Minimal underlying division rings of sets of points of a projective space

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Abstract

Let V be a vector space over a division ring K. Let P be a spanning set of points in \( \Sigma := \text{PG}(V) \). Denote by \( K(P) \) the family of sub-division rings \( F \) of \( K \) having the property that there exists a basis \( B_F \) of \( V \) such that all points of \( P \) are represented as \( F \)-linear combinations of \( B_F \). We prove that when \( K \) is commutative, then \( K(P) \) admits a least element. When \( K \) is not commutative, then, in general, \( K(P) \) does not admit a minimal element. However we prove that under certain very mild conditions on \( P \), any two minimal elements of \( K(P) \) are conjugate in \( K \), and if \( K \) is a quaternion division algebra then \( K(P) \) admits a minimal element.

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1. Introduction

A motivation for this paper arises from the investigation of non-full projective embeddings of point-line geometries (called lax embeddings in the literature; see Van Maldeghem [5, 8.6], for instance). Let \( e : \mathcal{S} \rightarrow \text{PG}(V) \) be a non-full projective embedding of a point-line geometry \( \mathcal{S} \),
for a (possibly infinite dimensional) vector space $V$ defined over a given division ring $K$. It may happen that $\text{PG}(V)$ admits a subgeometry $\text{PG}(V_1)$ defined over a sub-division ring $K_1$ of $K$ such that $e(P) \subseteq \text{PG}(V_1)$ and the mapping $e$, regarded as an embedding of $\mathcal{S}$ in $\text{PG}(V_1)$, is full. In other words, the embedding $e: \mathcal{S} \to \text{PG}(V)$ is obtained from a full embedding $e_1: \mathcal{S} \to \text{PG}(V_1)$ simply by extending the underlying division ring $K_1$ of $e_1$. In short: $e$ is a scalar extension of $e_1$. If this is the case, we think of $e$ as being essentially the same as $e_1$. However, if we would like to identify $e$ with $e_1$, we should require that $e$ is not a scalar extension of two essentially different full embeddings. More explicitly, the following should hold for any two sub-division rings $K_1$ and $K_2$ of $K$:

1) Suppose that, for $i = 1, 2$, the lax embedding $e$ is a scalar extension of a full embedding $e_i: \mathcal{S} \to \text{PG}(V_i)$, where $V_i$ is the set of vectors of $V$ that can be expressed as $K_1$-linear combinations of vectors of $B_i$, for a suitable basis $B_i$ of $V$. Then there exists an automorphism $\sigma$ of $K$ and a $\sigma$-semilinear mapping $f_\sigma: V \to V$ such that $K_1^{\sigma} = K_2$, $f_\sigma(B_1) = B_2$ and $f_\sigma e = e$.

It is not too difficult to see that the restrictions put in (1) force $\sigma$ to act on $K_1$ as an inner automorphism of $K$, namely there is an element $\lambda \in K \setminus \{0\}$ such that $x^\sigma = x^\lambda$ for every $x \in K_1$. Therefore $K_2 = K_1^\lambda$. So, (1) entails the following:

2) If $e$ is obtained as a scalar extension from two full embeddings $e_1$ and $e_2$ defined over two sub-division rings $K_1$ and $K_2$ of $K$, then $K_1$ and $K_2$ are conjugate in $K$; in particular, when $K$ is commutative, $K_1 = K_2$.

The case considered above, where $e$ is a scalar extension of a full embedding, is not the unique interesting case. It may also happen that $K$ admits a sub-division ring $K_1$minimal with respect to the property that $e$ can be obtained from an embedding $e_1$ by extending $K_1$ to $K$, but $e_1$ is non-full. This situation naturally leads to consider the following generalization of (2):

3) Let $K_1$ and $K_2$ be sub-division rings of $K$ such that $e$ can be regarded as a scalar extension of embeddings defined over $K_1$ and $K_2$ respectively, with $K_1$ and $K_2$ minimal with respect to this property. Then $K_1$ and $K_2$ are conjugate in $K$ (in particular, when $K$ is commutative, $K_1 = K_2$).

In this paper we want to understand if (3) is true, or when it is true. The second author of this paper has made a first attempt to answer this question in [4]. However the way he chose allowed him to obtain definite (actually affirmative) conclusions only in certain cases, where $K$ is commutative and with the help of some hypotheses on $\mathcal{S}$ and $e$. In this paper we will follow an approach rather different from [4] (but we will turn back to [4] in Section 8).

We firstly simplify our question by replacing the mapping $e$ by its image. Accordingly, throughout this paper $P$ is a set of points of $\Sigma := \text{PG}(V)$ such that $P$ spans $\Sigma$. Denoting by $\mathcal{L}_\Sigma$ the set of lines of $\Sigma$, we put $\mathcal{L}(P) := \{L \cap P \mid |L \cap P| > 2, L \in \mathcal{L}_\Sigma\}$. The collinearity graph of the partial linear space $(P, \mathcal{L}(P))$ will be denoted by $\Gamma(P)$.

We need to state a few more conventions in order to go on. We will write $F \subseteq K$ to say that $F$ is a sub-division ring of $K$ (a subring of $K$, for short) and we put $F^* := F \setminus \{0\}$. For every $\lambda \in K^*$ and every subring $F$ of $K$, we put $F^\lambda := \lambda^{-1} F \lambda$ and we say that $F^\lambda$ is the subring conjugate to $F$ by $\lambda$. 
Given a subring $F \leq K$ and a subset $X$ of $V$, we denote by $\langle X \rangle_F$ the set of linear combinations of vectors of $X$ with all scalars in $F$, also putting $\langle X \rangle = \langle X \rangle_K$, for short. We denote by $[X]$ the set of points of $\Sigma = \text{PG}(V)$ represented by non-zero vectors of $X$. Given a set $S$ of points of $\Sigma$, we denote by $(S)_\Sigma$ the subspace of $\Sigma$ spanned by $S$.

Given a subring $F \leq K$ and a basis $B$ of $V$, a set $S$ of points of $\Sigma$ is said to be $(F, B)$-rational if $S \subseteq [(B)_F]$. We say that $S$ is $F$-rational if it is $(F, B)$-rational for some basis $B$ of $V$. We denote by $K(S)$ the partially ordered set of subrings $F \leq K$ such that $S$ is $F$-rational, with inclusion as the ordering relation. Clearly, $K(S)$ is closed under taking conjugates in $K$. Indeed, if $S$ is $(F, B)$-rational for a subring $F \leq K$ and a basis $B$ of $V$, then $S$ is $(F^\lambda, B\lambda)$-rational for every $\lambda \in K^*$ (compare Section 2, Lemma 2.3).

