has polynomial growth then \( \Phi \) does not have any exceptional minimal sets.

**Proof.** Since \( \Gamma \) is finitely presented it is the fundamental group of some compact manifold \( B \) [8, p. 143]. Letting \( U \) be the universal covering space of \( B \) and using the construction of Section 3 we obtain a \( C^2 \) foliation of a compact manifold \( M \) which is a circle bundle over \( B \). The proof now is the same as that of (1.2) except that we make a slight change in the definition of the asymptotic homology class. Assume that for the covering \( U \to B \) we have selected a fixed collection of fundamental domains which cover \( U \) (each of which has finite diameter and a piecewise smooth boundary). Let \( \Delta_R \) be the union of all the fundamental domains which intersect \( D_R \) (in \( U \)) and define

\[
A_\omega(\eta) = \lim_{k \to \infty} \frac{1}{\text{vol} \, \Delta_{R_k}} \int_{\Delta_{R_k}} \omega \cdot j^*_x \eta
\]

where \( \eta \) is a closed \( m-1 \) form on \( M \) (\( m = \dim M \)) and \( R_k \to \infty \) is a sequence such that \( A_\omega \) is well-defined on a collection of closed \( m-1 \) forms which generate \( H^{m-1}(M; \mathbb{R}) \).

**Corollary 5.2.** Let \( \Gamma \) and \( \Phi \) be as in (5.1). If \( \Phi \) has no finite orbits then \( \Phi \) is topologically conjugate to a group of isometries.

**Proof.** The statement is an easy consequence of the arguments used in the proof of Theorem 9 of [21].

**6. Foliations Transverse to Circle Bundles**

We suppose that \( M \) is a compact manifold which is a bundle over another compact manifold \( B \) and with fiber \( S^1 \) and that \( \mathcal{F} \) is a codimension one foliation which is transverse to the fibers. In this case our results take the following form. The proof is the same as that in the preceding section.

**Theorem 6.1.** If \( \mathcal{F} \) is a \( C^2 \) codimension one foliation which is transverse to the fibers of an \( S^1 \) bundle over a compact manifold \( B \) and if \( \pi_1(B) \) has polynomial growth then \( \mathcal{F} \) does not have any exceptional minimal sets.

**Remark.** This result generalizes Theorem 3 of [19] and explains the contrast between Sackstader's example [20] and the results of Reeb [17] which were mentioned in the introduction.
7. A Weak Closing Lemma

It is natural to ask to what extent the $\mathcal{C}^r (r \geq 0)$ closing lemma is valid for codimension one foliations of compact manifolds, that is, given a point $x$ contained in a minimal set of the foliation is it possible to make a $\mathcal{C}^r$ small perturbation of the foliation so that $x$ is contained in a compact leaf of the new foliation. That this is not always possible is shown by the following examples of Hirsch [6]. Let $\mathfrak{g}$ be the three dimensional real Lie algebra generated by $X, Y, Z$ with relations of the form

\[
[X, Y] = aZ \quad a > 0 \\
[X, Z] = bX \quad b > 0 \\
[Y, Z] = -bY
\]

Let $G$ be the simply connected Lie group having $\mathfrak{g}$ as Lie algebra. When $a > 0, \mathfrak{g}$ is simple and when $a = 0, \mathfrak{g}$ is solvable, but in either case $G$ has a uniform discrete subgroup $\Gamma$ and we consider the foliation $\mathcal{F}$ on $G/\Gamma$ whose tangent bundle is spanned by the vector fields $Y$ and $Z$. When $G$ is solvable $G/\Gamma$ is a torus bundle over the circle and any loop in a fiber is null-homologous in $G/\Gamma$ [13]. When $G$ is simple, for appropriate $\Gamma$, $G/\Gamma$ is a circle bundle over a compact 2-manifold and the fibers are seen to be null-homologous, (for example using the Gysin exact sequence). Thus, in either case, we have examples where there exists a null-homologous closed transversal. Hence, by the Poincaré–Bendixson theorem for codimension one foliations any perturbation which is still transverse to the given transversal has no minimal set which (a) intersects the transversal and (b) contains a leaf having nonexponential growth. In the case where $G/\Gamma$ is a bundle with circle fibers, no perturbation which remains transverse to the fibers can have a minimal set containing leaves with polynomial growth (in particular, there will be no compact leaves). Thus, the general closing lemma is not valid.

(In [6], Hirsch actually shows the foliations in question are structurally stable and hence sufficiently small perturbations fail to have compact leaves since the original has none.)

In the above examples, however, we note the leaves of the foliations have exponential growth. The following result may be thought of as a weak closing lemma.

**Proposition 7.1.** Let $\mathcal{F}$ be a codimension one foliation of a compact manifold $M$ which is of class $\mathcal{C}^{1+}$. Assume also that the leaves of $\mathcal{F}$ have polynomial growth. Then either $\mathcal{F}$ has a compact leaf or there is a foliation $\mathcal{G}$ close to $\mathcal{F}$ which does have a compact leaf.
Proof. Suppose that $\mathcal{F}$ has no compact leaves. Then by (1.5) there is a continuous vector field $X$ transverse to $\mathcal{F}$ having a continuous flow which takes leaves into leaves. The form which is 1 on $X$ and annihilates vectors tangent to $\mathcal{F}$ is closed and by a $C^0$ small perturbation (among closed nonsingular one forms) as in [14] we may obtain a closed one form with rational periods. The foliation determined by the perturbed form has all its leaves compact [14] and our conclusion follows.

Since the existence of a nonsingular closed one form which is not a multiple of any form with rational periods (on a compact manifold $M$) implies that rank $H^1(M; \mathbb{R}) > 1$ the above argument also yields the following.

**Proposition 7.2.** If $\mathcal{F}$ is as in (7.1) and rank $H^1(M; \mathbb{R}) \leq 1$ then $\mathcal{F}$ has a compact leaf.

If $M$ has dimension three and $\mathcal{F}$ has no compact leaves then there are no null-homotopic closed transversals by results of Novikov [12]. Thus the above results together with (1.4) imply

**Corollary 7.3.** Let $M$ be a compact manifold of dimension 3 and such that $\pi_1(M)$ has polynomial growth. If $\mathcal{F}$ is a codimension one foliation (of $M$) of class $C^0$ then $\mathcal{F}$ has a compact leaf or else there exists a $C^0$ small perturbation of $\mathcal{F}$ which does.

**Corollary 7.4.** If $M$ and $\mathcal{F}$ are as in (7.3) and rank $H^1(M; \mathbb{R}) \leq 1$ then $\mathcal{F}$ has a compact leaf.

**References**