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Characterization of balanced coherent configurations

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ABSTRACT

Let G be a group acting on a finite set Ω. Then G acts on $\Omega \times \Omega$ by its entry-wise action and its orbits form the basis relations of a coherent configuration (or shortly scheme). Our concern is to consider what follows from the assumption that the number of orbits of G on $\Omega_i \times \Omega_j$ is constant whenever Ω_i and Ω_j are orbits of G on Ω. One can conclude from the assumption that the actions of G on Ω_i's have the same permutation character and are not necessarily equivalent. From this viewpoint one may ask how many inequivalent actions of a given group with the same permutation character there exist. In this article we will approach to this question by a purely combinatorial method in terms of schemes and investigate the following topics: (i) balanced schemes and their central primitive idempotents, (ii) characterization of reduced balanced schemes.

1. Introduction

Let G be a group acting on a finite set Ω with its orbits $\Omega_1, \ldots, \Omega_n$ and its permutation character $\pi = \sum_{i=1}^{n} \pi_i$ where $\pi_i(g) := |\{\alpha \in \Omega_i \mid \alpha^g = \alpha\}|$ for $g \in G$. One may think what happens if $\pi_i = \pi_j$ for all $1 \leqslant i, j \leqslant n$ and can say that the number of orbits of G on $\Omega_i \times \Omega_j$ by its entry-wise action is constant for all $1 \leqslant i, j \leqslant n$, which motivates us to define the following concepts whose terminology is due to [3].

Definition 1.1. Let V be a finite set and $\mathcal{R}$ a set of non-empty binary relations on V. The pair $\mathcal{C} = (V, \mathcal{R})$ is called a *coherent configuration* (for short *scheme*) on V if the following conditions hold:

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(C1) $\mathcal{R}$ forms a partition of the set $V \times V$.
(C2) $\Delta_V := \{(v, v) \mid v \in V\}$ is a union of certain relations from $\mathcal{R}$.
(C3) For every $R \in \mathcal{R}$, $R^t := \{(v, u) \mid (u, v) \in R\} \in \mathcal{R}$.
(C4) For every $R, S, T \in \mathcal{R}$, the size of $\{w \in V \mid (u, w) \in R,\ (w, v) \in S\}$ does not depend on the choice of $(u, v) \in T$ and is denoted by c_{RS}^T.

We say that the elements of V are *points* and those of $\mathcal{R}$ are *basis relations.*

Let $\mathcal{C} = (V, \mathcal{R})$ be a scheme and $\emptyset \neq X \subseteq V$. We say that X is a *fiber* of $\mathcal{C}$ if $\Delta_X = \{(x, x) \mid x \in X\} \in \mathcal{R}$. We denote by $\mathrm{Fib}(\mathcal{C})$ the set of all fibers of $\mathcal{C}$.

Definition 1.2. Let m, n and r be positive integers. We say that a scheme $\mathcal{C}$ is an (m, n, r)*-scheme* if the following conditions hold:

(i) $|\{R \in \mathcal{R} \mid R \subseteq X \times Y\}| = r$ for all $X, Y \in \mathrm{Fib}(\mathcal{C})$.
(ii) $|X| = m$ for all $X \in \mathrm{Fib}(\mathcal{C})$.
(iii) $|\mathrm{Fib}(\mathcal{C})| = n$.

A scheme $\mathcal{C}$ is called r*-balanced* if (i) holds, and *balanced* if it is r-balanced for some r. In Section 3 we will show that (i) implies (ii).

Example 1.3. (See [7, Section 12, p. 31].) Let $(X, \mathcal{B}, \mathcal{I})$ be a symmetric design with the set X of points, the set $\mathcal{B}$ of blocks and the incidence relation $\mathcal{I} \subseteq X \times \mathcal{B}$. Set $V = X \cup \mathcal{B}$ (disjoint union) and define the relations R_i $(i = 1, \ldots, 8)$ on V as follows.

$$R_1 = \Delta_X, \qquad R_2 = \Delta_{\mathcal{B}}, \qquad R_3 = (X \times X) \setminus \Delta_X, \qquad R_4 = (\mathcal{B} \times \mathcal{B}) \setminus \Delta_{\mathcal{B}},$$
$$R_5 = \mathcal{I}, \qquad R_6 = R_5^t, \qquad R_7 = (X \times \mathcal{B}) \setminus \mathcal{I}, \qquad R_8 = R_7^t.$$

It is known that $(V, \{R_i\}_{i=1}^8)$ is an $(m, 2, 2)$-scheme where $m = |X|$.

Let us return to the topic in the first paragraph. Note that the orbits of G on $\Omega \times \Omega$ form the basis relations of a scheme called the 2*-orbit scheme* of G on Ω and its fibers are $\Omega_1, \ldots, \Omega_n$. It is straightforward to check that $\pi_i \pi_j$ coincides with the permutation character of G on $X_i \times X_j$ for all $1 \leqslant i, j \leqslant n$. It is known that the number of orbits of G on $X_i \times X_j$ is equal to $[\pi_i \pi_j, 1_G]$[3], which coincides with $[\pi_i, \pi_j]$, since π_i's are real valued. Therefore, $\pi_i = \pi_j$ for all $1 \leqslant i, j \leqslant n$, if and only if the 2-orbit scheme of G on Ω is balanced.

We denote by $\mathcal{P}(\mathcal{C})$ the set of all central primitive idempotents of the adjacency algebra of $\mathcal{C}$ (see Section 2 for details). The following theorem shows a characterization of balanced schemes in terms of their central primitive idempotents.

Theorem 1.1. *Let $\mathcal{C}$ be a scheme. Then $\mathcal{C}$ is balanced if and only if for each $X \in \mathrm{Fib}(\mathcal{C})$ the mapping $\mathcal{P}(\mathcal{C}) \to \mathcal{P}(\mathcal{C}_X)$ $(P \mapsto P_X)$ is bijective with $n_P = |\mathrm{Fib}(\mathcal{C})| n_{P_X}$.*

One may conclude that $|\mathcal{P}(\mathcal{C})| = r$ if $\mathcal{C}$ is r-balanced and $r \leqslant 5$ (see Corollary 3.1). The following theorem deals with the converse argument for $r = 1, 2$.

Theorem 1.2. *Let $\mathcal{C} = (V, \mathcal{R})$ be a scheme. Then the following hold*:

(i) $|\mathcal{P}(\mathcal{C})| = 1$ *if and only if $\mathcal{C}$ is* 1*-balanced.*
(ii) $|\mathcal{P}(\mathcal{C})| = 2$ *if and only if $\mathcal{C}$ is* 2*-balanced or $\mathcal{C} = \mathcal{C}_1 \boxplus \mathcal{C}_2$ where $\mathcal{C}_i$ is i-balanced.*

[3] Here by 1_G and [,] we mean the principal character of G and the inner product of characters, respectively.

