J. Math. Anal. Appl. 359 (2009) 536–542

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Virtually stable maps and their fixed point sets

P. Chaoha [a,b,*], W. Atiponrat [a]

[a] Department of Mathematics, Faculty of Science, Chulalongkorn University, Bangkok 10330, Thailand
[b] National Centre of Excellence in Mathematics, PERDO, Bangkok 10400, Thailand

ARTICLE INFO

Article history:
Received 26 January 2009
Available online 11 June 2009
Submitted by T.D. Benavides

Keywords:
Fixed point set
Convergence set
Virtually nonexpansive
Virtually stable

ABSTRACT

We introduce the concept of virtually stable selfmaps of Hausdorff spaces, which generalizes virtually nonexpansive selfmaps of metric spaces introduced in the previous work by the first author, and explore various properties of their convergence sets and fixed point sets. We also prove that the fixed point set of a virtually stable selfmap satisfying a certain kind of homogeneity is always star-convex.

0. Introduction

In metric fixed point theory, the existence of fixed points of nonexpansive maps, as well as their generalizations, is extensively studied in the literatures (see [5] and [9] for more detail) while the structure of fixed point sets appeals only to relatively few authors. In particular, it is shown in [4] (and recently improved in [2]) that the fixed point sets of certain (asymptotically) nonexpansive selfmaps $f : X \to X$ are nonexpansive retracts of X. This fact certainly allows us to relate some geometric and topological structures (for examples, convexity, connectedness and path-connectedness) of the fixed point set of f to those of X. Unfortunately, such a fact is hardly true in general. Indeed, it is shown in [7] that, when X is a metric space and $f : X \to X$ is virtually nonexpansive (which includes both nonexpansive maps and asymptotically nonexpansive maps), the fixed point set of f is a retract of a certain subset, called the convergence set, of X instead. Therefore, topological properties of the fixed point set are naturally related to those of the convergence set rather than the domain of the map.

In Section 2 of this paper, the authors generalize the notion of virtually nonexpansive selfmaps of metric spaces to virtually stable selfmaps of Hausdorff spaces, and prove various properties of their convergence sets. We also show that the class of virtually stable maps includes both virtually nonexpansive maps and recurrent maps at the same time, and almost all properties of virtually nonexpansive maps remain valid for virtually stable maps. In particular, the fixed point set of a virtually stable selfmap of a regular space is a retract of its convergence set. This certainly allows us to relate topological structures of convergence sets to those of fixed point sets even in the setting where a metric is not possible. Towards the end of this section, we improve many results proved in [7] to work with virtually stable maps. Above all, we show that, under a mild condition, the convergence set of a virtually stable selfmap is always a G_δ-set.

We then devote the last section to explore the fixed point set of a virtually stable map. It was proved in [4] that the fixed point set of a certain kind of nonexpansive selfmaps of a closed and convex subset of a Banach space is always metrically convex. It was also proved in [6] that the fixed point set of a quasi-nonexpansive selfmap of a convex subset of a CAT(0) space is always convex. Since the notion of virtual stability generalizes both nonexpansiveness and quasi-nonexpansiveness,

* Corresponding author at: Department of Mathematics, Faculty of Science, Chulalongkorn University, Bangkok 10330, Thailand.
E-mail addresses: phichet.c@chula.ac.th (P. Chaoha), atiponrat@gmail.com (W. Atiponrat).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2009.06.015

it is natural to ask for a similar convexity result for a virtually stable map as well. Unfortunately, such a result is hardly obtained unless an extra assumption on a virtually stable map is assumed. We will show that if a virtually stable selfmap satisfies a certain kind of generalized homogeneity, its fixed point set is always star-convex and hence contractible in both topological real linear space and uniquely geodesic space settings.

1. Preliminaries

For a nonempty Hausdorff space X and a continuous selfmap $f: X \to X$, the *fixed point set* and the *convergence set* [7] of f are defined to be $F(f) = \{x \in X\colon\ f(x) = x\}$ and $C(f) = \{x \in X\colon$ the sequence $(f^n(x))$ converges$\}$, respectively.

It is clear that $F(f)$ is closed in X, $F(f) \subseteq C(f) \subseteq X$, $C(f)$ is f-invariant and $C(f) \neq \emptyset$ if and only if $F(f) \neq \emptyset$. Moreover, for each $x \in C(f)$, the continuity of f implies that $\lim_{n\to\infty} f^n(x) \in F(f)$ and hence we naturally obtain a well-defined map $f^\infty : C(f) \to F(f)$ given by $f^\infty(x) = \lim_{n\to\infty} f^n(x)$ for each $x \in C(f)$. Unfortunately, f^∞ is not continuous in general, otherwise it will always give a retraction from $C(f)$ onto $F(f)$.

For each $p \in F(f)$, we will let $C_p(f) = (f^\infty)^{-1}(p)$. It is immediate that

$$C(f) = \bigcup_{p \in F(f)} C_p(f),$$

where $C_p(f) \cap C_q(f) = \emptyset$ whenever $p \neq q$.

