JOURNAL OF ALGEBRA 9, 42-53 (1968)

Commutative Algebra Cohomology and Deformations of Lie and Associative Algebras

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Communicated by Saunders MacLane

Received May 8, 1967

INTRODUCTION

The theory of deformations of associative algebras or Lie algebras has heretofore been discussed in terms of the Hochschild (resp. Chevalley-Eilenberg) cohomology of these algebras. See Gerstenhaber [*4*] and the authors [*8*, *10*]. A geometric picture of the deformations is obtained by considering the variety $\mathscr{M}$ of all those bilinear maps ("products") of the underlying vector space into itself which satisfy the associativity condition (resp. the Jacobi identity). One associative (resp. Lie) algebra structure then represents a point m of $\mathscr{M}$, and deformations of the structure are represented by points of $\mathscr{M}$ near m. Thus the study of deformations of these algebras is a special case of a study of local geometric properties of varieties.

Harrison [*6*] has recently defined a cohomology theory for commutative algebras, which is particularly applicable to the coordinate rings of algebraic varieties. More specifically, let $\mathfrak{a}$ be an ideal in the algebra $A(W)$ of polynomial functions on a vector space W, and let $\mathscr{M}$ be the algebraic set of zeros of $\mathfrak{a}$. One can ask for conditions which imply that m is a simple point of $\mathscr{M}$. Harrison shows that, if $A = A(W)/\mathfrak{a}$, the vanishing of his module $H^2(A, k)$ implies that m is a simple point of $\mathscr{M}$; it also follows that the Zariski tangent space to $\mathscr{M}$ at m is equal to $Z^1(A, k)$, the space of 1-cocycles. (Here the base field k is an A-module through the evaluation homomorphism $A \to k$ at m.) If A has no

* Supported by NSF Grant GP 4503 at the University of Pennsylvania.

** Partial support received from an Office of Naval Research Contract at the University of Washington.

nilpotent elements, then $H^2(A, k) = 0$ is necessary and sufficient for m to be a simple point.

In the particular case when $\mathscr{M}$ is the variety of all associative (resp. Lie) structures on a vector space (and $\mathfrak{a}$ is the ideal generated by the polynomials defining the associativity condition, resp. the Jacobi identity), Harrison's work gives $H^2(A, k) = 0$ as a sufficient condition for a simple point. The work by Gerstenhaber and by the authors gave $H^3(L, L) = 0$ (vanishing of the "obstructions space") as a sufficient condition that the point $m \in \mathscr{M}$ representing the algebra L be a simple point of $\mathscr{M}$. This suggests the surprising conclusion that there should be a relation between the *associative* (resp. *Lie*) algebra cohomology module $H^3(L, L)$ and the *commutative* algebra cohomology module $H^2(A, k)$. We show that this is so.

More precisely, we prove that $H^2(A, k)$ is canonically isomorphic to a subspace of $H^3(L, L)$. A simple argument shows that $Z^2(L, L)$ (the space of "infinitesimal deformations") is identical with $H^1(A, k)$. Many of the theorems of [*4*, *8*, *9*, *10*] on deformations of algebras have thus become special cases of geometric results of Harrison.

Our proof is strictly nonhomological and depends on a detailed study of the equations defining $\mathscr{M}$. We give the proof within the graded Lie algebra framework of [*8*] and thus obtain, simultaneously, proofs for Lie algebras, associative algebras, and commutative algebras, as well as for similar theorems in several other cases, including homomorphisms of Lie and associative algebras.

1. Preliminaries

If V and W are vector spaces over a field k, then $A(V)$ will denote the k-algebra of all polynomial functions on V and $A(V, W)$ will denote the $A(V)$-module of all polynomial maps of V into W. (For polynomial functions and polynomial maps we refer the reader to [*1*].) To simplify our exposition, we shall assume throughout that the base field k is infinite. This allows us to identify $A(V)$ with the symmetric algebra $S(V')$ of the dual space V' of V. Similarly $A(V, W)$ can be identified with $S(V') \otimes_k W$. However, all of our theorems and proofs are valid, with only minor modifications, for the case of finite k. We shall leave their formulation to the reader.