With the above notation, the questions we address in this paper can be rephrased as follows:

(*) Is it true that, if $K$ is commutative, then $K(P)$ admits a least element? If $K$ is non-commutative, is it true that every member of $K(P)$ contains a minimal element of $K(P)$?

Are all minimal elements of $K(P)$ pairwise conjugate?

The next theorem, to be proved in Sections 3–5, answers the first of the above questions in the affirmative.

**Theorem 1.1.** If $K$ is commutative then the partially ordered set $K(P)$ admits a least element.

In order to state our second theorem we need a few more definitions. We say that a set $S$ of points of $\Sigma$ is closed if, for any subset $X \subseteq S$ and any point $x \in \langle X \rangle_\Sigma$, if there exists a projective line $L$ through $x$ such that $|L \cap S| > 1$ but $L$ is not contained in $\langle X \rangle_\Sigma$, then $x \in S$. Intersections of arbitrary families of closed sets are closed. We define the closure $\overline{S}$ of $S$ as the smallest closed subset of $\Sigma$ containing $S$. The closure $\overline{S}$ of $S$ can be constructed recursively, as follows: $\overline{S} = \bigcup_{n=0}^{\infty} S^{(n)}$ where $S^{(0)} := S$ and $S^{(n+1)}$ is formed by $S^{(n)}$ and all points $x$ such that $\{x\} = \langle X \rangle_\Sigma \cap \langle x_1, x_2 \rangle_\Sigma$ for a subset $X \subseteq S^{(n)}$ and distinct points $x_1, x_2 \in S^{(n)} \setminus \langle X \rangle_\Sigma$.

Turning to our given set of points $P$, let $\overline{P}$ be its closure. We say that $\Gamma(P)$ is quasi-connected if $\Gamma(\overline{P})$ admits at most one connected component of size greater than 1. (As we shall see in Section 3, Lemma 3.2, the connected components of $\Gamma(\overline{P})$ are complete graphs, but this fact has no relevance here.) It is clear from the above that every point of $\overline{P}$ is joined with a point of $P$ by a path of $\Gamma(\overline{P})$. Therefore, if $\Gamma(P)$ is connected then $\Gamma(\overline{P})$ is connected and then $\Gamma(P)$ is quasi-connected. We are now ready to state our second main theorem, to be proved in Sections 3 and 4.

**Theorem 1.2.** Let $K$ be non-commutative. Suppose that $\Gamma(P)$ is quasi-connected and that $K(P)$ admits minimal elements. Then any two minimal elements of $K(P)$ are conjugate in $K$.

The examples in Section 7 show that Theorems 1.1 and 1.2 are close to being the best possible. Examples 7.1 and 7.2 show that it may happen that the poset $K(P)$ does not have a least element, in the case when $K$ is non-commutative. The graph $\Gamma(P)$ is connected in either of these two examples. In Example 7.1 the poset $K(P)$ admits minimal elements (and they are conjugate, by Theorem 1.2). On the other hand, in Example 7.2, $K(P)$ does not have any minimal element. Finally, Example 7.3 shows that the hypothesis that $\Gamma(P)$ is quasi-connected cannot be dropped from Theorem 1.2.
One problem that remains open when $K$ is non-commutative is to find nice conditions on $K$ and $P$ which imply that every member of $K(P)$ contains a minimal element of $K(P)$. Theorem 1.3 below is proved in Section 6. It offers a first answer to this problem, but we hope that even better answers will be obtained in the future. Note that Example 7.1 of Section 7 indeed satisfies the hypotheses of this theorem. In order to state our theorem, we need some notation. We denote by $Z(K)$ the center of $K$ and by $\dim Z(K)(K)$ the dimension of $K$ as a $Z(K)$-vector space. It is well known that if $K$ is non-commutative then $\dim Z(K)(K) \geq 4$. If $\dim Z(K)(K) = 4$ then $K$ is called a quaternion division algebra.

**Theorem 1.3.** Suppose that $\Gamma(P)$ is quasi-connected and $K$ is a quaternion division algebra. Then every member of $K(P)$ contains a minimal element of $K(P)$.

In the last section of this paper we will compare our approach with that of [4]. A few interesting problems arise in this context.

### 2. Preliminaries

In this section we state a few lemmas and a definition, to be exploited later in this paper. We regard $V$ as a right vector space over the division ring $K$, thus writing $u\lambda$ to denote the multiplication of a vector $u \in V$ by a scalar $\lambda \in K$. As in the Introduction, $\Sigma := \text{PG}(V)$. The following is well known and straightforward.

**Lemma 2.1.** For any two bases $B_1$ and $B_2$ of $V$ and any subring $F$ of $K$, if $B_2 \subseteq \langle B_1 \rangle_F$, then $\langle B_1 \rangle_F = \langle B_2 \rangle_F$.

**Lemma 2.2.** Let $X, Y$ be sets of points of $\Sigma$ such that $\langle X \rangle_\Sigma \cap \langle Y \rangle_\Sigma \neq \emptyset$. Suppose that both $X$ and $Y$ are $(F, B)$-rational for a subring $F \subseteq K$ and a basis $B$ of $V$. Then $\langle X \rangle_\Sigma \cap (\langle Y \rangle_\Sigma \cap \langle (B)_F \rangle) \neq \emptyset$. In particular, if $\langle X \rangle_\Sigma$ and $\langle Y \rangle_\Sigma$ meet in a single point, then that point belongs to $\langle (B)_F \rangle$.

**Proof.** If $z \in \langle X \rangle_\Sigma \cap \langle Y \rangle_\Sigma$ then $z \in \langle x_1, x_2, \ldots, x_k \rangle_\Sigma \cap \langle y_1, y_2, \ldots, y_h \rangle_\Sigma$ for suitable finite independent subsets $\{x_1, x_2, \ldots, x_k\} \subseteq X, \{y_1, y_2, \ldots, y_h\} \subseteq Y$. As both $X$ and $Y$ are $(F, B)$-rational, we can choose vectors $u_1, \ldots, u_k, v_1, \ldots, v_h \in \langle B \rangle_F$ such that $x_i = [u_i]$ for $i = 1, 2, \ldots, k$ and $y_j = [v_j]$ for $j = 1, 2, \ldots, h$. As $\{z\} = \langle x_1, x_2, \ldots, x_k \rangle_\Sigma \cap \langle y_1, y_2, \ldots, y_h \rangle_\Sigma$, there are scalars $t_1, \ldots, t_k, s_1, \ldots, s_h \in K$ such that $w := \sum_{i=1}^{k} u_i t_i = \sum_{j=1}^{h} v_j s_j$ represents $z$. The $(k + h)$-tuple of scalars $(t_1, \ldots, t_k, s_1, \ldots, s_h)$ is a non-trivial solution of the following vector equation:

$$\sum_{i=1}^{k} u_i t_i = \sum_{j=1}^{k} v_j s_j.$$  

(1)