We have the following constructions of balanced schemes (see Sections 3, 4 for the details):

(i) Let U be a union of fibers of $\mathcal{C}$. Then the restriction of $\mathcal{C}$ to U is r-balanced if $\mathcal{C}$ is r-balanced.
(ii) If $\mathcal{C}_i$ $(i=1,2)$ is an (m_i,n_i,r_i)-scheme, then $\mathcal{C}_1\otimes\mathcal{C}_2$ is an (m_1m_2,n_1n_2,r_1r_2)-scheme.

We say that a balanced scheme $\mathcal{C}$ is *reduced* if there exist no $X,Y\in \mathrm{Fib}(\mathcal{C})$ such that $\mathcal{C}_{X\cup Y}\simeq \mathcal{C}_X\otimes\mathcal{T}_2$ where $\mathcal{T}_2$ is a $(1,2,1)$-scheme (in Section 4 you will see another equivalent condition for a scheme to be reduced). Any r-balanced scheme is obtained by the restriction of the tensor product of a reduced r-balanced scheme and a 1-balanced scheme (see Theorem 3.5). Now we focus our attention on reduced balanced schemes. It seems a quite difficult problem to find possible n such that there exists a reduced (m,n,r)-scheme for given m and r. Actually, D.G. Higman asked if there exists a reduced $(m,3,3)$-scheme for some m (see [8, Section 8, p. 229]). Furthermore, H. Wielandt conjectured that a transitive permutation group of prime degree p has at most two inequivalent transitive representations of degree p (see [1]), though it can be solved by the classification of finite simple groups.

Theorem 1.3. *Let $\mathcal{C}$ be a reduced (m,n,r)-scheme and p a prime. Then we have the following*:

(i) *If $m<2r$, then $n=1$.*
(ii) *If $p\nmid m$ and $\mathcal{C}_X$ is p-valanced*[4] *for some $X\in\mathrm{Fib}(\mathcal{C})$, then $n=1$.*

The preceding theorem is applied to characterize (m,n,r)-schemes up to $m\leqslant 11$ as follows.

Theorem 1.4. *Let m,n,r be positive integers and $m\leqslant 11$. Then a reduced (m,n,r)-scheme can exist only if $n\leqslant 2$.*

Let us show the organization of this article. In Section 2 we prepare some terminologies related to schemes. Section 3 is devoted to balanced schemes. First we investigate the features of balanced schemes. Indeed, we shall characterize a balanced scheme in terms of its central primitive idempotents and we prove Theorem 1.1. Secondly we shall characterize schemes with at most two central primitive idempotents and we prove Theorem 1.2. In Section 4 we shall extend the notion of inequivalent permutation representations to schemes. Namely, we shall define reduced (m,n,r)-schemes and then introduce some examples and known constructions of them to support our theory. Finally in Section 5, first we prove Theorem 1.3, secondly we shall enumerate reduced (m,n,r)-schemes for $m\leqslant 11$ in order to prove Theorem 1.4.

2. Preliminaries

According to [3] we prepare some terminologies related to schemes. For the remainder of this section we assume that $\mathcal{C}=(V,\mathcal{R})$ is a scheme. One can see that $V=\bigcup_{X\in\mathrm{Fib}(\mathcal{C})}X$ (disjoint union) and

$$\mathcal{R}=\bigcup_{X,Y\in\mathrm{Fib}(\mathcal{C})}\mathcal{R}_{X,Y}\quad\text{(disjoint union)},\tag{1}$$

where $\mathcal{R}_{X,Y}:=\{R\in\mathcal{R}\mid R\subseteq X\times Y\}$. We shall denote $\mathcal{R}_{X,X}$ by $\mathcal{R}_X$.

Let $X,Y\in\mathrm{Fib}(\mathcal{C})$ and R be a non-empty union of basis relations in $\mathcal{R}_{X,Y}$. For $(x,y)\in R$ we set $R_{out}(x)=\{u\mid (x,u)\in R\}$ and $R_{in}(y)=\{v\mid (v,y)\in R\}$. The size of $R_{out}(x)$ and that of $R_{in}(y)$ does not depend on the choice of $x\in X$ and $y\in Y$, respectively; so we shall denote them by d_R and e_R, respectively. It is easy to see that

$$|X|d_R=|R|=|Y|e_R.\tag{2}$$

[4] See Section 2 for the definition.

We define the multi-set $d_{X,Y} := \{d_R \mid R \in \mathcal{R}_{X,Y}\}$. For $\mathcal{D} \subseteq \mathcal{R}$ we define $d_{\mathcal{D}} := \sum_{R\in\mathcal{D}} d_R$ as well as $e_{\mathcal{D}} := \sum_{R\in\mathcal{D}} e_R$. For instance $d_{\mathcal{R}_{X,Y}} = |Y|$ and $e_{\mathcal{R}_{X,Y}} = |X|$.

Note that $d_R = e_R$ for each $R \in \mathcal{R}$ if and only if $|X| = |Y|$ for all $X, Y \in \mathrm{Fib}(\mathcal{C})$. A scheme $\mathcal{C}$ is called *half-homogeneous* if the latter condition holds. If $\mathcal{C}$ is a half-homogeneous scheme, then d_R $(= e_R)$ is called the *degree* or the *valency* of R. Given a prime p a half-homogeneous scheme $\mathcal{C}$ is called *p-valenced* if the degree of each basis relation of $\mathcal{C}$ is a power of p.

A basis relation $R \in \mathcal{R}$ is called *thin* if $d_R = e_R = 1$ and a scheme $\mathcal{C}$ is called a *homogeneous scheme* or (*association scheme*) if $|\mathrm{Fib}(\mathcal{C})| = 1$ or equivalently, if $\Delta_V \in \mathcal{R}$ (for more details regarding association schemes we refer to [13]). Given $X \in \mathrm{Fib}(\mathcal{C})$ the pair $\mathcal{C}_X = (X, \mathcal{R}_X)$ is a homogeneous scheme called the *homogeneous component* of $\mathcal{C}$ corresponding to X.

For each $R \in \mathcal{R}$ we define a $\{0,1\}$-matrix A_R whose rows and columns are simultaneously indexed by the elements of V such that the (u,v)-entry of A_R is one if and only if $(u,v) \in R$. Then A_R is called the *adjacency matrix* of R. Note that the subspace of $\mathrm{Mat}_V(\mathbb{C})$ spanned by $\{A_R \mid R \in \mathcal{R}\}$ is a subalgebra called the *adjacency algebra* of $\mathcal{C}$ and denoted by $\mathcal{A}(\mathcal{C})$. Obviously,

(C′1) $\mathcal{A}(\mathcal{C})$ contains the identity matrix I_V and the all-one matrix J_V.
(C′2) $A_{R^t} = A_R^t$ for every $R \in \mathcal{R}$ where A_R^t is the transpose of A_R.
(C′3) For every $R, S \in \mathcal{R}$, $A_R A_S = \sum_{T\in\mathcal{R}} c_{RS}^T A_T$.