2. Cohomology of Commutative Algebras

Let A be a commutative k-algebra and let E be an A-module. We refer the reader to Harrison's paper [*6*] for the definition of the cohomology modules $H^n(A, E)$ ($n = 1, 2, 3$). (The "correct" definition of the higherdimensional cohomology modules $H^n(A, E)$ $(n > 3)$ is not yet clear.) We shall need the following properties of these cohomology modules:

2.1. *Let the ideal* $\mathfrak{a}$ *of* A *annihilate the* A*-module* E. *Then there is a natural exact sequence*

$$0 \to H^1(A/\mathfrak{a}, E) \to H^1(A, E) \to \mathrm{Hom}_A(\mathfrak{a}, E) \to H^2(A/\mathfrak{a}, E) \to H^2(A, E).$$

This is just Theorem 2 of [*6*].

2.2. *Let* S *be a multiplicative subset of* A *and let* $S^{-1}A$ *denote the ring of fractions of* A *with respect to* S. *Let* E *be an* $S^{-1}A$*-module and consider* E *as an* A*-module via the canonical homomorphism* $A \to S^{-1}A$. *Then the canonical homomorphism* $H^2(S^{-1}A, E) \to H^2(A, E)$ *is an isomorphism.*

This is Theorem 16 of [*6*].

2.3. *Let* A *be a polynomial algebra over* k *and let* E *be an* A*-module. Then* $H^2(A, E) = 0$.

This is included in Theorem 11 of [*6*].

2.4. *Let the commutative algebra* A *over the perfect field* k *be a local ring with maximal ideal* $\mathfrak{m}$. *Then* A *is a regular local ring if and only if* $H^2(A, A/\mathfrak{m}) = 0$.

This is Theorem 19 of [*6*].

2.5. *Let* A *be a finitely-generated commutative* k*-algebra, let* N *denote the nilradical of* A *and let* $\varphi : A \to k$ *be a* k*-algebra homomorphism. Let* X *be the affine algebraic set corresponding to the affine* k*-algebra* A/N *and let* $x \in X$ *correspond to* φ. *If* $H^2(A, k) = 0$, *then* x *is a simple point of* X.

Using localization (i.e., 2.2), this is an immediate consequence of Theorem 24 of [*6*]. The hypothesis that k be a perfect field made in Theorem 24 is not necessary in this case. All one needs is that $H^2(k, k) = 0$, and it is trivial to check that this holds for any field k. We remark that the converse of 2.5 is not valid.

Remark. Let A and N be as in 2.5. It follows from 2.5 and 2.4 that, if $H^2(A, k) = 0$, then $H^2(A/N, k) = 0$. However, it is not true in general that the canonical map $H^2(A/N, k) \to H^2(A, k)$ is an injection.

3. Polynomial Mappings and the Harrison Cohomology

For the purposes of this paper we shall need an explicit description of the commutative algebra cohomology modules associated with a polynomial mapping of vector spaces. Such a description is given by Theorem 3.1 below, which can be considered as a geometric interpretation of the exact sequence of 2.1.

Let V and W be finite-dimensional vector spaces over k and let W' denote the dual space of W. The pairing of elements $w \in W$ and $w' \in W'$ is denoted by $\langle w', w\rangle$. This pairing extends to a pairing of the $A(V)$-modules $A(V, W')$ and $A(V, W)$ into $A(V)$ defined as follows: if $Q \in A(V, W')$ and $P \in A(V, W)$, then $\langle Q, P\rangle \in A(V)$ is defined by $\langle Q, P\rangle (v) = \langle Q(v), P(v)\rangle$. This pairing defines $A(V, W')$ as the dual space of the $A(V)$-module $A(V, W)$.

THEOREM 3.1. *Let $P : V \to W$ be a polynomial mapping, let $m \in V$ with $P(m) = 0$, and consider k as an $A(V)$-module by means of the evaluation homomorphism $\varphi_m : A(V) \to k$ at m. Let $\eta : A(V, W') \to A(V)$ be the $A(V)$-module homomorphism defined by $\eta(Q) = \langle Q, P\rangle$, let $\mathfrak{a} = \mathrm{Im}(\eta)$ and let $\mathfrak{r} = \mathrm{Ker}\,(\eta)$. Let*

$$Z = \{w \in W \mid \langle Q(m), w\rangle = 0 \textit{ for every } Q \in \mathfrak{r}\}$$

and let B denote the image of the differential $d_mP : V \to W$. Then P canonically determines an isomorphism of the quotient space Z/B onto the Harrison cohomology module $H^2(A(V)/\mathfrak{a}, k)$. Furthermore, the kernel of d_mP is canonically isomorphic to $H^1(A(V)/\mathfrak{a}, k)$.