When (X, d) is a metric space, a continuous selfmap $f: X \to X$ is said to be

(i) *nonexpansive* if $d(f(x), f(y)) \leqslant d(x, y)$ for any $x, y \in X$;
(ii) *quasi-nonexpansive* if $d(f(x), p) \leqslant d(x, p)$ for any $x \in X$ and $p \in F(f)$;
(iii) *asymptotically nonexpansive* if there is a sequence $(k_n) \subseteq \mathbb{R}^+$ converging to 1 such that $d(f^n(x), f^n(y)) \leqslant k_n d(x, y)$ for any $x, y \in X$ and $n \in \mathbb{N}$;
(iv) *asymptotically quasi-nonexpansive* if there is a sequence $(k_n) \subseteq \mathbb{R}^+$ converging to 1 such that $d(f^n(x), p) \leqslant k_n d(x, p)$ for any $x \in X$, $p \in F(f)$ and $n \in \mathbb{N}$;
(v) *uniformly lipschitzian* [5] if there is $k > 0$ such that $d(f^n(x), f^n(y)) \leqslant k d(x, y)$ for any $x, y \in X$ and $n \in \mathbb{N}$;
(vi) *virtually nonexpansive* [7] if the family of iterates $\{f^n\colon\ n \in \mathbb{N}\}$ of f is equicontinuous on $C(f)$, or equivalently on $F(f)$;
(vii) *periodic* if there exists $n \in \mathbb{N}$ such that $f^n = 1_X$;
(viii) *recurrent* if for each $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that $d(f^n(x), x) < \epsilon$ for all $x \in X$.

It is proved in [7] that nonexpansive maps, quasi-nonexpansive maps, asymptotically nonexpansive maps and asymptotically quasi-nonexpansive maps are virtually nonexpansive, and f^∞ is always continuous whenever f is virtually nonexpansive. Therefore, the fixed point set of a virtually nonexpansive map is always a retract of the convergence set.

By a topological real linear space, we mean a vector space $(X, +, \cdot)$ over $\mathbb{R}$ equipped with a Hausdorff topology that makes the addition $+: X \times X \to X$ and the scalar multiplication $\cdot: \mathbb{R} \times X \to X$ continuous. Notice that a topological real linear space may not be metrizable. Also, for $x, y \in X$ and $t \in [0, 1]$, the linear combination $tx + (1-t)y$ represents the point on the line segment joining x and y. We say that a subset A of X is *convex* if $tx + (1-t)y \in A$ for all $x, y \in A$ and $t \in [0, 1]$. Also, for any $x_0 \in X$, we say that A is *x_0-star-convex* if $tx + (1-t)x_0 \in A$ for all $x \in A$ and $t \in [0, 1]$.

A metric space (X, d) is called a *geodesic space* if each pair of points $x, y \in X$ can be joined by a geodesic segment—the image of an isometry $c: [0, d(x, y)] \to X$ with $c(0) = x$ and $c(d(x, y)) = y$. When there is exactly one geodesic segment joining x and y for each $x, y \in X$, we will call X a *uniquely geodesic space* and use $[x, y]$ to represents the geodesic segment joining x and y. We will also denote the point $z \in [x, y]$ such that $d(z, y) = td(x, y)$ by $tx \oplus (1-t)y$. We say that a subset A of a uniquely geodesic space X is *convex* if $[x, y] \subseteq A$ for all $x, y \in A$. Also, for any $x_0 \in X$, we say that A is *x_0-star-convex* if $[x, x_0] \subseteq A$ for all $x \in A$. For more details about geodesic geometry, we refer to [3].

2. Virtually stable maps and their convergence sets

In this section, we will extend the notion of virtually nonexpansive maps of metric spaces, introduced in [7], to virtually stable maps of Hausdorff spaces and study some topological properties of their convergence sets. Let X be a nonempty Hausdorff space and $f: X \to X$ a continuous selfmap whose fixed point set is nonempty.

Definition 2.1. A fixed point x of f is said to be *virtually f-stable* if for each neighborhood U of x, there exist a neighborhood V of x and an increasing sequence (k_n) of positive integers such that $f^{k_n}(V) \subseteq U$ for all $n \in \mathbb{N}$. We simply call f *virtually stable* if every fixed point of f is virtually stable.

Moreover, we will call a fixed point x of f *uniformly virtually f-stable* with respect to an increasing sequence (k_n) of positive integers if for each neighborhood U of x, there exists a neighborhood V of x such that $f^{k_n}(V) \subseteq U$ for all $n \in \mathbb{N}$. When every fixed point of f is uniformly virtually f-stable with respect to the same sequence, we will simply call f *uniformly virtually stable*.

Remark 2.2. From the definition above, a selfmap $f : X \to X$ for which each fixed point is uniformly virtually f-stable may not be uniformly virtually stable.

Example 2.3. Periodic maps and virtually nonexpansive maps are uniformly virtually stable. An f-stable fixed point in [1] is also virtually f-stable.