A scheme is called *trivial* if all its fibers are singletons. We denote a trivial scheme on n points by $\mathcal{T}_n$. Note that $\mathcal{A}(\mathcal{T}_n) \cong \mathrm{Mat}_n(\mathbb{C})$ and it is easy to see that a scheme is trivial if and only if it is 1-balanced.

By $\mathrm{Fib}^*(\mathcal{C})$ we mean the set of all non-empty unions of fibers of $\mathcal{C}$. Given $U \in \mathrm{Fib}^*(\mathcal{C})$ we set $\mathcal{R}_U := \{R_U \mid R \in \mathcal{R}\}$ where $R_U = R \cap (U \times U)$. Then the pair $\mathcal{C}_U = (U, \mathcal{R}_U)$ is a scheme on U called the *restriction* of $\mathcal{C}$ to U. Note that $\mathcal{C}_U$ is homogeneous whenever $U \in \mathrm{Fib}(\mathcal{C})$.

Given $U, U' \in \mathrm{Fib}^*(\mathcal{C})$ we define $\mathcal{A}_{U,U'}$ to be the subspace of $\mathcal{A}$ spanned by the set $\{A_R \mid R \in \mathcal{R}, R \subseteq U \times U'\}$.

A basis relation S of $\mathcal{C}$ is called *symmetric* if $S^t = S$ and $\mathcal{C}$ is called *symmetric* if each basis relation of $\mathcal{C}$ is symmetric; and $\mathcal{C}$ is called *commutative* if $c_{RS}^T = c_{SR}^T$ for all $R, S, T \in \mathcal{R}$. This is equivalent to $A_R A_S = A_S A_R$ for all $R, S \in \mathcal{R}$. It is known that symmetric schemes are commutative and that the converse does not hold. Furthermore, one can see that a commutative scheme is a homogeneous one.

Lemma 2.1. *(See [7, (4.2)], [13, Theorem 4.5.1].) If $\mathcal{C} = (V, \mathcal{R})$ is a homogeneous scheme and $|\mathcal{R}| \leqslant 5$, then $\mathcal{C}$ is commutative.*

Given $R, S \in \mathcal{R}$ the *complex product* of them is defined to be $RS = \{T \in \mathcal{R} \mid c_{RS}^T > 0\}$ and the *relational product* $R \circ S$ is defined as follows.

$$R \circ S := \{(u,v) \mid \exists w \in V;\ (u,w) \in R,\ (w,v) \in S\}.$$

Note that $R \circ S = \bigcup_{T\in RS} T$ and $d_{R\circ S} = d_{RS}$.

Lemma 2.2. *Let $\mathcal{C}$ be a scheme and $X, Y, Z \in \mathrm{Fib}(\mathcal{C})$. Then for all $R \in \mathcal{R}_{X,Y}$, $S \in \mathcal{R}_{Y,Z}$ and $T \in \mathcal{R}_{X,Z}$ the following hold*:

(i) $d_R d_S = \sum_{T\in\mathcal{R}_{X,Z}} c_{RS}^T d_T$.
(ii) $c_{RS}^T d_T = c_{TS^t}^R d_R = c_{R^tT}^S d_S$ *and* $\mathrm{lcm}(d_R, d_S) \mid c_{RS}^T d_T$.
(iii) $d_R = \sum_{S\in\mathcal{R}_{Y,Z}} c_{RS}^T$, $e_R = \sum_{S\in\mathcal{R}_{Y,Z}} c_{R^tT}^S$, $c_{RS}^T \leqslant \min\{d_R, e_S\}$ *and* $R\mathcal{R}_{Y,Z} = \mathcal{R}_{X,Z}$.
(iv) $d_R \delta_{SR^t} = c_{RS}^{\Delta_X}$ *and* $e_R \delta_{SR^t} = c_{SR}^{\Delta_Y}$ *where δ denotes the Kronecker's delta.*
(v) $d_S \leqslant d_{RS} \leqslant d_R d_S$ *and* $e_R \leqslant e_{RS} \leqslant e_R e_S$.
(vi) *If $d_R = 2$, then $RR^t = \{\Delta_X, S\}$ where $S \in \mathcal{R}_X$ is symmetric with $d_S \leqslant 2$.*
(vi) $|RS| \leqslant \gcd(d_R, d_S)$.

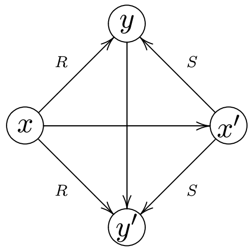

Fig. 1.

Proof. The proof is done by the same procedure as [13, Lemma 1.4.2, 1.4.3, 1.5.2, 1.5.6]. □

Lemma 2.3. *Let $S \in \mathcal{R}_{X,Y}$ and $L_S := \{R \in \mathcal{R}_X \mid RS = \{S\}\}$. Then*

$$d_{L_S} \mid \gcd\bigl(|X|, e_S\bigr).$$

Proof. Let $y \in Y$ and $x \in S_{in}(y)$. The condition $RS = \{S\}$ shows that $\bigcup_{R \in L_S} R_{in}(x) \subseteq S_{in}(y)$ and $\bigcup_{R \in L_S} R$ is an equivalence relation on X. Since $y \in Y$ and $x \in S_{in}(y)$ are arbitrarily taken, all equivalence classes have the same size d_{L_S}. It follows that d_{L_S} divides both d_S and $|X|$. □

Lemma 2.4. *Let $X, Y \in \mathrm{Fib}(\mathcal{C})$ with $X \neq Y$ and $R, S \in \mathcal{R}_{X,Y}$ with $R \neq S$. Then $T \in R^t R \cap S^t S$ for some $T \in \mathcal{R}_Y$ with $T \neq \Delta_Y$ if and only if $c_{RS^t}^{T'} \geqslant 2$ for some $T' \in \mathcal{R}_X$.*

Proof. Let us prove the necessity. By the assumption $c_{R^t R}^{T} \neq 0$ and $c_{S^t S}^{T} \neq 0$. Taking $(y, y') \in T$ (of course $y \neq y'$) there exist $x, x' \in X$ such that $(x, y), (x, y') \in R$ and $(x', y), (x', y') \in S$. On the other hand, there exists $T' \in \mathcal{R}_X$ such that $(x, x') \in T'$. It follows that $c_{RS^t}^{T'} \geqslant 2$ (see Fig. 1). Sufficiency follows from Fig. 1 since $c_{RS^t}^{T'} \geqslant 2$ implies that $y \neq y'$. □

Let $U, U' \in \mathrm{Fib}^*(\mathcal{C})$ such that $U \cap U' = \emptyset$ and $V = U \cup U'$. Then we say that $\mathcal{C}$ is the *internal direct sum* of $\mathcal{C}_U$ and $\mathcal{C}_{U'}$ if $|\mathcal{R}_{X,Y}| = 1$ for all $X, Y \in \mathrm{Fib}(\mathcal{C})$ with $X \subseteq U$ and $Y \subseteq U'$. In this case we shall write $\mathcal{C} = \mathcal{C}_U \boxplus \mathcal{C}_{U'}$.