See [1] for the differential of a polynomial mapping.

Proof. Since $A(V)$ is a polynomial algebra, 2.3 implies that $H^2(A(V), k) = 0$. We note that the ideal $\mathfrak{a}$ annihilates the $A(V)$-module k. Hence, following 2.1, we have an exact sequence

$$0 \to H^1(A(V)/\mathfrak{a}, k) \to H^1(A(V), k) \xrightarrow{\sigma} \mathrm{Hom}_{A(V)}\,(\mathfrak{a}, k) \xrightarrow{\tau} H^2(A(V)/\mathfrak{a}, k) \to 0. \tag{3.2}$$

Consequently, $H^2(A(V)/\mathfrak{a}, k)$ is isomorphic to the quotient space $\mathrm{Hom}_{A(V)}\,(\mathfrak{a}, k)/\sigma(H^1(A(V), k))$. We shall prove the main conclusion of 3.1 by defining a linear isomorphism $\psi : \mathrm{Hom}_{A(V)}\,(\mathfrak{a}, k) \to Z$ such that $\psi(\sigma(H^1(A(V), k))) = B$.

The homomorphism η determines an $A(V)$-module isomorphism $A(V, W')/\mathfrak{r} \to \mathfrak{a}$. Let

$$\alpha : \mathrm{Hom}_{A(V)}\,(\mathfrak{a}, k) \to \mathrm{Hom}_{A(V)}\,(A(V, W')/\mathfrak{r}, k)$$

be the induced isomorphism and let

$$\beta : \mathrm{Hom}_{A(V)}\,(A(V, W')/\mathfrak{r}, k) \to \mathrm{Hom}_{A(V)}\,(A(V, W'), k)$$

be the canonical monomorphism. Since $A(V, W')$ is canonically isomorphic to $A(V) \otimes_k W'$, it follows that there is a canonical isomorphism γ of

$\mathrm{Hom}_{A(V)}(A(V, W'), k)$ with $\mathrm{Hom}_k(W', k)$, the bi-dual of W; we identify W with its bi-dual $\mathrm{Hom}_k(W', k)$ in the standard manner. If $w \in W$ and $Q \in A(V, W')$, then $\gamma^{-1}(w)(Q) = \langle Q(m), w\rangle$. Now let $\psi : \mathrm{Hom}_{A(V)}(\mathfrak{a}, k) \rightarrow W$ denote the composite homomorphism $\gamma \circ \beta \circ \alpha$; ψ is a monomorphism. Furthermore, one checks easily from the definitions that the image of ψ is precisely Z.

The cohomology module $H^1(A(V), k)$ is, by definition, the vector space of all φ_m-derivations of $A(V)$ into k. It is well-known that there is a canonical isomorphism θ of V onto $H^1(A(V), k)$. For each φ_m-derivation d of $A(V)$ into k, let $\sigma(d)$ denote the restriction of d to $\mathfrak{a}$; then $\sigma(d) \in \mathrm{Hom}_{A(V)}(\mathfrak{a}, k)$ and $\sigma : H^1(A(V), k) \rightarrow \mathrm{Hom}_{A(V)}(\mathfrak{a}, k)$ is the map which occurs in the exact sequence (3.2). An elementary argument shows that the composite map

$$V \xrightarrow{\theta} H^1(A(V), k) \xrightarrow{\sigma} \mathrm{Hom}_{A(V)}(\mathfrak{a}, k) \xrightarrow{\psi} W$$

is just the differential $d_mP : V \rightarrow W$ of the polynomial map P at m. Since θ is an isomorphism, the image of $\psi \circ \sigma$ is just the image B of d_mP. Consequently, ψ determines an isomorphism of $\mathrm{Hom}_{A(V)}(\mathfrak{a}, k)/\sigma(H^1(A(V), k))$ onto Z/B. Moreover, it follows easily from (3.2) and the remarks above that the kernel of d_mP is canonically isomorphic to $H^1(A(V)/\mathfrak{a}, k)$. This completes the proof of Theorem 3.1.