Theorem 2.4. *A recurrent selfmap of a metric space is uniformly virtually stable.*

Proof. Let (X, d) be a metric space and $f : X \to X$ a recurrent map. Since f is recurrent, the set $\{k \in \mathbb{N}\colon\ d(f^k(x), x) < \frac{1}{n}$ for all $x \in X\}$ is infinite for each $n \in \mathbb{N}$. Hence, there is a strictly increasing sequence of natural numbers (N_n) such that $d(f^{N_n}(x), x) < \frac{1}{n}$ for all $x \in X$ and $n \in \mathbb{N}$. Let $x \in F(f)$, $m \in \mathbb{N}$ and $y \in B_d(x, \frac{1}{2(m+1)})$. Then, for each $n \geqslant 2(m+1)$, we have

$$d\big(f^{N_n}(y), x\big) \leqslant d\big(f^{N_n}(y), y\big) + d(y, x) \leqslant \frac{1}{n} + \frac{1}{2(m+1)} \leqslant \frac{1}{2(m+1)} + \frac{1}{2(m+1)} < \frac{1}{m}.$$

Also, by continuity, there is a neighborhood U of x such that $f^{N_i}(U) \subseteq B_d(x, \frac{1}{m})$ for $i = 1, \dots, 2m+1$. Hence, $B_d(x, \frac{1}{2(m+1)}) \cap U$ is a neighborhood of x with the property that $f^{N_n}(B_d(x, \frac{1}{2(m+1)}) \cap U) \subseteq B_d(x, \frac{1}{m})$ for all $n \in \mathbb{N}$. Therefore, each fixed point x is uniformly virtually f-stable with respect to the same sequence (N_n) and f is uniformly virtually stable as desired. $\square$

Remark 2.5. According to the previous theorem, even in metrizable settings, the notion of virtual stability is generally weaker than the notion of virtual nonexpansiveness.

Theorem 2.6. *Suppose X is a regular space. If f is a virtually stable, then f^∞ is continuous and hence $F(f)$ is a retract of $C(f)$.*

Proof. Let $x \in C(f)$ and U a neighborhood of $f^\infty(x)$ in $F(f)$. Since X is regular, so is $F(f)$. Then, there is a neighborhood W of $f^\infty(x)$ in X such that $W \cap F(f) \subseteq \overline{W} \cap F(f) \subseteq U$. Now, by virtual stability, there exist a neighborhood V of $f^\infty(x)$ in X and a strictly increasing sequence (k_n) of positive integers such that $f^{k_n}(V) \subseteq W$ for all $n \in \mathbb{N}$. Since V is a neighborhood of $f^\infty(x)$, there is $N \in \mathbb{N}$ such that $f^N(x) \in V$. Let $A = f^{-N}(V) \cap C(f)$. Then A is a neighborhood of x in $C(f)$ such that

$$\begin{aligned} f^\infty(A) &= \Big\{\lim_{n\to\infty} f^n(a)\colon\ a \in A\Big\} \\ &= \Big\{\lim_{n\to\infty} f^n\big(f^N(a)\big)\colon\ a \in A\Big\} \\ &\subseteq \Big\{\lim_{n\to\infty} f^n(a)\colon\ a \in V \cap C(f)\Big\} \\ &= \Big\{\lim_{n\to\infty} f^{k_n}(a)\colon\ a \in V \cap C(f)\Big\} \\ &\subseteq \overline{W} \cap F(f) \\ &\subseteq U. \end{aligned}$$

Thus f^∞ is continuous and $F(f)$ is a retract of $C(f)$. $\square$

According to the previous theorem, if f is a virtually stable selfmap of a regular space, the (path-)connectedness of $C(f)$ implies the (path-)connectedness of $F(f)$. However, the converse may not be true by considering the following example.

Example 2.7. Let $X = \{-1, 1\}$ and $f(x) = x^2$. Clearly, f is a nonexpansive (and hence virtually stable) selfmap of X with $F(f) = \{1\}$ and $C(f) = X = \{-1, 1\}$.

To obtain the converse of the above statement, an additional assumption is required. For instance, we have:

Lemma 2.8. *Suppose $C_p(f)$ is (path-)connected for all $p \in F(f)$. If $F(f)$ is (path-)connected, then so is $C(f)$.*

Proof. Since $C_p(f) \cup F(f)$ is (path-)connected for each $p \in F(f)$, the result then follows immediately. $\square$

Therefore, we obtain the following theorem:

Theorem 2.9. *Suppose f is a virtually stable selfmap of a regular space and $C_p(f)$ is (path-)connected for all $p \in F(f)$. Then $F(f)$ is (path-)connected if and only if is $C(f)$ (path-)connected.*

Please also note that the (path-)connectedness assumption on each $C_p(f)$ in the previous theorem is not necessary as we consider the following examples:

Example 2.10. Let $X = \mathbb{R}^2 - \bigcup_{n\in\mathbb{Z}}(\{\frac{1}{2^n}\} \times [0,1])$ and $f : X \to X$ be defined by $f(x, y) = (\frac{x}{2}, y)$. It is easy to see that f is nonexpansive (hence virtually stable), and both $F(f) = \{0\} \times \mathbb{R}$ and $C(f) = X$ are (path-)connected. Notice also that, for each $p \in F(f)$, $C_p(f) \cong \mathbb{R} - \{\frac{1}{2^n}\colon n \in \mathbb{Z}\}$ is not (path-)connected if $p \in \{0\} \times [0,1]$, and $C_p(f) \cong \mathbb{R}$ is (path-)connected otherwise.