Let $\mathcal{C}_i = (V_i, \mathcal{R}_i)$ $(i = 1, 2)$ be schemes. We set

$$\mathcal{R}_1 \otimes \mathcal{R}_2 = \{R_1 \otimes R_2 \mid R_1 \in \mathcal{R}_1,\ R_2 \in \mathcal{R}_2\},$$

where $R_1 \otimes R_2 = \{((u_1, u_2), (v_1, v_2)) \mid (u_1, v_1) \in R_1,\ (u_2, v_2) \in R_2\}$. Then $\mathcal{C} = (V_1 \times V_2, \mathcal{R}_1 \otimes \mathcal{R}_2)$ is a scheme called the *tensor product* of $\mathcal{C}_1$ and $\mathcal{C}_2$ and denoted by $\mathcal{C}_1 \otimes \mathcal{C}_2$. One can see that $\mathrm{Fib}(\mathcal{C}) = \mathrm{Fib}(\mathcal{C}_1) \times \mathrm{Fib}(\mathcal{C}_2)$.

An *isomorphism* from $\mathcal{C}_1$ to $\mathcal{C}_2$ is defined to be a bijection $\psi : V_1 \cup \mathcal{R}_1 \to V_2 \cup \mathcal{R}_2$ such that for all $u, v \in V_1$ and $R \in \mathcal{R}_1$, $(u, v) \in R$ if and only if $(\psi(u), \psi(v)) \in \psi(R)$. We say that $\mathcal{C}_1$ is *isomorphic* to $\mathcal{C}_2$ and denote it by $\mathcal{C}_1 \simeq \mathcal{C}_2$ if there exists an isomorphism from $\mathcal{C}_1$ to $\mathcal{C}_2$.

Let $\mathcal{A}$ be the adjacency algebra of $\mathcal{C}$. Since $\mathcal{A}$ is closed under the complex conjugate transpose map, $\mathcal{A}$ is semisimple. By the Wedderburn theorem $\mathcal{A}$ is isomorphic to a direct sum of full matrix algebras over $\mathbb{C}$:

$$\mathcal{A} = \bigoplus_{P \in \mathcal{P}(\mathcal{C})} \mathcal{A}P \cong \bigoplus_{P \in \mathcal{P}(\mathcal{C})} \mathrm{Mat}_{n_P}(\mathbb{C}), \tag{3}$$

where n_P is a positive integer and $\mathrm{Mat}_{n_P}(\mathbb{C})$ is the full matrix algebra of complex $n_P \times n_P$ matrices. A comparison of dimensions of the left- and right-hand sides of (3) shows that

$$|\mathcal{R}| = \sum_{P \in \mathcal{P}(\mathcal{C})} n_P^2. \tag{4}$$

Obviously $\mathcal{C}$ is commutative if and only if $n_P = 1$ for each $P \in \mathcal{P}(\mathcal{C})$, since $\mathcal{P}(\mathcal{C})$ is a basis of the center of $\mathcal{A}(\mathcal{C})$. For each $P \in \mathcal{P}(\mathcal{C})$ we set $m_P := \mathrm{rank}(P)/n_P$. Then

$$|V| = \sum_{P \in \mathcal{P}(\mathcal{C})} m_P n_P. \tag{5}$$

The numbers m_P and n_P are called the *multiplicity* and the *degree* of P. Set $P_0 = \sum_X J_X/|X|$ where X runs over $\mathrm{Fib}(\mathcal{C})$ and $J_X = \sum_{R \in \mathcal{R}_X} A_R$. Then $P_0 \in \mathcal{P}(\mathcal{C})$, which is called *principal*. It is known that

$$(m_{P_0}, n_{P_0}) = \bigl(1, \bigl|\mathrm{Fib}(\mathcal{C})\bigr|\bigr). \tag{6}$$

Below for $X \in \mathrm{Fib}^*(\mathcal{C})$ and $P \in \mathcal{P}(\mathcal{C})$ put $P_X = PI_X$ and set

$$\mathcal{P}_X(\mathcal{C}) = \bigl\{P \in \mathcal{P}(\mathcal{C}) \bigm| P_X \neq 0\bigr\} \quad \text{and} \quad \mathrm{Supp}(P) = \bigl\{X \in \mathrm{Fib}(\mathcal{C}) \bigm| P_X \neq 0\bigr\}.$$

Theorem 2.5. *(See [4, Proposition 2.1].) Let $\mathcal{C} = (V, \mathcal{R})$ be a scheme. Then the following hold*:

(i) *For each $X \in \mathrm{Fib}^*(\mathcal{C})$ the mapping $P \mapsto P_X$ induces a bijection between $\mathcal{P}_X(\mathcal{C})$ and $\mathcal{P}(\mathcal{C}_X)$.*
(ii) *For all $P \in \mathcal{P}(\mathcal{C})$ and $X \in \mathrm{Supp}(P)$, $n_P = \sum_{X \in \mathrm{Supp}(P)} n_{P_X}$ and $m_P = m_{P_X}$.*

Lemma 2.6. *Let $\mathcal{C} = (V, \mathcal{R})$ be a scheme. Then the following hold*:

(i) *$\mathcal{P}(\mathcal{C}) = \mathcal{P}_X(\mathcal{C})$ for each $X \in \mathrm{Fib}(\mathcal{C})$ if and only if $\mathrm{Supp}(P) = \mathrm{Fib}(\mathcal{C})$ for each $P \in \mathcal{P}(\mathcal{C})$.*
(ii) *$\mathrm{Supp}(P) \neq \emptyset$ for each $P \in \mathcal{P}(\mathcal{C})$, and*

$$\mathcal{P}(\mathcal{C}) = \bigcup_{X \in \mathrm{Fib}(\mathcal{C})} \mathcal{P}_X(\mathcal{C}). \tag{7}$$

Besides, $\mathcal{P}(\mathcal{C}) = \mathcal{P}_U(\mathcal{C}) \cup \mathcal{P}_{U'}(\mathcal{C})$ where $U, U' \in \mathrm{Fib}^*(\mathcal{C})$ with $U \cap U' = \emptyset$ and $V = U \cup U'$.

Proof. (i) Let $X \in \mathrm{Fib}(\mathcal{C})$ and $P \in \mathcal{P}(\mathcal{C})$. Then $P \in \mathcal{P}_X(\mathcal{C})$ if and only if $X \in \mathrm{Supp}(P)$. This completes the proof.