3.2. *Remarks.* (a) The $A(V)$-module $\mathfrak{r}$ can be considered as the "module of relations" determined by the polynomial map $P : V \rightarrow W$. Consider, in particular, the case $W = k^n$. Then $P = (P_1, \dots, P_n)$, where $P_1, \dots, P_n \in A(V)$. If we identify the dual space W' with k^n by means of the usual pairing of k^n and k^n into k, then, for $Q = (Q_1, \dots, Q_n) \in A(V, W')$, we have

$$\langle Q, P\rangle = Q_1P_1 + \cdots + Q_nP_n .$$

Thus $\mathfrak{r}$ consists of all n-tuples $(Q_1, \dots, Q_n)$ of polynomial functions such that $Q_1P_1 + \cdots + Q_nP_n = 0$. This is the usual definition of the module of relations determined by an n-tuple $(P_1, \dots, P_n)$ of polynomial functions.

(b) The element $m \in V$ determines a pairing of the $A(V)$-modules $A(V, W')$ and W into k defined by $\langle Q, w\rangle = \langle Q(m), w\rangle$ for $Q \in A(V, W')$ and $w \in W$. With respect to this pairing, Z is the submodule of W which is orthogonal to the module of relations $\mathfrak{r}$.

(c) It is easy to check that the ideal $\mathfrak{a}$ of $A(V)$ is generated by all functions of the form $f \circ P$, where $f \in W'$.

3.3. We continue with the notation of Theorem 3.1. Let C be a vector subspace of W which is supplementary to Z and let $\pi_C : W \rightarrow C$ and let $\pi_Z : W \rightarrow Z$ be the projection operators corresponding to the direct sum decomposition $W = Z + C$. The following lemma is crucial in a number of problems involving "deformations" of Lie and associative algebras. For example, it includes as a special case, Lemma 19.2 of [8], which was the crucial lemma in the construction of "Kuranishi families" of deformations. An infinite-dimensional analogue of this lemma is basic in Kuranishi's construction of locally complete families of complex structures on a compact manifold. See [5, pp. 122-123, in particular, Lemma 2.3].

LEMMA 3.4. *There exists a Zariski-open subset U of V containing m such that, if $v \in U$ and $\pi_Z(P(v)) = 0$, then $P(v) = 0$.*

Proof. The $A(V)$-module $\mathfrak{r}$ is finitely-generated; let $Q_1, \ldots, Q_n$ be a set of generators of $\mathfrak{r}$. For each $v \in V$, let the linear map $\varphi_v : W \to k^n$ be defined by

$$\varphi_v(w) = (\langle Q_1(v), w\rangle, \ldots, \langle Q_n(v), w\rangle).$$

By definition of Z and C the restriction of φ_m to C is injective. Hence there exists a Zariski-open subset U of V containing m such that, if $v \in U$, then the restriction of φ_v to C is injective. We have $\langle Q(v), P(v)\rangle = 0$ for every $Q \in \mathfrak{r}$, $v \in V$. Thus $\varphi_v(P(v)) = 0$ for every $v \in V$. Now let $v \in U$ with $\pi_Z(P(v)) = 0$. Then

$$\begin{aligned} 0 &= \varphi_v(P(v)) \\ &= \varphi_v(\pi_Z(P(v)) + \pi_C(P(v))) \\ &= \varphi_v(\pi_C(P(v))). \end{aligned}$$

Since $v \in U$, this implies that $\pi_C(P(v)) = 0$, hence that $P(v) = 0$.

4. THE DEFORMATION EQUATION

As indicated in [8], the set of all Lie (resp. associative) multiplications on a vector space can be described as the set of all solutions of a "deformation equation" in a graded Lie algebra. In this section we shall study the deformation equation from the point of view set forth in Section 3. We assume, henceforth, that the base field k is not of characteristic 2. However, all our results are valid for characteristic 2; see [8] for the necessary modifications in this case.