Example 2.11. Let $X = \mathbb{R}^2 - \{0\}$ and $f : X \to X$ be defined by $f(x, y) = (-x, -|y|)$. It is easy to verify that f is nonexpansive with $F(f) = \{(0, y)\colon y < 0\}$ and $C(f) = \{(0, y)\colon y \neq 0\}$. Therefore, $F(f)$ is (path-)connected while $C(f)$ is not. Notice also that $C_p(f)$ is not (path-)connected for any $p \in F(f)$.

We will now explore some topological structures of the convergence set of a virtually stable selfmap. The results directly involve the structure of the sequence associated to such a map.

Lemma 2.12. *For any $p \in F(f)$ and $x \in X$, we have $x \in C_p(f)$ if and only if $(f^n(x))$ has a subsequence $(f^{n_k}(x))$ converging to p with $\sup\{n_{k+1} - n_k\colon k \in \mathbb{N}\} < \infty$.*

Proof. ($\Rightarrow$) Obvious.

($\Leftarrow$) Suppose $\lim_{k\to\infty} f^{n_k}(x) = p$ and $h = \sup\{n_{k+1} - n_k\colon k \in \mathbb{N}\} < \infty$. Since f is continuous, we have $p = f^i(p) = f^i(\lim_{k\to\infty} f^{n_k}(x)) = \lim_{k\to\infty} f^{n_k+i}(x)$ for all $0 \leqslant i \leqslant h$. To show that $\lim_{n\to\infty} f^n(x) = p$, we let U be a neighborhood of p. Since $\lim_{k\to\infty} f^{n_k+i}(x) = p$ for all $0 \leqslant i \leqslant h$, there exists $N \in \mathbb{N}$ such that $f^{n_k+i}(x) \in U$ for all $0 \leqslant i \leqslant h$ and $k \geqslant N$. Now, for each $m \geqslant n_{N+1}$, there exists $k \geqslant N + 1$ such that $n_{N+1} \leqslant n_k \leqslant m \leqslant n_{k+1}$ and hence $m = n_k + i$ for some $0 \leqslant i \leqslant h$. Therefore, we have $f^m(x) \in U$ for all $m \geqslant n_{N+1}$ which implies that $x \in C_p(f)$. □

Remark 2.13. According to the above lemma, we immediately have $x \in C(f)$ if and only if $(f^n(x))$ has a subsequence $(f^{n_k}(x))$ converging to a fixed point of f with $\sup\{n_{k+1} - n_k\colon k \in \mathbb{N}\} < \infty$.

Theorem 2.14. *Let p be a uniformly virtually f-stable fixed point with respect to the sequence (k_n) and $x \in X$. Suppose there exist $r, h \in \mathbb{N}$ such that $k_{i+r} = k_i + h$ for all $i \in \mathbb{N}$. Then $x \in C_p(f)$ if and only if the sequence $(f^{r_n}(x))$ converges to p where (r_n) is a strictly increasing sequence of natural numbers satisfying: for each $n \in \mathbb{N}$, there exists $m \in \mathbb{N}$ such that for all $i \in \mathbb{N}$, $k_i + r_n = k_{i+m-1} + r_1$.*

Proof. ($\Rightarrow$) Obvious.

($\Leftarrow$) To show that $\lim_{n\to\infty} f^n(x) = p$, let U be a neighborhood of p. Then by virtual stability, there exists a neighborhood V of p such that $f^{k_n}(V) \subseteq U$ for all $n \in \mathbb{N}$. Since $\lim_{n\to\infty} f^{r_n}(x) = p$, there exists $N \in \mathbb{N}$ such that $f^{r_n}(x) \in V$ for all $n \geqslant N$. Thus $f^{k_i+r_n}(x) \in U$ for all $i \in \mathbb{N}$ and $n \geqslant N$. However, by the assumption, there exists $M \in \mathbb{N}$ such that $k_i + r_N = k_{i+M-1} + r_1$ for all $i \in \mathbb{N}$. It follows that $f^{k_{i+M-1}+r_1}(x) = f^{k_i+r_N}(x) \in U$ for all $i \geqslant M$, and hence, $\lim_{n\to\infty} f^{k_n+r_1}(x) = p$. Again, by the assumption $k_i + h = k_{i+r}$, it is easy to see that

$$\sup\{k_{n+1} - k_n\colon n \in \mathbb{N}\} = \sup\{k_{n+1} - k_n\colon 1 \leqslant n \leqslant r\} < \infty.$$