(ii) Let $P \in \mathcal{P}(\mathcal{C})$ such that $\mathrm{Supp}(P) = \emptyset$. Then for all $X \in \mathrm{Fib}(\mathcal{C})$, $PI_X = 0$ and then $P = PI_V = \sum_{X \in \mathrm{Fib}(\mathcal{C})} PI_X = 0$, a contradiction. Therefore, $\mathrm{Supp}(P) \neq \emptyset$. Let $P \in \mathcal{P}(\mathcal{C})$, as $\mathrm{Supp}(P) \neq \emptyset$, there exists $X \in \mathrm{Fib}(\mathcal{C})$ such that $PI_X \neq 0$. This means that $P \in \mathcal{P}_X(\mathcal{C})$ and the proof of (7) is completed.

Let $P \in \mathcal{P}(\mathcal{C})$. Then $P \in \mathcal{P}_X(\mathcal{C})$ for some $X \in \mathrm{Fib}(\mathcal{C})$. Since $V = U \cup U'$, $X \subseteq U$ or $X \subseteq U'$. It follows that $P \in \mathcal{P}_U(\mathcal{C})$ or $P \in \mathcal{P}_{U'}(\mathcal{C})$. This completes the proof. □

Proposition 2.7. *(See [8, p. 223], [7, p. 22 (8.1)].) Let $\mathcal{C} = (V, \mathcal{R})$ be a scheme. Then the following hold*:

(i) *Let $X, Y \in \mathrm{Fib}^*(\mathcal{C})$ such that $X \cap Y = \emptyset$ and $V = X \cup Y$. Then*

$$\dim_{\mathbb{C}}(\mathcal{A}_{X,Y}) = \sum_{P \in \mathcal{P}_X(\mathcal{C}) \cap \mathcal{P}_Y(\mathcal{C})} n_{P_X} n_{P_Y}.$$

(ii) *For all $X, Y \in \mathrm{Fib}(\mathcal{C})$, $|\mathcal{R}_{X,Y}| = \sum_{P \in \mathcal{P}_X(\mathcal{C}) \cap \mathcal{P}_Y(\mathcal{C})} n_{P_X} n_{P_Y}$.*

Lemma 2.8. *Let $\mathcal{C} = (V, \mathcal{R})$ be a scheme with the adjacency algebra $\mathcal{A}(\mathcal{C})$. If $U, U' \in \mathrm{Fib}^*(\mathcal{C})$ such that $U \cap U' = \emptyset$, then*

$$\left|\mathrm{Fib}(\mathcal{C}_U)\right|\left|\mathrm{Fib}(\mathcal{C}_{U'})\right| \leqslant \dim_{\mathbb{C}}(\mathcal{A}_{U,U'}). \tag{8}$$

Furthermore, the equality holds if and only if $\mathcal{C}_{U\cup U'} = \mathcal{C}_U \boxplus \mathcal{C}_{U'}$.

Proof. The proof is a direct consequence of Proposition 2.7(i) and the definition of direct sum. □

Lemma 2.9. *Let $\mathcal{C} = (V, \mathcal{R})$ be a scheme with the principal idempotent P_0 and let $U, U' \in \mathrm{Fib}^*(\mathcal{C})$ such that $U \cap U' = \emptyset$ and $V = U \cup U'$. Then $\mathcal{C} = \mathcal{C}_U \boxplus \mathcal{C}_{U'}$ if and only if $\mathcal{P}_U(\mathcal{C}) \cap \mathcal{P}_{U'}(\mathcal{C}) = \{P_0\}$.*

Proof. Let us prove the sufficiency first. It is clear that $P_0 \in \mathcal{P}_U(\mathcal{C}) \cap \mathcal{P}_{U'}(\mathcal{C})$. By Lemma 2.8 and Proposition 2.7(i) we have

$$\left|\mathrm{Fib}(\mathcal{C}_U)\right|\left|\mathrm{Fib}(\mathcal{C}_{U'})\right| = \sum_{P\in\mathcal{P}_U(\mathcal{C})\cap\mathcal{P}_{U'}(\mathcal{C})} n_{P_U} n_{P_{U'}}.$$

Since $n_{P_{0U}} = |\mathrm{Fib}(\mathcal{C}_U)|$ and $n_{P_{0U'}} = |\mathrm{Fib}(\mathcal{C}_{U'})|$, it follows that

$$\mathcal{P}_U(\mathcal{C}) \cap \mathcal{P}_{U'}(\mathcal{C}) = \{P_0\}. \tag{9}$$

Conversely, if $\mathcal{P}_U(\mathcal{C}) \cap \mathcal{P}_{U'}(\mathcal{C}) = \{P_0\}$, then by Proposition 2.7(i),

$$\dim_{\mathbb{C}}(\mathcal{A}_{U,U'}) = \left|\mathrm{Fib}(\mathcal{C}_U)\right|\left|\mathrm{Fib}(\mathcal{C}_{U'})\right|.$$

It follows from Lemma 2.8 that $\mathcal{C} = \mathcal{C}_U \boxplus \mathcal{C}_{U'}$. □

3. Characterization of balanced schemes

Proof of Theorem 1.1. First we prove the necessity. Let $X, Y \in \mathrm{Fib}(\mathcal{C})$. By Proposition 2.7, $|\mathcal{R}_{X,Y}| = \sum_{P\in\mathcal{P}_X(\mathcal{C})\cap\mathcal{P}_Y(\mathcal{C})} n_{P_X} n_{P_Y}$. By the Cauchy–Schwarz inequality we have

$$\begin{aligned}|\mathcal{R}_{X,Y}|^2 = \left(\sum_{P\in\mathcal{P}_X(\mathcal{C})\cap\mathcal{P}_Y(\mathcal{C})} n_{P_X} n_{P_Y}\right)^2 &\leqslant \sum_{P\in\mathcal{P}_X(\mathcal{C})\cap\mathcal{P}_Y(\mathcal{C})} n_{P_X}^2 \sum_{P\in\mathcal{P}_X(\mathcal{C})\cap\mathcal{P}_Y(\mathcal{C})} n_{P_Y}^2 \\ &\leqslant \sum_{P\in\mathcal{P}_X(\mathcal{C})} n_{P_X}^2 \sum_{P\in\mathcal{P}_Y(\mathcal{C})} n_{P_Y}^2 \\ &= |\mathcal{R}_X||\mathcal{R}_Y| = |\mathcal{R}_{X,Y}|^2.\end{aligned}$$

This implies that

$$\left(\sum_{P\in\mathcal{P}_X(\mathcal{C})\cap\mathcal{P}_Y(\mathcal{C})} n_{P_X} n_{P_Y}\right)^2 = \sum_{P\in\mathcal{P}_X(\mathcal{C})} n_{P_X}^2 \sum_{P\in\mathcal{P}_Y(\mathcal{C})} n_{P_Y}^2.$$

It follows that $\mathcal{P}_X(\mathcal{C}) = \mathcal{P}_Y(\mathcal{C})$ and thus applying Lemma 2.6(i) we have $\mathcal{P}(\mathcal{C}) = \mathcal{P}_X(\mathcal{C})$. Consequently, the mapping $\mathcal{P}(\mathcal{C}) \to \mathcal{P}(\mathcal{C}_X)$ $(P \mapsto P_X)$ is well defined and bijective by Theorem 2.5. Since the equality holds in the Cauchy–Schwarz inequality, we have $\langle n_{P_X} \mid P \in \mathcal{P}(\mathcal{C})\rangle = \alpha\langle n_{P_Y} \mid P \in \mathcal{P}(\mathcal{C})\rangle$.