A *graded Lie algebra* over k is given by a graded vector space $E = \bigoplus_{n \in Z} E_n$, together with a bilinear map $(x, y) \to [x, y]$ of $E \times E$ into E such that the following conditions are satisfied for every $x \in E_p$, $y \in E_q$, and $z \in E_r$:

(1) $[x, y] \in E_{p+q}$;

(2) $[x, y] = -(-1)^{pq}[y, x]$;

(3) $(-1)^{pr}[x, [y, z]] + (-1)^{pq}[y, [z, x]] + (-1)^{qr}[z, [x, y]] = 0.$

If characteristic $k = 3$, we also require $[x[x, x]] = 0$ for p odd (this follows from (3) if characteristic $k \neq 3$).

Equation (3) is called the graded Jacobi identity.

Let $D : E \to E$ be a homogeneous derivation of degree 1 (see [*8*, p. 7] for definition) such that $D \circ D = 0$. The equation

$$Dx + \tfrac{1}{2}[x, x] = 0, \qquad x \in E_1 \tag{4.1}$$

is called the "deformation equation".

For each $x \in E_1$, we define a homogeneous linear map $D_x : E \to E$ of degree 1 by $D_x(y) = Dy + [x, y]$. It follows from the graded Jacobi identity that D_x is a homogeneous derivation of degree 1. Now, let $m \in E_1$ satisfy the deformation equation (4.1). In this case the graded Jacobi identity implies that $D_m \circ D_m = 0$. Consequently, (E, D_m) is a cochain complex. We denote by $H^n(E, D_m)$ (resp. $Z^n(E, D_m)$, $B^n(E, D_m)$) the nth cohomology module (resp. module of n-cocycles, n-coboundaries) of (E, D_m).

We now assume that each summand E_j is finite-dimensional. Let the polynomial map $P : E_1 \to E_2$ be defined by $P(x) = Dx + \frac{1}{2}[x, x]$. The deformation equation (4.1) now reads $P(x) = 0$. Let $m \in E_1$ satisfy $P(m) = 0$. We now adopt the notation of Theorem 3.1 with $V = E_1$, $W = E_2$, and P and m as above. Since m satisfies (4.1), it follows that

$$P(m + x) = D_m x + \tfrac{1}{2}[x, x].$$

Thus the differential $d_m P$ is just D_m. It follows from Theorem 3.1 that $Z^1(E, D_m)$, the kernel of D_m, is equal to the Harrison cohomology module $H^1(A(E_1)/\mathfrak{a}, k)$. (Here $\mathfrak{a}$ is the ideal of $A(E_1)$ generated by all functions $\eta \circ P$, where η is a linear function on E_2.) Furthermore, $B^2(E_1, D_m)$, the image of $D_m = d_m P$, is equal to B.

We now wish to show that $Z \subset Z^2(E, D_m)$. For each linear function η on E_3, we define a polynomial mapping (in fact, a linear mapping) f_η of E_1 into E_2', the dual space of E_2, by $f_\eta(x) = \eta \circ D_x$. We note that the graded Jacobi identity implies that $D_x(P(x)) = 0$ for every $x \in E_1$. Hence $\langle f_\eta(x), P(x) \rangle = \eta \circ D_x(P(x)) = 0$ for every $x \in E_1$. It follows that f_η belongs to $\mathfrak{r}$, the module of relations, for every $\eta \in E_3'$. Let $z \in Z$. Then, since $f_\eta \in \mathfrak{r}$, $\eta(D_m z) = \langle f_\eta(m), z \rangle = 0$. Since this holds for every $\eta \in E_3'$, this implies that $D_m z = 0$, hence that $z \in Z^2(E, D_m)$. Thus $Z \subset Z^2(E, D_m)$.

Since $H^2(A(E_1)/\mathfrak{a})$ is canonically isomorphic to Z/B, the injection $Z/B \subset Z^2(E, D_m)/B$ gives a canonical injection of $H^2(A(E_1)/\mathfrak{a}, k)$ into $H^2(E, D_m)$. Furthermore, it follows immediately from the definitions that $H^1(A(E_1)/\mathfrak{a}, k)$ is identical to $Z^1(E, D_m)$. Thus we have proved

THEOREM 4.2. *Let the notation be as above. Then $H^2(A(E_1)/\mathfrak{a}, k)$ is canonic-*

ally isomorphic to a vector subspace of $H^2(E, D_m)$ *and* $H^1(A(E_1)/\mathfrak{a}, k)$ *is identical to* $Z^1(E, D_m)$.

