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Note

Global behavior of the nonlinear difference equation $x_{n+1} = f(x_{n-s}, x_{n-t})$ ☆

Taixiang Sun [a,*], Hongjian Xi [b]

[a] *Department of Mathematics, Guangxi University, Nanning, Guangxi 530004, PR China*

[b] *Department of Mathematics, Guangxi College of Economics, Nanning, Guangxi 530004, PR China*

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Abstract

In this note we consider a nonlinear difference equation of the form

$$x_{n+1} = f(x_{n-s}, x_{n-t}), \quad n = 0, 1, \ldots,$$

under some certain assumptions, where $s, t \in \{0, 1, 2, \ldots\}$ with $s < t$ and the initial values x_{-t}, $x_{-t+1}, \ldots, x_0 \in (0, +\infty)$. We prove that the length of its finite semicycle is less than or equal to t and give sufficient conditions under which every positive solution of this equation converges to the positive equilibrium. Some known results are included and improved.

Keywords: Difference equation; Equilibrium; Cycle length

1. Introduction

Nonlinear difference equations of order greater than one are of paramount importance in applications. Such equations also appear naturally as discrete analogues and as numerical solutions of differential and delay differential equations which model various diverse

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* Corresponding author.

E-mail address: stxhql@gxu.edu.cn (T. Sun).

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phenomena in biology, ecology, physiology, physics, engineering and economics. Some nonlinear difference equations, especially the boundedness, global attractivity, oscillatory and some other properties of second order nonlinear difference equations have been investigated by many authors, see [1–3].

Amleh et al. [4] studied the characteristics of the difference equation

$$x_{n+1} = p + \frac{x_{n-1}}{x_n}. \tag{1}$$

They confirmed conjecture $x.y$ 4 in [5] and obtained that the solutions of Eq. (1) with positive initial conditions are globally asymptotically stable provided that $p > 1$.

Fan et al. [6] investigated nonlinear difference equation of the form

$$x_{n+1} = f(x_n, x_{n-k}) \tag{2}$$

under some certain assumptions. They showed that the length of finite semicycle of Eq. (2) is less than or equal to k and give sufficient conditions under which every positive solution of this equation converges to the unique positive equilibrium.

To be motivated by the above studies, in this note, we consider the more general equation

$$x_{n+1} = f(x_{n-s}, x_{n-t}), \quad n = 0, 1, 2, \ldots, \tag{3}$$

where $s, t \in \{0, 1, 2, \ldots\}$ with $s < t$, the initial values $x_{-t}, x_{-t+1}, \ldots, x_0 \in R_+ \equiv (0, +\infty)$ and f satisfies the following hypotheses:

- (H_1) $f \in C(E \times E, (0, +\infty))$ with $\inf_{(u,v)\in E\times E} f(u, v) \in E$, where $E \in \{(0, +\infty), [0, +\infty)\}$;
- (H_2) $f(u, v)$ is decreasing in u and increasing in v;
- (H_3) Eq. (3) has the unique positive equilibrium, denoted by $\bar{x}$.

First we give some definitions which can be found in [6].

Definition 1. The trivial solution of Eq. (3) is the solution $\{x_n\}_{n=-t}^{\infty}$ with $x_n = \bar{x}$ for all $n \geqslant -k$.

Definition 2. If $x_n - \bar{x} \geqslant 0$ for all $n \in \{r, r + 1, \ldots, s\}$ with $x_{r-1} - \bar{x} < 0$ and $x_{s+1} - \bar{x} < 0$, then the terms x_n such that $n \in \{r, r + 1, \ldots, s\}$ form a positive semicycle of length $s - r + 1$. Similarly, if $x_n - \bar{x} < 0$ for all $n \in \{r, r + 1, \ldots, s\}$ with $x_{r-1} - \bar{x} \geqslant 0$ and $x_{s+1} - \bar{x} \geqslant 0$, then the terms x_n such that $n \in \{r, r + 1, \ldots, s\}$ form a negative semicycle of length $s - r + 1$.

2. Main results

Theorem 1. *Assume that* (H_1)–(H_3) *hold and* $\{x_n\}_{n=-t}^{\infty}$ *is a nontrivial positive solution of Eq.* (3).