Therefore, the previous lemma implies that $\lim_{n\to\infty} f^n(x) = p$. □

Corollary 2.15. *Suppose (X, d) is a metric space, f is virtually nonexpansive and $p \in F(f)$. Then $C_p(f) = \{x \in X\colon (f^n(x))$ has a subsequence converging to $p\}$.*

Proof. Following all notations from the previous theorem, since f is virtually nonexpansive, we can set $k_n = n$, $h = 1$ and $r = 1$. Now if there is a subsequence $(f^{r_n}(x))$ converging to p, we clearly have $r_n = i + (r_n - r_1 + 1) - 1 + r_1 - i = k_{i+(r_n-r_1+1)-1} + r_1 - k_i$, for all $i, n \in \mathbb{N}$. The result then follows from the previous theorem. □

Lemma 2.16. *Suppose (X, d) is a metric space, p is a uniformly virtually f-stable fixed point with respect to a sequence (nh) for some $h \in \mathbb{N}$, and $x \in C_p(f)$. Then for each $\epsilon > 0$, there exists $\delta > 0$ such that $f^{nh}(B_d(x, \delta)) \subseteq B_d(f^{nh}(x), \epsilon)$ for all $n \in \mathbb{N}$.*

Proof. Let $\epsilon > 0$. By uniform virtual stability, there exists $r \in (0, \frac{\epsilon}{2})$ such that $f^{nh}(B_d(p, r)) \subseteq B_d(p, \frac{\epsilon}{2})$ for all $n \in \mathbb{N}$. Since $x \in C_p(f)$, there exists $N \in \mathbb{N}$ such that $f^{nh}(x) \in B_d(p, r)$ for all $n \geqslant N$. By the continuity of $f^h, \ldots, f^{Nh}$, there exists $\delta > 0$ such that $f^{Nh}(B_d(x, \delta)) \subseteq B_d(p, r)$ and $f^{nh}(B_d(x, \delta)) \subseteq B_d(f^{nh}(x), \epsilon)$ for $n = 1, \ldots, N$. Now for each $n \in \mathbb{N}$, if $n \leqslant N$, we are done by the property of δ above. Otherwise, says $n = N + i$ for some $i \in \mathbb{N}$, we immediately have

$$f^{nh}\big(B_d(x, \delta)\big) = f^{(N+i)h}\big(B_d(x, \delta)\big) \subseteq f^{ih}\big(B_d(p, r)\big) \subseteq B_d\left(p, \frac{\epsilon}{2}\right) \subseteq B_d\big(f^{nh}(x), \epsilon\big). \qquad \square$$

When f is a uniformly virtually stable map with respect to a sequence (nh) for some $h \in \mathbb{N}$, the previous lemma immediately implies that f^h is indeed a virtually nonexpansive map and hence, when X is complete, $C(f^h)$ is a G_δ-set by Theorem 2.2 in [8]. However, the next theorem guarantees the same result for $C(f)$.

Theorem 2.17. *If (X, d) is a complete metric space and f is a uniformly virtually stable map with respect to a sequence (nh) for some $h \in \mathbb{N}$, then $C(f)$ is a G_δ-set.*

Proof. By the previous lemma, for every $x \in C(f)$ and $m \in \mathbb{N}$, there exists $\delta_{x,m} > 0$ such that $f^{nh}(B_d(x, \delta_{x,m})) \subseteq B_d(f^{nh}(x), \frac{1}{m})$ for all $n \in \mathbb{N}$. Let $K = \bigcap_{m\in\mathbb{N}} \bigcup_{x\in C(f)} B_d(x, \delta_{x,m})$ which is clearly a G_δ-set. Notice that $C(f) \subseteq K$. To show that $K \subseteq C(f)$, we let $k \in K$. Then for each $n \in \mathbb{N}$, there exist $x \in C(f)$ and $\delta_{x,4n} > 0$ such that $d(k, x) < \delta_{x,4n}$. It follows that $d(f^{mh}(k), f^{mh}(x)) < \frac{1}{4n}$ for all $m \in \mathbb{N}$. Also, since $x \in C(f)$, there is $p_n \in F(f)$ and $N_n \in \mathbb{N}$ such that $d(f^{mh}(x), p_n) < \frac{1}{4n}$ for all $m > N_n$. Then for all $m > N_n$, we have $d(f^{mh}(k), p_n) \leqslant d(f^{mh}(k), f^{mh}(x)) + d(f^{mh}(x), p_n) < \frac{1}{4n} + \frac{1}{4n} = \frac{1}{2n}$ and hence $d(f^{m'h}(k), f^{mh}(k)) \leqslant d(f^{m'h}(k), p_n) + d(p_n, f^{mh}(k)) < \frac{1}{2n} + \frac{1}{2n} = \frac{1}{n}$ for all $m', m > N_n$. So, $(f^{nh}(k))$ is a Cauchy sequence and converges, by the completeness of X, says to some $p \in X$. We now claim that the sequence (p_n) above also converges to p. For each $n \in \mathbb{N}$ and $l \geqslant n$, we have $d(f^{mh}(k), p_l) < \frac{1}{2l}$ for all $m \geqslant N_l$. Also, since $\lim_{n\to\infty} f^{nh}(k) = p$, there is $M_l \in \mathbb{N}$ such that $M_l > N_l$ and $d(f^{M_l h}(k), p) < \frac{1}{2l}$. Therefore, $d(p_l, p) \leqslant d(p_l, f^{M_l h}(k)) + d(f^{M_l h}(k), p) < \frac{1}{2l} + \frac{1}{2l} = \frac{1}{l} \leqslant \frac{1}{n}$, that is $\lim_{n\to\infty} p_n = p$. Now, since $F(f)$ is closed, we must have $p \in F(f)$ and by Lemma 2.12, $k \in C(f)$ as desired. □