All vectors $u_1, \ldots, u_k, v_1, \ldots, v_h$ belong to $\langle B \rangle_F$. So, if we replace $u_1, \ldots, u_k, v_1, \ldots, v_h$ by their expressions as $F$-linear combinations of vectors of $B$ then (1) is turned into a finite system of linear equations with coefficients in $F$, of which $t_1, \ldots, t_k, s_1, \ldots, s_h$ is a solution. Hence this system has a solution in $F$. So $t_1, \ldots, t_k, s_1, \ldots, s_h$ can be chosen in $F$. For such a choice of $t_1, \ldots, t_k, s_1, \ldots, s_h$ we have $w \in \langle B \rangle_F$. \hfill $\Box$

For the rest of this section $S$ is a given set of points of $\Sigma$. We do not assume that $S$ spans $\Sigma$. 


Definition. Let $F$ be a subring of $K$ and $u \in V \setminus \{0\}$ such that $[u] \in S$. Given a basis $B$ of $V$, if $S$ is $(F, B)$-rational and $u \in \langle B \rangle_F$, then we say that $S$ is $(F, B)$-rational with respect to $u$. We say that $S$ is $F$-rational with respect to $u$ if $S$ is $(F, B)$-rational with respect to $u$ for some basis $B$ of $V$.

Clearly, if $S$ is $(F, B)$-rational then it is $(F, B)$-rational with respect to $u$ for every $u \in \langle B \rangle_F \setminus \{0\}$ such that $[u] \in S$. The following is also clear:

Lemma 2.3. Let $B$ be a basis of $V$, $\lambda \in K^*$, $F \subseteq K$ and $u \in \langle B \rangle_F \setminus \{0\}$ be such that $[u] \in S$. Then $S$ is $(F, B)$-rational with respect to $u$ if and only if $S$ is $(F^\lambda, B\lambda)$-rational with respect to $u\lambda$.

Therefore:

Corollary 2.4. Let $\lambda \in K^*$, $F \subseteq K$ and $u \in V \setminus \{0\}$ be such that $[u] \in S$.

(1) If $S$ is $F$-rational with respect to $u$, then $S$ is $F^\lambda$-rational with respect to $u\lambda$.

(2) If $S$ is $F$-rational, then $S$ is also $F^\lambda$-rational.

3. Proofs of Theorems 1.1 and 1.2 in the case that $\Gamma(\overline{P})$ is connected

Throughout this section, $\Sigma = \text{PG}(V)$ for a $K$-vector space $V$ and $P$ is a set of points of $\Sigma$ such that $\langle P \rangle_\Sigma = \Sigma$ and $\overline{P}$ is the closure of $P$, as in the Introduction. We assume that $\Gamma(\overline{P})$ is connected.

As remarked in the paragraph before the statement of Theorem 1.2, we have $\overline{P} = \bigcup_{n=0}^{\infty} P^{(n)}$ where $P^{(0)} = P$ and $P^{(n+1)}$ is $P^{(n)}$ together with all points $x$ such that $\{x\} = \langle x \rangle_\Sigma \cap \langle x_1, x_2 \rangle_\Sigma$ for a subset $X \subseteq P^{(n)}$ and distinct points $x_1, x_2 \in P^{(n)} \setminus (P^{(n)} \cap \langle X \rangle_\Sigma)$. By applying Lemma 2.2 to $X, \{x_1, x_2\}$ and $x$ we see that, if $P^{(n)}$ is $(F, B)$-rational, then $x \in [\langle B \rangle_F]$. Therefore $K(P^{(n+1)}) = K(P^{(n)})$ for every $n = 0, 1, 2, \ldots$. Hence,


Thus we may safely assume that $\overline{P} = P$. So, for the rest of this section we suppose that $P$ is closed. Accordingly, the graph $\Gamma := \Gamma(P)$ ($= \Gamma(\overline{P})$) is connected.

Henceforth, given two points $x, y$ of $\Sigma$, we put $xy := \langle x, y \rangle_\Sigma$, for short.

Lemma 3.2. Under the above assumptions, the partial linear space $(P, \mathcal{L}(P))$ is an irreducible projective space. In particular, $\Gamma(P)$ is a complete graph.

Proof. Let $(x, y, z)$ be a path of $\Gamma(P)$, with $xy \neq yz$. Then there exist points $x_1 \in (xy \setminus \{x, y\}) \cap P$ and $z_1 \in (yz \setminus \{y, z\}) \cap P$. The points $x, z, x_1, z_1$ are coplanar. Hence the lines $xz$ and $x_1z_1$ meet in a point, say $y_1$. We have $y_1 \in P$, as $P$ is assumed to be closed. Therefore $xz \cap P \in \mathcal{L}(P)$. Thus, we have proved that $\Pi := (P, \mathcal{L}(P))$ is a linear space and that it satisfies Pasch’s axiom (see [3, Definition 2.1.1, Axiom (L3)]). All lines of $\Pi$ have size at least 3, by definition. Hence $\Pi$ is an irreducible projective space (see [3, Section 2.7]). □

Suppose $\dim(\Sigma) \geq 2$ and let $K_\Pi$ be the underlying division ring of the projective space $\Pi := (P, \mathcal{L}(P))$ (see [3, Chapter 8] and note that $\Pi$ inherits Desargues property from $\Sigma$). Then
every member of $K(P)$ contains a copy of $K_{II}$. However, as $K_{II}$ is only determined up to isomorphisms, we are not allowed to conclude that $K_{II}$ is the least element of $K(P)$. Indeed, we are not even allowed to regard $K_{II}$ as a member of $K(P)$, since $K_{II}$ is an abstract object whereas $K(P)$ consists of actual subrings of $K$. Thus, we must push our investigation further.

Given a vector $u_0 \in V \setminus \{0\}$ such that $[u_0] \in P$, we denote by $K(P,u_0)$ the set of subrings $F \leq K$ such that $P$ is $F$-rational with respect to $u_0$, ordered by the inclusion relation. Note that $K(P,u_0) \neq \emptyset$, since $K \in K(P,u_0)$.

**Proposition 3.3.** Let $u_0 \in V \setminus \{0\}$ be such that $p_0 = [u_0] \in P$. Then $K(P,u_0)$ admits a least element.

**Proof.** Put $\mathcal{F} := K(P,u_0)$ for short and define $K_0 := \bigcap_{F \in \mathcal{F}} F$. We shall prove that $K_0$ is indeed the least element of $\mathcal{F}$.