COROLLARY 4.3. *Let* k *be algebraically-closed and let* $\mathcal{M}$ *be the affine algebraic set of all solutions of the deformation equation (4.1). Let* $m \in \mathcal{M}$ *be such that* $H^2(E, D_m) = 0$. *Then* m *is a simple point of* $\mathcal{M}$ *and the tangent space of* $\mathcal{M}$ *at* m *is equal to* $Z^1(E, D_m)$.

The proof of the first conclusion follows immediately from 4.2 and 2.5. The second conclusion follows easily from 4.2 and the definition of the tangent space of an affine algebraic set. The restriction that k be algebraically-closed is purely for technical reasons. (Otherwise one must replace $\mathcal{M}$ by the set of all solutions of the deformation equation in the graded Lie algebra $E \otimes_k K$, where K denotes an algebraically-closed field containing k.)

We remark that Corollary 4.3 is identical with Theorem 23.3 of [*8*]. Furthermore, Theorem 23.4 of [*8*] on the existence of "Kuranishi families" of deformations is an easy consequence of Theorem 4.2 and Lemma 3.2.

5. APPLICATIONS

(1) *Associative algebras* (See [*3*, *4*].) Let V be a finite-dimensional vector space over k. For each integer $p \geqslant -1$, we define E_p to be the vector space of all $(p+1)$-linear maps of V into V; we set $E_p = \{0\}$ if $p < -1$ and let $E = \bigoplus_{p \in Z} E_p$. If $\varphi \in E_p$ and $\psi \in E_q$, then, following Gerstenhaber [*3*], we define $\varphi \circ \psi \in E_{p+q}$, the composition product of φ and ψ, by

$$\varphi \circ \psi(v_0, \ldots, v_{p+q})$$
$$= \sum_{i=0}^{p} (-1)^{i(q+1)} \varphi(v_0, \ldots, v_{i-1}, \psi(v_i, \ldots, v_{i+q}), v_{i+q+1}, \ldots, v_{p+q}).$$

We define $[\varphi, \psi] = \varphi \circ \psi - (-1)^{pq} \psi \circ \varphi$. Then, as shown by Gerstenhaber, this product defines on E the structure of a graded Lie algebra.

Let $D = 0$. Since $\frac{1}{2}[\varphi, \varphi] = \varphi \circ \varphi$ for $\varphi \in E_1$, the deformation equation (4.1) becomes $\varphi \circ \varphi = 0$. A trivial computation shows that $\varphi \circ \varphi = 0$ if and only if φ is associative. Thus the solutions of the deformation equation are precisely the associative multiplications on V.

Let m be an associative multiplication on V and $L = (V, m)$ be the corresponding associative algebra (not necessarily with identity). We consider L as a two-sided L-module in the obvious manner. Let $C(L, L) = \bigoplus_n C^n(L, L)$ denote the Hochschild complex of L with coefficients in L. (See [7].) We note that $C^{n+1}(L, L) = E_n$. Furthermore, if $\varphi \in E_n$, a trivial computation shows that $\delta\varphi = (-1)^n D_m\varphi$, where δ denotes the Hochschild coboundary operator. Thus the Hochschild cohomology space $H^{n+1}(L, L)$ is identical with $H^n(E, D_m)$. Let $\mathfrak{a}$ be as in Section 4. It follows from Theorem 4.2 that, in this case, $H^2(A(E_1)/\mathfrak{a}, k)$ is canonically isomorphic to a vector subspace of $H^3(L, L)$. Furthermore, $H^1(A(E_1)/\mathfrak{a}, k)$ is identical to $Z^2(L, L)$. As an immediate consequence of Corollary 4.3 we have

PROPOSITION 5.1. *Let $\mathcal{M}$ be the affine algebraic set of all associative multiplications on a finite-dimensional vector space over an algebraically-closed field. Let $m \in \mathcal{M}$ and let $L = (V, m)$ be the corresponding associative algebra. If $H^3(L, L) = 0$, then m is a simple point of $\mathcal{M}$ and the tangent space of $\mathcal{M}$ at m is equal to $Z^2(L, L)$.*

This result explains, in a sense, the interpretation of elements of $H^3(L, L)$ as "obstructions to integrating infinitesimal deformations of L." See Section 5 of [4].