However, $\alpha = 1$ since $|\mathcal{R}_X| = |\mathcal{R}_Y|$. Hence, $n_{P_X} = n_{P_Y}$ for all $P \in \mathcal{P}(\mathcal{C})$. Therefore, by Theorem 2.5 and Lemma 2.6(ii),

$$n_P = \sum_{X \in \mathrm{Supp}(P)} n_{P_X} = \sum_{X \in \mathrm{Fib}(\mathcal{C})} n_{P_X} = \left|\mathrm{Fib}(\mathcal{C})\right| n_{P_X}.$$

Now let us prove the sufficiency. Given $X, Y \in \mathrm{Fib}(\mathcal{C})$ the assumption along with Theorem 2.5 assert that $\mathcal{P}(\mathcal{C}) = \mathcal{P}_X(\mathcal{C}) = \mathcal{P}_Y(\mathcal{C})$ and $n_{P_X} = n_{P_Y}$ for each $P \in \mathcal{P}(\mathcal{C})$. On the other hand, by Proposition 2.7(ii), we have

$$|\mathcal{R}_{X,Y}| = \sum_{P \in \mathcal{P}_X(\mathcal{C}) \cap \mathcal{P}_Y(\mathcal{C})} n_{P_X} n_{P_Y} = \sum_{P \in \mathcal{P}(\mathcal{C})} n_{P_X}^2 = |\mathcal{R}_X|.$$

Hence, $\mathcal{C}$ is balanced. □

Corollary 3.1. *Let $\mathcal{C}$ be an r-balanced scheme. If $\mathcal{C}_X$ is commutative for some $X \in \mathrm{Fib}(\mathcal{C})$, then so is $\mathcal{C}_X$ for all $X \in \mathrm{Fib}(\mathcal{C})$, and $|\mathcal{P}(\mathcal{C})| = r$. In particular, the latter holds whenever $r \leqslant 5$.*

Proof. Let $X \in \mathrm{Fib}(\mathcal{C})$. Since $\mathcal{C}_X$ is commutative, $|\mathcal{P}(\mathcal{C}_X)| = |\mathcal{R}_X| = r$. By Theorem 1.1, $|\mathcal{P}(\mathcal{C})| = |\mathcal{P}(\mathcal{C}_X)| = r$. In particular, if $r \leqslant 5$, then by Lemma 2.1, $\mathcal{C}_X$ is commutative and thus $|\mathcal{P}(\mathcal{C})| = r$. □

Proof of Theorem 1.2(i). Let $X \in \mathrm{Fib}(\mathcal{C})$. By Theorem 2.5, $|\mathcal{P}(\mathcal{C}_X)| = 1$. On the other hand, $\mathcal{C}_X = (X, \mathcal{R}_X)$ is a homogeneous scheme, so $|X| = m_{P_{0X}} n_{P_{0X}} = 1$, by (6). Hence, every fiber of $\mathcal{C}$ is a singleton and thus $\mathcal{C}$ is trivial. Conversely, the adjacency algebra of a trivial scheme is the full matrix algebra and thus it has only one central primitive idempotent. □

In order to prove Theorem 1.2(ii), we need the following theorem.

Theorem 3.2. *Let $\mathcal{C} = (V, \mathcal{R})$ be a scheme. If $\mathcal{C}$ is homogeneous, then $|\mathcal{P}(\mathcal{C})| = 2$ if and only if $|\mathcal{R}| = 2$. If $\mathcal{C}$ is not homogeneous and $\mathcal{P}(\mathcal{C}) = \{P_0, P_1\}$ with $P_0 \neq P_1$, then the following hold:*

(i) *$X \notin \mathrm{Supp}(P_1)$ if and only if $|X| = 1$.*

(ii) $|\mathcal{R}_X| = \begin{cases} 2 & \text{if } X \in \mathrm{Supp}(P_1), \\ 1 & \text{if } X \notin \mathrm{Supp}(P_1). \end{cases}$

(iii) *$|\mathcal{R}_{X,Y}| = 2$ for all $X, Y \in \mathrm{Supp}(P_1)$.*

(iv) *$n_{P_1} = |\mathrm{Supp}(P_1)|$ and $|X| = 1 + m_{P_1}$ for each $X \in \mathrm{Supp}(P_1)$.*

Proof. For the first part we refer to [7, (4.2)].

(i) Since $I_V = P_0 + P_1$, $P_1 = \sum_{X \in \mathrm{Fib}(\mathcal{C})} (I_X - J_X/|X|)$. Let $X \in \mathrm{Fib}(\mathcal{C})$. Then $X \notin \mathrm{Supp}(P_1)$ if and only if $0 = P_1 I_X = I_X - J_X/|X|$ if and only if $|X| = 1$.

(ii) If $X \in \mathrm{Supp}(P_1)$, then $P_1 I_X \neq 0$ and by Theorem 2.5, $|\mathcal{P}(\mathcal{C}_X)| = 2$. Since $\mathcal{C}_X = (X, \mathcal{R}_X)$ is homogeneous, it follows from the first part of this theorem that $|\mathcal{R}_X| = 2$. If $X \notin \mathrm{Supp}(P_1)$, then by (i), we have $|X| = 1$. It follows that $|\mathcal{R}_X| = 1$.

(iii) Let $X, Y \in \mathrm{Supp}(P_1)$. Then by (ii), $|\mathcal{R}_Y| = |\mathcal{R}_X| = 2$ and then by the first part of this theorem, $\mathcal{P}_X(\mathcal{C}) \cap \mathcal{P}_Y(\mathcal{C}) = \mathcal{P}(\mathcal{C})$. Therefore, Proposition 2.7(ii) implies that $|\mathcal{R}_{X,Y}| = 2$.