Let $P_0$ be a basis of $\Sigma$ contained in $P$ and containing $p_0$. For every point $p \in P_0 \setminus \{p_0\}$, let $r_p$ denote a point of $(p_0 p \setminus \{p_0, p\}) \cap P$ (which exists by Lemma 3.2). For every $p \in P_0 \setminus \{p_0\}$, let $v_p \in V \setminus \{0\}$ denote the unique representative of the point $p$ such that $\langle u_0 + v_p \rangle = r_p$. Put $v_{p_0} = u_0$ and $B_0 := \{v_p \mid p \in P_0\}$.

Let $F$ be an arbitrary element of $\mathcal{F}$ and $B$ be a basis of $V$ such that $P$ is $(F,B)$-rational with respect to $u_0$. For every point $p$ of $P_0$, let $w_p$ denote a representative of $p$ contained in $(B,F)$. We can choose $w_{p_0} = u_0$. Notice that $w_p, p \in P_0 \setminus \{p_0\}$, is uniquely determined up to a factor in $F^*$. We put $w_p = v_p \lambda_p$ for some $\lambda_p \in K^*$. By Lemma 2.1, $\{\{w_p \mid p \in P_0\}\} = (B)_F$. Combining this with $r_p = \langle u_0 + v_p \rangle = P \subseteq (B)_F$, it follows that $\lambda_p \in F^*$. So, since $w_p$ is uniquely determined up to a factor in $F^*$, we may suppose that $\lambda_p = 1$. Hence, $B_0 = \{w_p \mid p \in P_0\}$ and $\langle B_0 \rangle_F = (B)_F$. It follows that $P$ is $(F,B_0)$-rational with respect to $u_0$ for every $F \in \mathcal{F}$.

Now, let $p$ be an arbitrary point of $P$. Then $p = \langle \tilde{v}_1 \lambda_1 + \tilde{v}_2 \lambda_2 + \cdots + \tilde{v}_k \lambda_k \rangle$ for some $\lambda_1, \lambda_2, \ldots, \lambda_k \in K^*$ and some finite subset $\{\tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_k\}$ of $B_0$. For every $F \in \mathcal{F}$, $P$ is $(F,B_0)$-rational. Hence $\lambda_i \lambda_j^{-1} \in F$ for all $F \in \mathcal{F}$ and all $i, j \in \{1, \ldots, k\}$. This implies that $\lambda_i \lambda_j^{-1} \in K_0$ for all $i, j \in \{1, \ldots, k\}$. So, $p \in \langle B_0 \rangle_{K_0}$. As a consequence, $P$ is $(K_0,B_0)$-rational. Notice also that $u_0 = \langle B_0 \rangle_{K_0}$ since $u_0 \in B_0$. \hfill $\Box$

**Definition.** If a member $F$ of $K(P)$ is the least element of $K(P,u_0)$ for some $u_0 \in V \setminus \{0\}$ such that $[u_0] \in P$, then we say that $F$ is nearly minimal.

**Corollary 3.4.** All nearly minimal elements of $K(P)$ are isomorphic to $K_{II}$. All minimal elements of $K(P)$ are nearly minimal. More explicitly, if $F \in K(P,u_0)$ is minimal in $K(P)$, then $F$ is the smallest element of $K(P,u_0)$.

**Proof.** Let $F_0$ be the least element of $K(P,u_0)$. The division ring $F_0$ contains a copy $F$ of $K_{II}$. As $[u_0] \in P$, modulo replacing $F$ by one of its conjugates in $F_0$ we may assume that $F \in K(P,u_0)$. Therefore $F = F_0$, by the minimality of $F$ in $K(P,u_0)$. This proves the first claim of the corollary. The remaining claims are obvious. \hfill $\Box$

We warn that the converse of the above corollary is false in general. Indeed, as shown in Example 7.2, nearly minimal elements of $K(P)$ might be non-minimal.

**Proposition 3.5.** If $F_1$ and $F_2$ are nearly minimal members of $K(P)$, then there exists a scalar $\lambda \in K^*$ such that $F_2 = F_1^\lambda$. 
Proof. Let \( u_1 \) and \( u_2 \) be vectors of \( V \setminus \{0\} \) such that \( p_1 := [u_1] \in P \) and \( p_2 := [u_2] \in P \). Suppose that \( F_i \) is the least element of \( K(P, u_i) \), for \( i = 1, 2 \). Let \( B_1 \) and \( B_2 \) be bases of \( V \) such that \( P \) is \((F_1, B_1)\)-rational with respect to \( u_1 \). Hence, \( u_1 \in \langle B_1 \rangle_{F_1} \). Since \([u_1], [u_2] \in P\), there exist \( \lambda_1, \lambda_2 \in K^* \) such that \( u_1 \lambda_1 \in \langle B_2 \rangle_{F_2} \) and \( u_2 \lambda_2 \in \langle B_1 \rangle_{F_1} \). Since \( P \) is \((F_1, B_1)\)-rational with respect to \( u_2 \lambda_2 \), it is also \((F_1^{\lambda_2^{-1}}, B_1 \lambda_2^{-1})\)-rational with respect to \( u_2 \). Hence,

\[
F_2 \subseteq F_1^{\lambda_2^{-1}}.
\]  
(2)

In a similar way, one shows that

\[
F_1 \subseteq F_2^{\lambda_1^{-1}}.
\]  
(3)

Now, let \( P_0 \) be a basis of \( \Sigma \) contained in \( P \) and containing \( p_2 \). For every point \( p \in P_0 \setminus \{p_2\} \), let \( r_p \) denote a point of \((p_2 p \setminus \{p_2, p\}) \cap P \) (see Lemma 3.2). For every point \( p \in P_0 \setminus \{p_2\} \), let \( v_p \in V \setminus \{0\} \) denote the unique representative of the point \( p \) such that \( \langle u_2 + v_p \rangle = r_p \). Put \( v_{p_2} = u_2 \) and \( B_0 = \{v_p \mid p \in P_0 \} \). In the proof of Proposition 3.3, we showed that if \( F \) is a subring of \( K \) and \( B \) is a basis of \( V \) such that \( P \) is \((F, B)\)-rational with respect to \( u_2 \), then \( \langle B_0 \rangle_F = \langle B \rangle_F \).

In particular, \( \langle B_0 \rangle_{F_2} = \langle B_2 \rangle_{F_2} \) and \( \langle B_0 \rangle_{F_1^{\lambda_2^{-1}}} = \langle B_1 \lambda_2^{-1} \rangle_{F_1^{\lambda_2^{-1}}} \). (Recall that \( P \) is \((F_1^{\lambda_2^{-1}}, B_1 \lambda_2^{-1})\)-rational with respect to \( u_2 \), as we have noticed above.)