(2) *Lie algebras.* (See [10].) Let V be a finite-dimensional vector space over k. For each $p \geqslant -1$, let E_p be the vector space of all alternating $(p+1)$-linear maps of V into V. Set $E_p = \{0\}$ for $p < -1$. If $\varphi \in E_p$ and $\psi \in E_q$, we define $\varphi \barwedge \psi \in E_{p+q}$ by

$$\varphi \barwedge \psi(v_0, \ldots, v_{p+q}) = \sum_\sigma \operatorname{sgn}(\sigma)\, \varphi(\psi(v_{\sigma(0)}, \ldots, v_{\sigma(q)}), v_{\sigma(q+1)}, \ldots, v_{\sigma(p+q)})$$

where the sum is taken over all permutations σ of $\{0, \ldots, p+q\}$ such that $\sigma(0) < \cdots < \sigma(q)$ and $\sigma(q+1) < \cdots < \sigma(p+q)$. We set

$$[\varphi, \psi] = \varphi \barwedge \psi - (-1)^{pq}\, \psi \barwedge \varphi.$$

This product defines on $E = \bigoplus_{p\in Z} E_p$ the structure of a graded Lie algebra.

Let $D = 0$. The deformation equation becomes $\varphi \barwedge \varphi = 0$ for $\varphi \in E_1$. One checks easily that $\varphi \barwedge \varphi = 0$ if and only if φ satisfies the Jacobi identity. Thus the Lie multiplications on V are precisely the solutions of the deformation equation.

Let $L = (V, m)$ be a Lie algebra. Consider L as an L-module via the adjoint representation. As in (1), a straightforward computation shows that $H^n(E, D_m)$ is identical with the Lie algebra cohomology space $H^{n+1}(L, L)$. If $\mathfrak{a}$ is as in Section 4, then Theorem 4.2 shows that $H^2(A(E_1)/\mathfrak{a}, k)$ is canonically isomorphic to a subspace of $H^3(L, L)$. Similarly, $H^1(A(E_1)/\mathfrak{a}, k)$ is identical with $Z^2(L, L)$. Proposition 5.1 is equally valid for Lie algebras.

(3) *Commutative algebras.* Let E be as in (1). If T denotes the tensor algebra of V, then E can be canonically identified with $\operatorname{Hom}_k(T, V)$. We now define a new structure of graded associative algebra on T in which the

multiplication is given by the "shuffle" product of [*2*]. Let T^+ denote the ideal in T generated by all homogeneous elements of positive degree. Let $F = \bigoplus_n F_n$ be the graded subspace of E consisting of all $f \in E$ ($= \mathrm{Hom}(T, V)$) which vanish on $(T^+)^2$. Then a lengthy computation shows that F is a graded subalgebra of the graded Lie algebra E.

Let $D = 0$. The solutions of the deformation equation $\frac{1}{2}[m, m] = 0$ for the graded Lie algebra F are precisely the commutative and associative multiplications on V. Let $L = (V, m)$ be a commutative algebra. Consider L as an L-module in the obvious way. It follows immediately from the definitions of [*6*] that $H^n(F, D_m)$ is identical with the Harrison cohomology space $H^{n+1}(L, L)$ $(n = 0, 1, 2)$. It follows from Theorem 4.2 that $H^2(AF_1)/\mathfrak{a}, k)$ is canonically isomorphic to a vector subspace of the commutative algebra cohomology module $H^3(L, L)$ and that $H^1(A(F_1)/\mathfrak{a}, k)$ is identical with $Z^2(L, L)$. Proposition 5.1 is also valid for commutative algebras.