(iv) Let $X \in \mathrm{Supp}(P_1)$. By (ii), $|\mathcal{R}_X| = 2$ and thus by Lemma 2.1, $\mathcal{C}_X$ is commutative. By Theorem 2.5 we have

$$n_{P_1} = \sum_{X \in \mathrm{Supp}(P_1)} n_{P_{1X}} = \left|\mathrm{Supp}(P_1)\right|.$$

Thus (5) implies that $|X| = 1 + m_{P_1}$. □

Proof of Theorem 1.2(ii). Let $\mathcal{P}(\mathcal{C}) = \{P_0, P_1\}$ and set $U := \bigcup_{X\in \mathrm{Supp}(P_1)} X$ and $U' := V \setminus U$. If $X \in \mathrm{Supp}(P_1)$ and $Y \notin \mathrm{Supp}(P_1)$, then $|\mathcal{R}_{X,Y}| = 1$, since $|Y| = 1$ by Theorem 3.2. Note that $U \neq \emptyset$, since $\mathcal{C}$ is not trivial. If $U' \neq \emptyset$, then $\mathcal{C} = \mathcal{C}_U \boxplus \mathcal{C}_{U'}$ whereas if $U' = \emptyset$, then $\mathcal{C} = \mathcal{C}_U$. Note that by Theorem 3.2(ii), (iii), $\mathcal{C}_U$ is 2-balanced and $\mathcal{C}_{U'}$ is 1-balanced. Conversely, by Lemma 2.9 and Corollary 3.1, $|\mathcal{P}(\mathcal{C})| = |\mathcal{P}(\mathcal{C}_1 \boxplus \mathcal{C}_2)| = |\mathcal{P}(\mathcal{C}_1)| + |\mathcal{P}(\mathcal{C}_2)| - 1 = |\mathcal{P}(\mathcal{C}_2)| = 2$. □

Corollary 3.3. *Any balanced scheme is half-homogeneous, and any two homogeneous component of it are isomorphic as algebras over $\mathbb{C}$.*

Proof. (i) Let $X \in \mathrm{Fib}(\mathcal{C})$ and consider the scheme $\mathcal{C}_X = (X, \mathcal{R}_X)$. It follows from Theorem 1.1 that the mapping $\mathcal{P}(\mathcal{C}) \to \mathcal{P}(\mathcal{C}_X)$ $(P \mapsto P_X)$ is bijective with $n_P = |\mathrm{Fib}(\mathcal{C})| n_{P_X}$. By (5) and Theorem 2.5(ii), the size of X is computed as follows.

$$|X| = \sum_{P\in\mathcal{P}(\mathcal{C})} n_{P_X} m_{P_X} = \frac{1}{|\mathrm{Fib}(\mathcal{C})|} \sum_{P\in\mathcal{P}(\mathcal{C})} n_P m_P = \frac{|V|}{|\mathrm{Fib}(\mathcal{C})|}.$$

Hence, the size of each fiber is constant and thus $\mathcal{C}$ is half-homogeneous.

(ii) By Theorem 1.1, $n_{P_X} = n_{P_Y}$ for all $X, Y \in \mathrm{Fib}(\mathcal{C})$ and $P \in \mathcal{P}(\mathcal{C})$. It follows from (3), $\mathcal{A}_X = \bigoplus_{P\in\mathcal{P}(\mathcal{C})} \mathrm{Mat}_{n_{P_X}}(\mathbb{C}) \cong \bigoplus_{P\in\mathcal{P}(\mathcal{C})} \mathrm{Mat}_{n_{P_Y}}(\mathbb{C}) = \mathcal{A}_Y$. □

Given a scheme $\mathcal{C}$ we define a relation $E_{\mathcal{C}}$ on $\mathrm{Fib}(\mathcal{C})$ as follows.

$$E_{\mathcal{C}} := \bigl\{(X, Y) \in \mathrm{Fib}(\mathcal{C}) \bigm| \exists R \in \mathcal{R}_{X,Y};\ d_R = e_R = 1\bigr\}. \tag{10}$$

Lemma 3.4. *$E_{\mathcal{C}}$ is an equivalence relation on* $\mathrm{Fib}(\mathcal{C})$.

Proof. For each $X \in \mathrm{Fib}(\mathcal{C})$, Δ_X is a thin basis relation in $\mathcal{R}_X$ and thus $E_{\mathcal{C}}$ is reflexive. If $R \in \mathcal{R}_{X,Y}$ is thin, then $R^t \in \mathcal{R}_{Y,X}$ is also thin and then $E_{\mathcal{C}}$ is symmetric. Let $X, Y, Z \in \mathrm{Fib}(\mathcal{C})$ and $R \in \mathcal{R}_{X,Y}$, $S \in \mathcal{R}_{Y,Z}$ such that $d_R = d_S = 1$ and $e_R = e_S = 1$. It follows from Lemma 2.2(v) that RS is a thin basis relation in $\mathcal{R}_{X,Z}$ and thus $E_{\mathcal{C}}$ is transitive. □

Theorem 3.5. *Any balanced scheme $\mathcal{C}$ is isomorphic to a restriction of the scheme $\mathcal{C}_U \otimes \mathcal{T}_n$ where U is the union of fibers belonging to a transversal of $E_{\mathcal{C}}$ and $n = |\mathrm{Fib}(\mathcal{C})|$.*

Proof. Let $I_n := \{1, \dots, n\}$ and $E_n := \{e_{ij} \mid 1 \leqslant i, j \leqslant n\}$ where $e_{ij} = \{(i, j)\}$. Then $\mathcal{T}_n = (I_n, E_n)$. Let $\{X_1, \dots, X_s\}$ be a transversal of $E_{\mathcal{C}}$ and suppose that for each $i \in \{1, 2, \dots, s\}$, $E_{\mathcal{C}}(X_i) = \{X_{i1}, X_{i2}, \dots, X_{im_i}\}$ where $X_{i1} := X_i$ and X_{ij}'s are distinct fibers. In this case, $V = \bigcup_{i=1}^{s} \bigcup_{j=1}^{m_i} X_{ij}$. For all $i \in \{1, \dots, s\}$ and $j \in \{1, \dots, m_i\}$, there exists $R_{ij} \in \mathcal{R}_{X_{i1},X_{ij}}$ with $d_{R_{ij}} = 1$. Therefore, there exists a bijection $R_{ij} : X_i \to X_{ij}$, $(x_i \mapsto x)$ where x is the unique element of X_{ij} such that $(x_i, x) \in R_{ij}$. Indeed, $R_{ij}(X_i) = X_{ij}$. Thus, for each $x \in V$, there exist unique $i \in \{1, \dots, s\}$ and $j \in \{1, \dots, m_i\}$ such that $R_{ij}(X_i) = X_{ij}$ and $x \in R_{ij}(X_i)$. Assuming that $U = \bigcup_{i=1}^{s} X_i$ we define the map ψ as follows.

$$\psi : V \cup \mathcal{R} \longrightarrow (U \times I_n) \cup (\mathcal{R}_U \otimes E_n).$$

$$x \longmapsto (x_i, j); \qquad R_{ij}(x_i) = x,$$

$$R \longmapsto R_{ij} R R_{kl}^t \otimes e_{jl}; \quad R \in \mathcal{R}_{X_{ij},X_{kl}}.$$

Note that ψ is injective, since R_{ij} is a bijection for all $i \in \{1, \dots, s\}$ and $j \in \{1, \dots, m_i\}$. Let $(x, y) \in R$ and $R \in \mathcal{R}_{X_{ij},X_{kl}}$. Then there exists $(x_i, y_k) \in X_i \times X_k$ such that $R_{ij}(x_i) = x$ and $R_{kl}(y_k) = y$. This means that $(x_i, y_k) \in R_{ij} R R_{kl}^t$. It follows that, $(\psi(x), \psi(y)) = ((x_i, j), (y_k, l)) \in \psi(R)$. This completes the proof. □

The following is an immediate consequence of the preceding theorem.