Since \( F_2 \subseteq F_1^{\lambda_2^{-1}} \), we have \( \langle B_0 \rangle_{F_2} \subseteq \langle B_0 \rangle_{F_1^{\lambda_2^{-1}}} \) whence \( \langle B_2 \rangle_{F_2} \subseteq \langle B_1 \lambda_2^{-1} \rangle_{F_1^{\lambda_2^{-1}}} \). As \( u_1 \lambda_1 \in \langle B_2 \rangle_{F_2} \), \( u_1 \lambda_1 \in \langle B_1 \lambda_2^{-1} \rangle_{F_1^{\lambda_2^{-1}}} \), namely \( u_1 \lambda_1 \lambda_2 \in \langle B_1 \rangle_{F_1} \). Since also \( u_1 \in \langle B_1 \rangle_{F_1} \), this implies that

\[
\lambda_1 \lambda_2 \in F_1.
\]  
(4)

From Eqs. (2)–(4),

\[
F_1 \subseteq F_2^{\lambda_1^{-1}} \subseteq F_1^{(\lambda_1 \lambda_2)^{-1}} = F_1.
\]

Hence, \( F_1 = F_2^{\lambda_1^{-1}} \). \( \square \)

The conclusion of Theorem 1.2 in the case that \( \Gamma(\bar{P}) \) is connected immediately follows from Corollary 3.4 and Proposition 3.5. If \( K \) is commutative, we have the following corollary.

**Corollary 3.6.** Suppose that \( K \) is commutative and \( \Gamma(\bar{P}) \) is connected. Then \( K(P) \) admits a smallest element.

**Proof.** If \( F \in K(P) \), then \( F \supseteq F_0 \) for some nearly minimal element \( F_0 \) of \( K(P) \). By Proposition 3.5 and the fact that \( K \) is commutative, all nearly minimal elements of \( K(P) \) are equal to each other. This proves the corollary. \( \square \)
4. Completing the proof of Theorem 1.2

In the previous section we have shown that the conclusion of Theorem 1.2 holds when \( \Gamma(\overline{P}) \) is connected. In this section we assume that \( \Gamma(P) \) is quasi-connected but \( \Gamma(\overline{P}) \) is not connected. As in the previous section, we may assume that \( \overline{P} = P \).

Lemma 4.1. One of the following holds:

1. the set \( P \) is a basis of \( \Sigma \);
2. the set \( P \) is the disjoint union of an independent set \( S \) of points of \( \Sigma \) and an irreducible projective subgeometry \( P' \) of dimension at least 1 of \( \Sigma \) such that, if \( B' \) is a basis of \( P' \), then \( B' \cup S \) is a basis of \( \Sigma \).

Proof. Let \( S \) be the set of points \( p \in P \) such that \( \{ p \} \) is a connected component of \( \Gamma(P) \) and put \( P' := P \setminus S \). As \( \Gamma(P) \) is quasi-connected and \( P \) is assumed to be closed, either \( S = P \) or \( P' \) is an irreducible projective subgeometry of dimension at least 1 of \( \Sigma \) (Lemma 3.2). In order to prove the lemma we only need to show that, if \( X \) is an independent subset of \( P' \) (possibly \( X = \emptyset \)), then \( S \cup X \) is independent in \( \Sigma \).

Suppose the contrary and let \( Y \) be a maximal independent subset of \( S \cup X \). Hence \( (Y)_\Sigma = (S \cup X)_\Sigma \). Moreover \( (S \cup X) \setminus Y \neq \emptyset \), since we are assuming that \( S \cup X \) is dependent. Clearly, \( S \cap X = \emptyset \). Pick a point \( p \in (S \cup X) \setminus Y \) and let \( Y_0 \) be the smallest subset of \( Y \) such that \( p \in (Y_0)_\Sigma \). Then \( Y_0 = S_0 \cup X_0 \) where \( S_0 := Y_0 \cap S \) and \( X_0 := Y_0 \cap X \). Suppose first that \( S_0 \neq \emptyset \) and choose \( q \in S_0 \). Then \( p \in (Y_0)_\Sigma = \bigcup (q \cdot x \mid x \in (Y_0 \setminus \{ q \})_\Sigma) \). (Note that \( |Y_0| > 1 \), otherwise \( p \in (Y_0)_\Sigma \) would force \( p \in Y_0 \), contrary to the assumption that \( p \notin Y \).) By the minimality of \( Y_0 \), neither \( q \) nor \( p \) belong to \( (Y_0 \setminus \{ q \})_\Sigma \). Therefore, since \( pq \) meets \( (Y_0 \setminus \{ q \})_\Sigma \), the points \( p \) and \( q \) belong to the same connected component of \( \Gamma(P) \) (recall that \( P \) is assumed to be closed). However, \( \{ q \} \) is the connected component of \( \Gamma(P) \) containing \( q \), as \( q \in S_0 \subseteq S \). We have reached a contradiction.

Therefore \( S_0 = \emptyset \). Accordingly, \( Y_0 = X_0 \) and \( p \notin X \), since \( X \) is independent. Hence \( p \notin S \). We can now repeat the above argument but with \( q \in X_0 \). We obtain that \( p \) and \( q \) belong to the same connected component of \( \Gamma(P) \) and, once again, we reach a contradiction. Indeed now \( p \in S \), hence \( \{ p \} \) is the connected component of \( \Gamma(P) \) containing \( p \). \( \square \)

Theorem 1.2 now readily follows. In case (1) of Lemma 4.1, \( K(P) \) contains the prime subfield \( K_0 \) of \( K \). Hence \( K(P) \) admits a least element, namely \( K_0 \). In case (2) we have \( K(P) = K(P') \) and we are driven back to Section 3.

5. Proof of Theorem 1.1 in the general case

In this section we suppose that \( \Gamma := \Gamma(\overline{P}) \) is not connected. As in the previous two sections, we assume that \( P = \overline{P} \). Let \( \mathcal{C} \) be the set of all connected components of \( \Gamma(P) \). (By Lemma 3.2, these connected components are cliques, but this fact has no relevance for the following.)

Lemma 5.1. If \( C_1 \) and \( C_2 \) are nonempty subsets of \( \mathcal{C} \) such that \( C_1 \cap C_2 = \emptyset \), then \( \langle \bigcup_{C \in C_1} C \rangle_\Sigma \cap \langle \bigcup_{C' \in C_2} C' \rangle_\Sigma = \emptyset \).