(1) *If* $\{x_n\}_{n=-t}^{\infty}$ *contains an infinite semicycle, then it is the first semicycle*;

(2) *If* $\{x_n\}_{n=-t}^{\infty}$ *contains a finite semicycle, then its length is less than or equal to* t *with the exception of possibly the first semicycle.*

Proof. (1) Without loss of generality, we suppose that the solution $\{x_n\}_{n=-t}^{\infty}$ of Eq. (3) contains a positive infinite semicycle, then there exists a smallest integer $N \geqslant -t$ such that $x_n \geqslant \bar{x}$ for all $n \geqslant N$. If $N > -t$, then $x_{N-1} < \bar{x}$, thus

$$x_{N+t} = f(x_{N+t-1-s}, x_{N-1}) < f(\bar{x}, \bar{x}) = \bar{x}.$$

This is a contradiction. Therefore $N = -t$.

(2) Without loss of generality, we suppose that $\{x_{i+1}, x_{i+2}, \ldots, x_{i+l}\}$ is a positive finite semicycle of the solution $\{x_n\}_{n=-t}^{\infty}$ with $x_i < \bar{x}$ and $x_{i+l+1} < \bar{x}$. If $l > t$, then

$$\bar{x} \leqslant x_{i+t+1} = f(x_{i+t-s}, x_i) < f(\bar{x}, \bar{x}) = \bar{x}.$$

A contradiction, which implies $l \leqslant t$. Theorem 1 is proven. □

Let $a = \inf_{(u,v)\in E\times E} f(u, v) \in E$. Obviously, $a \geqslant 0$ and $x_n \geqslant a$ for all $n > 0$. In the sequence, we assume that

(H$_4$) The function $f(a, x)$ has only one fixed point in the interval $(a, +\infty)$, denoted by A;
(H$_5$) The function $f(a, x)/x$ is nonincreasing in $(a, +\infty)$ and $f(\bar{x}, x)/x$ is nonincreasing in E.

Theorem 2. *Assume that* (H$_1$)–(H$_5$) *hold and* $\{x_n\}_{n=-t}^{\infty}$ *is a positive solution of Eq.* (3), *then there exists a positive integer* N *such that*

$$f(A, a) \leqslant x_n \leqslant A \quad \textit{for } n \geqslant N.$$

Proof. Since $a = \inf_{(u,v)\in E\times E} f(u, v) \in E$, we have

$$\bar{x} = f(\bar{x}, \bar{x}) > f(\bar{x} + 1, \bar{x}) \geqslant a.$$

Claim 1. $f(A, a) < \bar{x} < A$.

Proof of Claim 1. If $A \leqslant \bar{x}$, then it follows from (H$_2$) and (H$_5$) that

$$A = f(a, A) > f(\bar{x}, A) \geqslant A,$$

this is a contradiction. Therefore $\bar{x} < A$.

Since $\bar{x} < A$, we have that

$$f(A, a) < f(\bar{x}, \bar{x}) = \bar{x}.$$

Claim 1 is proven. □

Claim 2. *For all* $n \geqslant t$, $x_{n+1} \leqslant x_{n-t}$ *if* $x_{n-t} > A$ *and* $x_{n+1} \leqslant A$ *if* $x_{n-t} \leqslant A$.

Proof of Claim 2. Obviously

$$x_{n+1} = f(x_{n-s}, x_{n-t}) \leqslant f(a, x_{n-t}).$$

If $x_{n-t} \leqslant A$, then $x_{n+1} \leqslant f(a, x_{n-t}) \leqslant f(a, A) = A$.

If $x_{n-t} > A$, then

$$\frac{f(a, x_{n-t})}{x_{n-t}} \leqslant \frac{f(a, A)}{A} = 1,$$

which implies $x_{n+1} \leqslant f(a, x_{n-t}) \leqslant x_{n-t}$. Claim 2 is proven. □

Claim 3. *For every $i \in \{0, 1, \dots, t\}$, there exists a positive integer N_i such that $x_{n(t+1)+i} \leqslant A$ for all $n \geqslant N_i$.*

Proof of Claim 3. Assume on the contrary that there exists some $i \in \{0, 1, \dots, t\}$ such that Claim 3 does not hold. Then it follows from Claim 2 that for all $n \geqslant 1$,

$$A < x_{(n+1)(t+1)+i} \leqslant x_{n(t+1)+i}.$$

Let $\lim_{n\to\infty} x_{n(t+1)+i} = A_i$, then $A_i \geqslant A$.

We know from Claim 2 that $\{x_n\}$ is bounded. Let

$$B = \limsup_{n\to\infty} x_{n(t+1)+i-s-1},$$

then $B \geqslant a$ and there exists a sequence $n_k \to \infty$ such that

$$\lim_{k\to\infty} x_{n_k(t+1)+i-s-1} = B.$$

By (3) we have that

$$x_{n_k(t+1)+i} = f(x_{n_k(t+1)+i-s-1}, x_{(n_k-1)(t+1)+i})$$

from which it follows that

$$A_i = f(B, A_i) \leqslant f(a, A_i) = A_i \frac{f(a, A_i)}{A_i} \leqslant A_i \frac{f(a, A)}{A} = A_i.$$

This with (H_2) and (H_4) implies $B = a$ and $A_i = A$. Therefore $\lim_{n\to\infty} x_{n(t+1)+i-s-1} = a$.