Corollary 2.18. *If (X, d) is a complete metric space and f is a virtually nonexpansive map, then $C(f)$ is a G_δ-set.*

Remark 2.19. When f is a uniformly lipschitzian selfmap of a complete metric space X, one can easily show that $C(f)$ is a closed subset of X.

3. Fixed point sets of virtually stable maps

In this section, we will explore the fixed point set of a virtually stable selfmap to obtain a similar convexity result to those in [4] and [6]. However, such a result is hardly obtained unless an extra assumption on the map is assumed. For example, the fixed point set of the virtually stable selfmap $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(x, y) = (x, \frac{y+x^2}{2})$ is simply the parabola $\{(x, x^2)\colon\ x \in \mathbb{R}^2\}$ which is not convex (not even star-convex). Since we will work in both topological real linear space and uniquely geodesic space settings simultaneously, we will use the notation $tx \oplus (1-t)y$ to denote both the linear combination $tx + (1-t)y$ in a topological linear space and the usual point $z \in [x, y]$ satisfying $d(z, y) = td(x, y)$ in a uniquely geodesic space. Notice that in both settings, for any $x, y \in X$ and $s, t \in [0, 1]$, $stx \oplus (1-st)y = s(tx \oplus (1-t)y) \oplus (1-s)y$. Moreover, when $x \neq y$, we have $sx \oplus (1-s)y = tx \oplus (1-t)y$ if and only if $s = t$.

Throughout this section, we let X be an x_0-star-convex subset of a topological real linear space or a uniquely geodesic space.

Definition 3.1. Let $f : X \to X$ and $\phi : [0, 1] \to [0, 1]$ be continuous selfmaps. We will call f a *ϕ-homogeneous map with respect to x_0* if f satisfies the following functional equation for all $t \in [0, 1]$ and $x \in X$:

$$f\big(tx \oplus (1-t)x_0\big) = \phi(t)f(x) \oplus \big(1 - \phi(t)\big)x_0.$$

Geometrically speaking, a ϕ-homogeneous map (with respect to x_0) is a certain kind of "geodesic preserving" map (with respect to x_0). When $x_0 = 0$, the above definition coincides with the definition of a ϕ-homogeneous map in [8]. If either $\phi \equiv 0$ or $\phi \equiv 1$, f must be constant. Moreover, even the case where ϕ is the identity map, the map f may still be highly nonlinear as we will see from the following examples.

Example 3.2. Any linear selfmap $f(x) = Ax + B$ of $\mathbb{R}$ (where $A, B \in \mathbb{R}$ and $A \neq 1$) is ϕ-homogeneous map with respect to $\frac{B}{1-A}$, where ϕ is the identity map.

Example 3.3. Consider $T : L^2[0, 1] \to L^2[0, 1]$ defined by $T(f)(x) = \sqrt{\int_0^x (f(\xi))^2\, d\xi}$. It is easy to see that T is a nonlinear ϕ-homogeneous map with respect to 0, where ϕ is the identity map.

We also note that the fixed point set of a ϕ-homogeneous map may not be convex or connected.

Example 3.4. Consider $f : [0, \infty) \to [0, \infty)$ defined by $f(x) = x^3$. It is easy to see that f is a ϕ-homogeneous map with respect to 0 where $\phi(t) = t^3$, $C(f) = [0, 1]$, and $F(f) = \{0, 1\}$ which is not 0-star-convex. Also, observe that f is not virtually stable.

The next proposition is useful and its proof is easily obtained.

Proposition 3.5. *For a ϕ-homogeneous map f, if ϕ is the identity map, then $F(f)$ is x_0-star-convex. Moreover, the converse holds when $F(f)$ has more than one point.*

Then next lemma improves Proposition 2.4 [8].