Proof. Suppose the contrary. Then there exist finite independent subsets \( X_1 \subseteq \bigcup_{C \in C_1} C \) and \( X_2 \subseteq \bigcup_{C' \in C_2} C' \) such that \( \langle X_1 \rangle_\Sigma \cap \langle X_2 \rangle_\Sigma \neq \emptyset \). Choose the above sets \( X_1 \) and \( X_2 \) so that
\[X_1 \cup X_2\] is as small as possible. Clearly, neither \(X_1\) nor \(X_2\) is empty. Moreover, \(X_1 \cap X_2 = \emptyset\). Let \(z \in X_2\). By elementary linear algebra (the so-called Grassmann dimensional relation) it follows that \(\dim(\langle X_1 \rangle_{\Sigma} \cap \langle X_2 \rangle_{\Sigma}) \geq \dim(\langle X_1 \rangle_{\Sigma} \cap \langle X_2 \rangle_{\Sigma}) - 1\). However, \(\langle X_1 \rangle_{\Sigma} \cap \langle X_2 \rangle_{\Sigma} = \emptyset\) by the minimality of \([X_1 \cup X_2]\). Hence \(\dim(\langle X_1 \rangle_{\Sigma} \cap \langle X_2 \rangle_{\Sigma}) = 0\), namely \(\langle X_1 \rangle_{\Sigma} \cap \langle X_2 \rangle_{\Sigma}\) is a singleton, say \(\langle X_1 \rangle_{\Sigma} \cap \langle X_2 \rangle_{\Sigma} = \{x\}\).

If \(|X_1| = 1\) then \(X_1 = \{x\}\). Clearly, \(x \notin X_2\). Hence \(|X_2| > 1\). Given a point \(y \in X_2\), we have \(\langle X_2 \rangle_{\Sigma} = \langle x \rangle_{\Sigma}.\) Therefore \(x \in \langle X_2 \rangle_{\Sigma}\), as \(x \in \langle X_2 \rangle_{\Sigma}\). However, \(\langle X_2 \rangle_{\Sigma} = \langle x \rangle_{\Sigma}\) is the union of the lines that contain \(y\) and meet \(\langle X \rangle_{\Sigma}\) in a point. Hence the line \(xy\) meets \(\langle X \rangle_{\Sigma}\) in a point. Therefore \(xy \cap P \in \mathcal{L}(P)\), as \(P = \overline{P}\) by assumption, contrary to the fact that \(x\) and \(y\) belong to distinct connected components of \(\Gamma\).

Let \(|X_1| > 1\). By the above we may assume that \(|X_2| > 1\). Given a point \(y_1 \in X_1\), we have \(x \in \langle X_1 \rangle_{\Sigma} = \langle x \rangle_{\Sigma}\). Hence there exists a point \(z_1 \in \langle X_1 \rangle_{\Sigma}\) such that \(x = y_1z_1\).

Similarly, given \(y_2 \in X_2\) there is a point \(z_2 \in \langle X_2 \rangle_{\Sigma}\) such that the line \(y_2z_2\) contains \(x\).

Clearly, \(\langle X_1 \rangle_{\Sigma} = \langle x \rangle_{\Sigma}\). Therefore either \(\langle X_1 \rangle_{\Sigma} \cap \langle X_2 \rangle_{\Sigma} = \emptyset\) or \(\langle X_1 \rangle_{\Sigma} \cap \langle X_2 \rangle_{\Sigma} = \emptyset\). Hence \(\langle X_1 \rangle_{\Sigma} \cap \langle X_2 \rangle_{\Sigma}\) containing neither \(y_1\) nor \(y_2\). In the first case, by the Grassmann relation we get that \(\dim(\langle X_1 \rangle_{\Sigma} \cap \langle X_2 \rangle_{\Sigma}) = \dim(\langle X_1 \rangle_{\Sigma} \cap \langle X_2 \rangle_{\Sigma}) - 2 = 0 - 2 = -2\). This is impossible, as no projective subspaces exist of dimension smaller than \(-1\). Hence \(\langle X_1 \rangle_{\Sigma} \cap \langle X_2 \rangle_{\Sigma}\) is a hyperplane of \(\langle X_1 \rangle_{\Sigma} \cap \langle X_2 \rangle_{\Sigma}\) containing neither \(y_1\) nor \(y_2\). Therefore the line \(y_1y_2\) meets \(\langle X_1 \rangle_{\Sigma} \cap \langle X_2 \rangle_{\Sigma}\) (actually, it meets the line \(z_1z_2\)) in a point which is of course distinct from \(y_1\) and \(y_2\). As \(P = \overline{P}\), this implies that \(y_1y_2 \cap P \in \mathcal{L}(P)\), contrary to the fact that \(y_1\) and \(y_2\) belong to distinct connected components of \(\Gamma\). \(\square\)

We can now complete the proof of Theorem 1.1. Assume now that \(K\) is commutative. For every \(C \in \mathbb{C}\) let \(F_C\) be the smallest element of \(K(C)\), where \(C\) is regarded as a set of points of the projective space \(\langle C \rangle_{\Sigma}\). Such a field exists by Corollary 3.6 and since now \(K\) is assumed to be commutative. Let \(K_0\) be the smallest subfield of \(K\) containing all subfields \(F_C, C \in \mathbb{C}\). Let \(V_C\) be the subspace of \(V\) such that \(\langle C \rangle_{\Sigma} = \text{PG}(V_C)\) and \(B_C\) be a basis of \(V_C\) such that \(C \subseteq \{\langle B_C \rangle_{F_C}\}\). Put \(B^* = \bigcup_{C \in \mathbb{C}} B_C\). We have \(V = \bigoplus_{C \in \mathbb{C}} V_C\), by Lemma 5.1 and since \(\langle \Sigma \rangle_{\Sigma} = \Sigma\). Hence \(B^*\) is a basis of \(V\). Clearly, \(P \subseteq \{\langle B^* \rangle_{K_0}\}\). Hence, \(P\) is \(K_0\)-rational.

Conversely, suppose that \(P\) is \(F\)-rational. Then \(C\) is also \(F\)-rational, for every \(C \in \mathbb{C}\). It follows that \(F\) contains \(F_C\). Hence \(F\) contains \(K_0\). Therefore, \(K_0\) is the least element of \(K(P)\).

### 6. Proof of Theorem 1.3

Throughout this section \(K\) is a quaternion division algebra. We put \(K_0 := Z(K)\). Given an element \(\lambda \in K\), we denote by \(K_0(\lambda)\) the smallest subring of \(K\) containing \(K_0 \cup \{\lambda\}\). Clearly, the division ring \(K_0(\lambda)\) is commutative for every \(\lambda \in K\). We put \(\deg(\lambda) := |K_0(\lambda) : K_0| = \dim_{K_0}(K_0(\lambda))\) and we call \(\deg(\lambda)\) the degree of \(\lambda\) over \(K_0\). It is well known that \(\deg(\lambda) = 2\) for every \(\lambda \in K \setminus K_0\).