Let

$$C = \limsup_{n\to\infty} x_{n(t+1)+i-2s-2},$$

then $\infty > C \geqslant a$ and there exists a sequence $l_k \to \infty$ such that

$$\lim_{k\to\infty} x_{l_k(t+1)+i-2s-2} = C.$$

Again by (3) we have that

$$x_{l_k(t+1)+i-s-1} = f(x_{l_k(t+1)+i-2s-2}, x_{(l_k-1)(t+1)+i-s-1})$$

from which it follows that

$$a = f(C, a) > f(C+1, a) \geqslant a.$$

This is a contradiction. Claim 3 is proven. □

Let $N = \max\{N_i\colon 0 \leqslant i \leqslant t\} + 2t$, then for all $n > N$ we have that

$$x_n \leqslant A$$

and

$$x_n = f(x_{n-s-1}, x_{n-t-1}) \geqslant f(A, a).$$

Theorem 2 is proven. □

Lemma 1 (see [6]). *Consider Eq.* (3). *Let* $I = [c, d]$ *be some interval of real numbers and* $f \in C(I \times I, I)$ *satisfy the following properties*:

(i) $f(u, v)$ *is decreasing in* u *and increasing in* v;
(ii) *If* $(x, y) \in [c, d]$ *is a solution of the system*

$$\begin{cases} x = f(y, x), \\ y = f(x, y), \end{cases}$$

then $x = y$.

Then Eq. (3) *has the unique positive equilibrium* $\bar{x}$ *and every solution of Eq.* (3) *converges to* $\bar{x}$.

By Theorem 2 and Lemma 1, we obtain the following theorem.

Theorem 3. *If* (H_1)–(H_5) *hold and the system*

$$\begin{cases} x = f(y, x), \\ y = f(x, y) \end{cases} \tag{4}$$

has the unique solution $(\bar{x}, \bar{x})$ *in* $[f(A, a), A] \times [f(A, a), A]$, *then every solution of Eq.* (3) *converges to* $\bar{x}$.

3. Applications

In this section, we shall give two applications of the above results.

Application 1. *Consider equation*

$$x_{n+1} = \frac{p + x_{n-t}}{q + x_{n-s}}, \quad n = 0, 1, \ldots, \tag{5}$$

where $s, t \in \{0, 1, 2, \ldots\}$ *with* $s < t$, *the initial conditions* $x_{-t}, \ldots, x_0 \in (0, +\infty)$ *and* $p, q \in (0, +\infty)$. *If* $q > 1$, *then every positive solution of Eq.* (5) *converges to the unique positive equilibrium.*

Proof. Let $E = [0, +\infty)$, it is easy to verify that (H_1)–(H_5) hold for Eq. (5). In addition, if

$$\begin{cases} x = \frac{p+x}{q+y}, \\ y = \frac{p+y}{q+x}, \end{cases} \tag{6}$$

then Eq. (6) has the unique solution

$$x = y = \bar{x} = \frac{1 - q + \sqrt{(1-q)^2 + 4p}}{2}.$$

It follows from Theorems 2 and 3 that every positive solution of Eq. (5) converges to the unique positive equilibrium $\bar{x} = \frac{1-q+\sqrt{(1-q)^2+4p}}{2}$. □

Application 2. *Consider equation*

$$x_{n+1} = p + \frac{x_{n-t}}{x_{n-s}}, \quad n = 0, 1, \ldots, \tag{7}$$

where $s, t \in \{0, 1, 2, \ldots\}$ *with* $s < t$*, the initial conditions* $x_{-t}, \ldots, x_0 \in (0, +\infty)$ *and* $p \in (0, +\infty)$*. If* $p > 1$*, then every positive solutions of Eq.* (7) *converges to the unique positive equilibrium.*

Proof. Let $E = (0, +\infty)$, it is easy to verify that (H_1)–(H_5) hold for Eq. (7). In addition, if

$$\begin{cases} x = p + \frac{x}{y}, \\ y = p + \frac{y}{x}, \end{cases} \tag{8}$$

then Eq. (8) has the unique solution

$$x = y = \bar{x} = p + 1.$$

It follows from Theorems 2 and 3 that every positive solution of Eq. (7) converges to the unique positive equilibrium $\bar{x} = p + 1$. □

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