Lemma 3.6. *If $f: X \to X$ is a non-constant ϕ-homogeneous map with respect to x_0, then $\phi(st) = \phi(s)\phi(t)$ for all $s, t \in [0, 1]$ and hence $\phi(1) = 1$.*

Proof. Since f is non-constant, there exists $x \in X$ be such that $f(x) \neq x_0$. Then, for any $s, t \in [0, 1]$, we have

$$\begin{aligned}\phi(st)f(x) \oplus \bigl(1 - \phi(st)\bigr)x_0 &= f\bigl(stx \oplus (1 - st)x_0\bigr)\\ &= f\bigl(s\bigl(tx \oplus (1 - t)x_0\bigr) \oplus (1 - s)x_0\bigr)\\ &= \phi(s)f\bigl(tx \oplus (1 - t)x_0\bigr) \oplus \bigl(1 - \phi(s)\bigr)x_0\\ &= \phi(s)\bigl(\phi(t)f(x) \oplus \bigl(1 - \phi(t)\bigr)x_0\bigr) \oplus \bigl(1 - \phi(s)\bigr)x_0\\ &= \phi(s)\phi(t)f(x) \oplus \bigl(1 - \phi(s)\phi(t)\bigr)x_0\end{aligned}$$

which implies $\phi(st) = \phi(s)\phi(t)$.

Now, by letting $s = t = 1$, we have $\phi(1) = [\phi(1)]^2$ which implies that $\phi(1)$ is either 0 or 1. However, if $\phi(1) = 0$, we must have $f(x) = f(1x \oplus (1 - 1)x_0) = \phi(1)f(x) \oplus (1 - \phi(1))x_0 = x_0$ which contradicts to the assumption. Therefore, $\phi(1) = 1$ as desired. □

Lemma 3.7. *If $f: X \to X$ is a non-constant ϕ-homogeneous map with respect to x_0, then $\phi(t) = t^\alpha$ for some $\alpha > 0$.*

Proof. First of all, we note that $\phi(t) > 0$ for all $t > 0$. If there is $0 < t < 1$ with $\phi(t) = 0$, we obtain $t_0 = \sup\{t\colon \phi(t) = 0\} > 0$ with $\phi(t_0) = 0$ by the continuity of ϕ. Clearly $t_0 < 1$ by the previous lemma. Since $\sqrt{t_0} > t_0$ and $\phi(t_0) = (\phi(\sqrt{t_0}))^2$ by the previous lemma, we must have $\phi(\sqrt{t_0}) = 0$ which contradicts to the definition of t_0.

Now, we obtain a well-defined function $g: (-\infty, 0] \to (-\infty, 0]$ given by $g(x) = \ln\phi(e^x)$. Clearly, g is continuous and, by the previous lemma, it satisfies the Cauchy functional equation $g(x + y) = g(x) + g(y)$. We now extend g to $G: \mathbb{R} \to \mathbb{R}$ by setting

$$G(x) = \begin{cases} g(x), & x \leqslant 0,\\ -g(-x), & x > 0.\end{cases}$$

It is easy to verify that G is continuous and still satisfying the same Cauchy functional equation. Then, it is well known that G must be of the form $G(x) = \alpha x$ for some $\alpha \in \mathbb{R}$. This immediately implies $g(x) = \alpha x$ and hence $\phi(t) = t^\alpha$. Since ϕ is a selfmap of $[0, 1]$ ($\alpha \geqslant 0$) and f is non-constant ($\alpha \neq 0$), we must have $\alpha > 0$ as desired. □

Theorem 3.8. *If $f: X \to X$ be a ϕ-homogeneous map with respect to x_0, then $C(f)$ is x_0-star-convex.*

Proof. If f is a constant function, we are done. Otherwise, from the previous lemma, $\phi(t) = t^\alpha$ for some $\alpha > 0$. Notice that, for each $t \in [0, 1]$, the sequence $(\phi^n(t)) = (t^{\alpha^n})$ always converges according to the following cases:

$$\lim_{n\to\infty} t^{\alpha^n} = \begin{cases} 0, & \alpha > 1 \text{ or } t = 0,\\ 1, & \alpha < 1 \text{ and } t \neq 0,\\ t, & \alpha = 1.\end{cases}$$

Now, suppose $x \in C(f)$. Then for each $t \in [0, 1]$, we clearly have

$$f^n\bigl(tx \oplus (1 - t)x_0\bigr) = f^{n-1}\bigl(t^\alpha f(x) \oplus \bigl(1 - t^\alpha\bigr)x_0\bigr) = \cdots = t^{\alpha^n} f^n(x) \oplus \bigl(1 - t^{\alpha^n}\bigr)x_0$$

and hence the sequence $(f^n(tx \oplus (1 - t)x_0))$ always converges. It follows that $tx \oplus (1 - t)x_0 \in C(f)$. Therefore, $C(f)$ is x_0-star-convex. □

The previous theorem shows that a ϕ-homogeneous map naturally behaves very well with respect to the geometric structure of its convergence set, however it is not difficult to see that this is not the case for its fixed point set (for example, consider $f(x) = x^2$ on $\mathbb{R}$).

The following results improve Theorem 3.3 in [8], and at the same time, partially extends Theorem 1.3 in [6] by giving a condition to a virtually stable map to ensure the star-convexity of its fixed point set.