Henceforth \(\lambda\) is a given element of \(K \setminus K_0\) and \(\alpha_\lambda\) is the \(K_0\)-linear mapping sending every \(t \in K\) to \(t^\lambda = \lambda^{-1} t \lambda\).

**Lemma 6.1.** The mapping \(\alpha_\lambda\) is a root of a polynomial \(q_\lambda(t)\) of the form
\[
q_\lambda(t) = t^3 + kt^2 - kt - 1
\]
for a suitable element \(k \in K_0\).
Proof. Put $K_1 := K_0(\lambda)$. Clearly, $\alpha_\lambda$ fixes all elements of $K_1$. By the Skolem–Noether theorem (see [1, Section 10.1]), there exists $\nu \in K^*$ such that $\nu \lambda \nu^{-1} = \lambda^\sigma$, where $\sigma : t \mapsto t^\sigma$ is the standard involution of the quaternion algebra $K$. Thus $\alpha_\lambda(\nu) = \lambda^{-1} \nu = \lambda^{-2} N(\lambda) \nu$, where $N(\lambda) = \lambda \lambda^\sigma$ is the norm of $\lambda$. It follows that, regarding $K$ as a $K_0$-vector space, the subspace $K_2$ of $K$ spanned by $\nu$ and $\lambda \nu$ is $\alpha_\lambda$-invariant, and $\alpha_\lambda(t) = \lambda^{-2} N(\lambda) t$ for every $t \in K_2$. So, $K = K_1 \oplus K_2$ (direct sum of $K_0$-vector spaces), and $K_1$ is the eigenspace of $\alpha_\lambda$ corresponding to the eigenvalue 1.

Put $\mu := \lambda^{-2} N(\lambda)$. Clearly $\mu \neq 1$, as $\lambda \notin K_0$. Also, $\mu \mu^\sigma = \lambda^{-2} N(\lambda) (\lambda^{-2} N(\lambda))^\sigma = 1$. Notice that $\mu$ is a root of the polynomial $t^2 - (\mu + \mu^\sigma) t + \mu \mu^\sigma = t^2 - (\mu + \mu^\sigma) t + 1$. Therefore $\alpha_\lambda^2 - (\mu + \mu^\sigma) \alpha_\lambda + 1$ induces the null mapping on $K_2$. On the other hand, $\alpha_\lambda - 1$ induces the null mapping on $K_1$. As $K = K_1 \oplus K_2$, the linear mapping $\alpha_\lambda$ is a root of the polynomial

$$q_\lambda(t) := (t - 1)(t^2 - (\mu + \mu^\sigma) t + 1) = t^3 + kt^2 - kt - 1,$$

where $k := -1 - \mu - \mu^\sigma \in K_0$. □

Lemma 6.2. For a sub-division ring $F$ of $K$, let $\lambda \in K^*$ be such that $F^{\lambda} \subseteq F$. Then $F^{\lambda} = F$.

Proof. Put $F_1 := F$ and $F_2 := F^{\lambda}$. For $i = 1, 2$, put $Z_i = F_i \cap K_0$. Then $Z_2 = Z_1^\lambda$. Hence $Z_1 = Z_2 = Z$, say. So, both $F_1$ and $F_2$ can be regarded as $Z$-vector spaces and $\alpha_\lambda$ induces a $Z$-linear mapping from $F_1$ to $F_2$. Let $\{e_j\}_{j \in J}$ be a basis of $F_1$ over $Z$. For every $j \in J$ and every $i = 0, 1, 2, \ldots$, put $e_{j,i} := \alpha_\lambda^i(e_j) = e_j^{\lambda^i}$. Then $\{e_{j,i}\}_{j \in J}$ is a basis of $\alpha_\lambda^i(F_1)$. Therefore, if $i > 0$ then $\{e_{j,i}\}_{j \in J}$ is a set of vectors of $F_2$. By Lemma 6.1,

$$q_\lambda(\alpha_\lambda) = \alpha_\lambda^3 + k \alpha_\lambda^2 - k \alpha_\lambda - 1 = 0.$$

So, if we compute $q_\lambda(\alpha_\lambda)$ at $e_j$ we obtain that

$$0 = \alpha_\lambda(e_j) = e_{j,3} + ke_{j,2} - ke_{j,1} - e_j.$$

Therefore:

$$e_j = e_{j,3} + k(e_{j,2} - e_{j,1}). \tag{5}$$

Since $e_{j,3} \in F_2 \subseteq F_1$, Eq. (5) forces $k(e_{j,2} - e_{j,1}) \in F_1$. If $e_{j,1} = e_{j,2}$ then $e_j = e_{j,3} \in F_2$. Otherwise, since $(e_{j,2} - e_{j,1})^{-1} \in F_2 \subseteq F_1$, we obtain that $k \in F_1$. Hence $k \in F_1 \cap K_0 = Z = F_2 \cap K_0$. Therefore $e_j \in F_2$ by (5). So, $e_j \in F_2$ in any case and for every $j \in J$. Consequently, $F_1 = F_2$. □

We are now ready to prove Theorem 1.3. Suppose that $\Gamma(P)$ is quasi-connected. As before, we may also assume that $P = \overline{P}$. If case (1) of Lemma 4.1 holds then $K(P)$ contains the prime subfield of $K$. In this case there is nothing to prove. Suppose that we have (2) of Lemma 4.1. Recall that in this case $K(P) = K(P')$. Hence, after replacing $P$ with $P'$, we may assume that $\Gamma(P)$ is connected and we can apply the results of Section 3. In particular, by Proposition 3.3, every member of $K(P)$ contains a nearly minimal element of $K(P)$. Theorem 1.3 now follows from the next proposition.