(4) *Homomorphisms of Lie algebras.* (See [*9*].) Let L and M be finite-dimensional Lie algebras. For each integer $p \geqslant 0$, let E_p be the vector space of all alternating p-linear maps of L into M. Set $E_p = \{0\}$ if $p < 0$ and let $E = \bigoplus_{p \in Z} E_p$. ($E$ depends only on the underlying vector spaces of L and M.) We denote the Lie algebra products on L and M by the usual bracket notation. We define a product on E, also denoted by [,], as follows: If $\varphi \in E_p$ and $\psi \in E_q$, then $[\varphi, \psi] \in E_{p+q}$ is given by

$$[\varphi, \psi](v_1, \ldots, v_{p+q}) = \sum_{\sigma} \mathrm{sgn}(\sigma)\, [\varphi(v_{\sigma(1)}, \ldots, v_{\sigma(p)}), \psi(v_{\sigma(p+1)}, \ldots, v_{\sigma(p+q)})],$$

where the sum is taken over all permutations σ of $\{1, \ldots, p + q\}$ such that $\sigma(1) < \cdots < \sigma(p)$ and $\sigma(p + 1) < \cdots < \sigma(p + q)$. This product defines on E the structure of a graded Lie algebra.

We define a homogeneous derivation $D : E \to E$ of degree 1 as follows: If $\varphi \in E_p$, then

$$D\varphi(v_1, \ldots, v_{p+1}) = \sum_{i<j} (-1)^{i+j} \varphi([v_i, v_j], v_1, \ldots, \hat{v}_i, \ldots, \hat{v}_j, \ldots, v_{p+1})$$

where the symbol ^ indicates that the argument underneath it is omitted. A trivial computation shows that a linear map $\rho : L \to M$ is a Lie algebra homomorphism if and only if ρ satisfies the deformation equation, $D\rho + \frac{1}{2}[\rho, \rho] = 0$.

Let $\rho : L \to M$ be a Lie algebra homomorphism. If ad_M denotes the adjoint representation of M, then $\mathrm{ad}_M \circ \rho$ defines a representation of L on M and thus defines M as an L-module. A straightforward computation shows that $H^n(E, D_\rho)$ is identical with the Lie algebra cohomology module $H^n(L, M)$. (Note that, in this case, there is no shift in dimension.)

Let the notation be as in Section 4 except that m is replaced by ρ. Then Theorem 4.2 implies that $H^2(A(E_1)/\mathfrak{a}, k)$ is canonically isomorphic to a vector subspace of $H^2(L, M)$ and that $H^1(A(E_1)/\mathfrak{a}, k)$ is identical with $Z^1(L, M)$. As an immediate consequence of Corollary 4.3 we have

PROPOSITION 5.2. *Let L and M be finite-dimensional Lie algebras over an algebraically-closed field and let $\mathcal{R}$ be the affine algebraic set of all homomorphisms of L into M. Let $\rho : L \to M$ be a homomorphism and consider M as an L-module via the representation $\mathrm{ad}_M \circ \rho$. If $H^2(L, M) = 0$ then ρ is a simple point of $\mathcal{R}$ and the tangent space of $\mathcal{R}$ at ρ is $Z^1(L, M)$.*

(5) *Homomorphisms of associative algebras.* (See [9].) Let L and M be associative algebras and let E_p ($p \geqslant 0$) be the vector space of all p-linear maps of L into M. Set $E_p = \{0\}$ if $p < 0$ and let $E = \bigoplus_{p \in Z} E_p$. If $\varphi \in E_p$ and $\psi \in E_q$, we define $\varphi\psi \in E_{p+q}$ by

$$(\varphi\psi)(v_1, \ldots, v_{p+q}) = \varphi(v_1, \ldots, v_p)\, \psi(v_{p+1}, \ldots, v_{p+q}).$$

We set

$$[\varphi, \psi] = \varphi\psi - (-1)^{qp}\, \psi\varphi.$$

An easy computation shows that this defines a graded Lie algebra structure on E.

Define the homogeneous derivation $D : E \to E$ of degree 1 as follows: If $\varphi \in E_n$, then

$$D\varphi(v_1, \ldots, v_{n+1}) = \sum_{i=1}^{n} (-1)^{i+1}\, \varphi(v_1, \ldots, v_i v_{i+1}, \ldots, v_{n+1}).$$

Then the solutions to the deformation equation are precisely the algebra homomorphisms of L into M. Let $\rho : L \to M$ be a homomorphism of algebras and consider M as an L-module via ρ. It now follows immediately from 4.2 and 4.3 that the results of (4) on homomorphisms of Lie algebras are also valid for homomorphisms of associative algebras.

We further remark that the obvious analogues of these theorems for homomorphisms of commutative algebras are valid.

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