Theorem 3.9. *If $f : X \to X$ is a virtually stable ϕ-homogeneous map with respect to x_0 that fixes more than one point, then ϕ is the identity map.*

Proof. First notice that f is non-constant. By the assumption, Lemmas 3.6 and 3.7, we let $x_1 \in F(f) - \{x_0\}$ and $\phi(t) = t^\alpha$ for some $0 < \alpha \neq 1$. It follows that $\phi(t_0) \neq t_0$ for some $t_0 \in (0, 1)$.

Case $\phi(t_0) > t_0$: It follows that $0 < \alpha < 1$, $\frac{1}{\alpha} > 1$ and hence $\lim_{n\to\infty}(t_0)^{(\frac{1}{\alpha})^n} = 0$. Now, let $t_n = (t_0)^{(\frac{1}{\alpha})^n}$ and consider the sequence $t_n x_1 \oplus (1 - t_n)x_0$ which clearly converges to x_0. Notice that, for each $n > 0$, we have $f(t_n x_1 \oplus (1 - t_n)x_0) = t_n^\alpha f(x_1) \oplus (1 - t_n^\alpha)x_0 = t_{n-1}x_1 \oplus (1 - t_{n-1})x_0$. Since $t_0 x_1 \oplus (1 - t_0)x_0 \neq x_0$, there is a neighborhood U of x_0 with the property that $t_0 x_1 \oplus (1 - t_0)x_0 \notin U$. However, by convergence, for any neighborhood V of x_0, there exists a large enough N such that $t_n x_1 \oplus (1 - t_n)x_0 \in V$ for all $n \geqslant N$. Then, we have

$$f^n(V) \ni f^n\big(t_n x_1 \oplus (1 - t_n)x_0\big) = f^{n-1}\big(t_{n-1}x_1 \oplus (1 - t_{n-1})x_0\big) = \cdots = t_0 x_1 \oplus (1 - t_0)x_0 \notin U$$

and hence $f^n(V)$ is not a subset of U for all $n \geqslant N$. Therefore, there is no strictly increasing sequence (k_n) of natural numbers satisfying $f^{k_n}(V) \subseteq U$ for all $n \in \mathbb{N}$, and this contradicts to the virtual f-stability at x_0.

Case $\phi(t_0) < t_0$: It follows that $\alpha > 1$, $\frac{1}{\alpha} < 1$ and hence $\lim_{n\to\infty}(t_0)^{(\frac{1}{\alpha})^n} = 1$. By letting $t_n = (t_0)^{(\frac{1}{\alpha})^n}$ and considering the sequence $t_n x_1 \oplus (1 - t_n)x_0$ which clearly converges to x_1, we can use the similar argument to obtain a contradiction to the virtual f-stability at x_1.

From both cases, we must have $\phi(t) = t$ for all $t \in [0, 1]$. □

Corollary 3.10. *If $f : X \to X$ is a virtually stable ϕ-homogeneous map with respect to x_0, then $F(f)$ is x_0-star-convex and hence contractible.*

Proof. If f has only one fixed point, we are done. Otherwise, by the previous theorem, ϕ is the identity map. Therefore, $F(f)$ is x_0-star-convex and hence contractible. □

Example 3.11. Consider $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x, y) = (x, \frac{y+|x|}{2})$. It is easy to verify that f is virtually stable ϕ-homogeneous with respect to 0 (where ϕ is the identity), but f is not nonexpansive. Moreover, $F(f)$ is the graph of $y = |x|$ which is 0-star-convex.

Acknowledgments

The authors are grateful to the editor and the anonymous referee(s) for their valuable comments and suggestions for preparing the final version of this manuscript.

References

[1] K. Athanassopoulos, Pointwise recurrent homeomorphisms with stable fixed points, Topology Appl. 153 (2006) 1192–1201.
[2] T.D. Benavides, P.L. Ramirez, Structure of the fixed point set and common fixed points of asymptotically nonexpansive mappings, Proc. Amer. Math. Soc. 129 (2001) 3549–3557.
[3] M. Bridson, A. Haefliger, Metric Spaces of Non-Positive Curvature, Springer-Verlag, Berlin, Heidelberg, New York, 1999.
[4] R.E. Bruck, Properties of fixed-point sets of nonexpansive mappings in Banach spaces, Trans. Amer. Math. Soc. 179 (1973) 251–262.
[5] K. Goebel, W.A. Kirk, Topics in Metric Fixed Point Theory, Cambridge Univ. Press, 1990.
[6] P. Chaoha, A. Phon-on, A note on fixed point sets in CAT(0) spaces, J. Math. Anal. Appl. 320 (2006) 983–987.
[7] P. Chaoha, Virtually nonexpansive maps and their convergence sets, J. Math. Anal. Appl. 326 (2007) 390–397.
[8] P. Chaoha, P. Chanthorn, Convergence and fixed point sets of generalized homogeneous maps, Thai J. Math. 5 (2) (2007) 281–289.
[9] M.A. Khamsi, W.A. Kirk, An Introduction to Metric Spaces and Fixed Point Theory, A Wiley–Interscience Publication, John Wiley & Sons, Inc., 2001.