Proposition 6.3. If $\Gamma(P)$ is connected then all nearly minimal elements of $K(P)$ are minimal.
Proof. Given a vector \( u \neq 0 \) such that \([u] \in P\), let \( F_1 \) be the least element of \( K(P,u) \) and suppose that \( F_1 \supseteq F \) for another member \( F \) of \( K(P) \). Then \( F \in K(P,u\lambda) \) for some \( \lambda \in K^* \). Let \( F_2 \) be the least element of \( K(P,u\lambda) \). Then \( F_2 = F_1^\lambda \) (see Proposition 3.5). Lemma 6.2 now implies that \( F_2 = F_1 \). \( \Box \)

7. Examples

Example 7.1. Denoting by \( \mathbb{Q} \) the field of rational numbers, let \( K = \{a + ib + jc + kd\}_{a,b,c,d \in \mathbb{Q}} \) be the division ring of rational quaternions, \( V \) a 2-dimensional \( K \)-vector space, \( u_1 \) and \( u_2 \) two given non-proportional vectors of \( V \) and \( P \) the quadruple of points of \( \Sigma := PG(V) \) represented by the vectors \( u_1, u_2, u_1 + u_2 \) and \( u_1 + u_2(i + j) \) of \( V \). We have \( (u_1 + u_2(i + j))i = u_1i + u_2i(i - j) \). Therefore \( P \) is both \( F^+ \)-rational and \( F^- \)-rational, where \( F^+ := \{a + b(i + j)\}_{a,b \in \mathbb{Q}} \) and \( F^- := \{a + b(i-j)\}_{a,b \in \mathbb{Q}} \). (Needless to say, \( F^+ \) and \( F^- \) are maximal subfields of \( K \).) However, \( F^+ \cap F^- = \mathbb{Q} \), but \( P \) is not \( \mathbb{Q} \)-rational. Hence \( F^+ \) and \( F^- \) are minimal in \( K(P) \).
They are conjugate, as claimed in Theorem 1.2. Indeed \( F^- = (F^+)^t \).

Example 7.2. Given a non-commutative division ring \( K_0 \), let \( K = K_0(t) \) be the division ring of rational functions over \( K_0 \) in the variable \( t \). (We refer the reader to [2, Chapter 2] for basics on non-commutative rings of polynomials or rational functions.) Let \( M \) be the set of rational functions of the following form: \( a_0t^{n_1}a_1t^{n_2}...a_{n-1}t^{n_n}a_n \) where \( a_0, a_n \in K_0 \) and either \( n = 0 \) or \( n > 1 \). \( K_0 \) is a subring of \( K \) and \( M \) is a subring of \( K \) containing a conjugate of \( F \). This set of subrings does not contain any minimal element. Indeed, if \( F' \) were such a minimal element then \( F' = g^{-1}Fg \) for some \( g \in K^* \). However, the infinite chain \( F \supset tFt^{-1} \supset t^2Ft^{-2} \supset \cdots \) yields a chain \( F' = g^{-1}Fg \supset g^{-1}tFt^{-1}g \supset g^{-1}t^2Ft^{-2}g \supset \cdots \). So, \( F' \) cannot be minimal in \( K(P) \).

Example 7.3. Given a non-commutative division ring \( K_0 \), let \( K = K_0(t_1, t_2, t_3) \) be the division ring of rational functions over \( K_0 \) in the variables \( t_1, t_2, t_3 \) and let \( V \) be a 4-dimensional \( K \)-vector space. Choose a basis \( \{e_1, e_2, e_3, e_4\} \) of \( V \) and, for \( i = 1, 2 \), put \( K_i := K_0(t_i) \) (the subring of rational functions with \( t_i \) as unique variable) and \( S_i := \{[e_i + e_{i+2}f]\}_{f \in K_i} \). Put \( P := S_1 \cup S_2 \). It is not difficult to see that, for any two elements \( g_1, g_2 \in K \), the subring \( K_1^{g_1} \cup K_2^{g_2} \) of \( K \) generated by \( K_1^{g_1} \cup K_2^{g_2} \) is a minimal element of \( K(P) \). However, these subrings are not pairwise conjugate. For instance, \( K_1 \cap K_2 \) and \( K_1 \cap K_2^3 \) are not conjugate in \( K \).

8. Global and local underlying fields of partial linear spaces

Suppose that \( K \) is commutative and let \( K_P \) be the least element of \( K(P) \) (Theorem 1.1). Given a subset \( L \subseteq L(P) \), put \( S = (P, L) \). By definition, \( S \) is a subgeometry of \( S(P) := (P, L(P)) \). It is quite natural to call \( K_P \) the global underlying field of \( S \). For every line \( L \in L \), let \( K_L \) be the least element of \( K(L) \) (which exists by Theorem 1.1 applied to \( L \)) and let \( K_S \) be the subfield of \( K_P \) generated by the family \( \{K_L\}_{L \in L} \). We call \( K_S \) the local underlying field of \( S \).
Clearly $K_S \leq K_{S(P)} \leq K_P$, where $K_{S(P)}$ is defined just as $K_S$ but with $L$ replaced by $L(P)$. However, neither of the equalities $K_S = K_{S(P)}$ or $K_{S(P)} = K_P$ holds in general, as the following two examples show. We have $K_S < K_{S(P)} = K_P$ in Example 8.1 and $K_{S(P)} < K_P$ in Example 8.2. Note that the considered partial linear space is connected in both examples.

**Example 8.1.** Let $\Sigma := \text{PG}(2, 4)$ (the projective plane over the field $\text{GF}(4)$ of order 4). We recall that a hyperoval of $\Sigma$ is a nonempty set $H$ of points of $\Sigma$ such that every line of $\Sigma$ meets $H$ in either 0 or exactly 2 points. Given a hyperoval $H$ of $\Sigma$, define $S = (P, L)$ as follows: $P$ is the set of points of $\Sigma$ exterior to $H$ and $L = \{L \cap P \mid L \in L_{\Sigma}, |L \cap H| = 2\}$. It is well known that $S$ is isomorphic to the symplectic generalized quadrangle $W(2)$ of order 2 (see [5, Chapter 2]). As all lines of $S$ have size 3, we have $K_L = \text{GF}(2)$ for every line $L \in L$. Hence $K_S = \text{GF}(2)$. However, $L(P)$ also contains the six lines of $\Sigma$ exterior to $H$. These lines have size 5, hence they can only be $\text{GF}(4)$-rational. It follows that $K_{S(P)} = K_P = \text{GF}(4)$.

**Example 8.2.** Let $\Sigma = \text{PG}(2, 4)$, as in Example 8.1. Given three lines $L_1, L_2, L_3$ of $\Sigma$ such that $L_1 \cap L_2 \cap L_3 = \emptyset$, put $T := L_1 \cup L_2 \cup L_3$ and let $P$ be the complement of $T$ in $\Sigma$. All lines $L \in L(P)$ have size 3. Hence $K_{S(P)} = \text{GF}(2)$. However, $K_P = \text{GF}(4)$, since $|P| = 9$ and only seven points exist in $\text{PG}(2, 2)$.

More examples with $K_S < K_P$ are given in [4]. The following problem now naturally arises: Find nice conditions on $S = (P, L)$ that imply $K_S = K_P$. In particular, find conditions on $P$ that force $K_{S(P)} = K_P$. Answers to this problem are given in [4], but it is unlikely that they are the best possible answers.

**References**