Corollary 3.6. *Let $\mathcal{C} = (V, \mathcal{R})$ be an (m,n,r)-scheme. Then $\mathcal{C} \simeq \mathcal{C}_X \otimes \mathcal{T}_n$ for $X \in \mathrm{Fib}(\mathcal{C})$ if and only if $E_{\mathcal{C}}$ is trivial, i.e., $E_{\mathcal{C}}$ has one equivalence class.*

4. Reduced (m, n, r)-schemes

Definition 4.1. An (m,n,r)-scheme $\mathcal{C}$ is called *reduced* if its equivalence relation $E_{\mathcal{C}}$ is discrete, i.e., all equivalence classes of $E_{\mathcal{C}}$ are singletons.

Remark 4.1. Note that by Corollary 3.6, a balanced scheme $\mathcal{C}$ is *reduced* if and only if there exist no $X, Y \in \mathrm{Fib}(\mathcal{C})$ such that $\mathcal{C}_{X\cup Y} \simeq \mathcal{C}_X \otimes \mathcal{T}_2$ where $\mathcal{T}_2$ is a $(1,2,1)$-scheme.

In [9], strongly regular designs of the second kind were introduced and shown to be equivalent to reduced $(m,2,3)$-schemes. Linked symmetric designs introduced in [1] are obviously identified with $(m,n,2)$-schemes (see [7, Section 12, p. 31]).

Remark 4.2. Let G act on the sets Ω_i, $i = 1,2$ with the same permutation characters. Recall that the action of G on Ω_1 is equivalent to that on Ω_2 if and only if $G_{\omega_1} = G_{\omega_2}$ for some $\omega_1 \in \Omega_1$ and $\omega_2 \in \Omega_2$ where $G_\omega = \{g \in G \mid \omega^g = \omega\}$. It follows that the 2-orbit scheme of G on $\Omega_1 \cup \Omega_2$ is reduced if and only if the actions are inequivalent.

Example 4.2. (See [12], [1, p. 6, Example (i)].) Let G be the split extension of the translation group of the vector space $\mathrm{GF}(2^t)^{2k}$ by the symplectic group $\mathrm{Sp}(2k, 2^t)$. Then G has 2^t pairwise inequivalent doubly transitive representations of degree 2^{2kt} with the same characters. If we denote them by (G, Ω_i), $i = 1,\dots,2^t$, then it follows from Remark 4.2 that the 2-orbit scheme of G on $\bigcup_{i=1}^{2^t} \Omega_i$ is a reduced $(2^{2kt}, 2^t, 2)$-scheme.

Example 4.3. Let $G = \mathrm{PGL}(t,q)$ and Ω_k the set of k-dimensional subspaces of the vector space $\mathrm{GF}(q)^t$. Let π_k denote the permutation character of G on Ω_k. Then it is known that (see [2, Chapter 4]) for each $k \leqslant \frac{t}{2}$ there exist irreducible characters $\chi_0, \chi_1, \dots, \chi_k$ of G with $\chi_0 = 1_G$ such that

$$\pi_{t-k} = \pi_k = \sum_{i=0}^{k} \chi_i. \tag{11}$$

Moreover, the action of G on Ω_k is inequivalent to that on Ω_{t-k} if $k < \frac{t}{2}$. Consequently, if r and t are positive integers such that $r-1 < \frac{t}{2}$, then by (11) and Remark 4.2, the 2-orbit scheme of $\mathrm{PGL}(t,q)$ on $\Omega_{r-1} \cup \Omega_{t-r+1}$ is a reduced $(\left[{t \atop r-1}\right]_q, 2, r)$-scheme, say $\mathcal{C}$. Moreover, as the actions of $\mathrm{PGL}(t,q)$ on both Ω_{r-1} and Ω_{t-r+1} are multiplicity free, both $\mathcal{C}_{\Omega_{r-1}}$ and $\mathcal{C}_{\Omega_{t-r+1}}$ are commutative and hence by Corollary 3.1, $|\mathcal{P}(\mathcal{C})| = r$.

Lemma 4.3. *Let $\mathcal{C}_i$ be an (m_i, n_i, r_i)-scheme for $i = 1,2$. Then $\mathcal{C}_1 \otimes \mathcal{C}_2$ is an (m_1m_2, n_1n_2, r_1r_2)-scheme. Furthermore, $\mathcal{C}_1 \otimes \mathcal{C}_2$ is reduced if and only if both $\mathcal{C}_1$ and $\mathcal{C}_2$ are reduced.*

Proof. The first statement is obtained by the definition of $\mathcal{C}_1 \otimes \mathcal{C}_2$. Let R_i be a basis relation of $\mathcal{C}_i$ for $i = 1,2$. Then $R_1 \otimes R_2$ is thin if and only if both R_1 and R_2 are thin. This implies that $\mathcal{C}_1 \otimes \mathcal{C}_2$ is reduced if and only if both $\mathcal{C}_1$ and $\mathcal{C}_2$ are reduced. □

Applying Lemma 4.3 for schemes given in Example 4.3 we can construct reduced r-balanced schemes with more than two fibers for each composite r. But, it seems quite difficult to construct reduced p-balanced schemes with more than two fibers where p is an odd prime. As mentioned in [8, Section 8, p. 229] it is still open whether or not a reduced $(m, 3, 3)$-scheme exists.

Problem 1. Given an odd prime p does there exist any reduced $(m, 3, p)$-scheme for some m?

The following problem is inspired from a conjecture by H. Wielandt on permutation representations (see [1], Remark 5.3 and Lemma 5.4).

Problem 2. If $\mathcal{C}$ is a reduced (p, n, r)-scheme for some r and prime p, then $n \leqslant 2$.

5. Enumeration of (m, n, r)-schemes for $m \leqslant 11$

Proof of Theorem 1.3(i). Let $\mathcal{C}$ be a reduced (m, n, r)-scheme and $X, Y \in \mathrm{Fib}(\mathcal{C})$ with $X \neq Y$. Then $2 \leqslant d_R$ for each $R \in \mathcal{R}_{X,Y}$ and

$$2|\mathcal{R}_{X,Y}| \leqslant \sum_{R\in\mathcal{R}_{X,Y}} d_R = m,$$

a contradiction. □

In order to prove Theorem 1.3(ii) we need the following lemma.

Lemma 5.1. *Let $\mathcal{C}$ be an (m, n, r)-scheme and $X, Y, Z \in \mathrm{Fib}(\mathcal{C})$. If $T \in \mathcal{R}_{X,Y}$ such that d_T is prime to $\prod_{R\in\mathcal{R}_{Y,Z}} d_R$, then $d_{Y,Z}$ coincides with $d_{X,Z}$ as multi-sets and $d_T \leqslant \min\{d_R \mid R \in \mathcal{R}_{Y,Z}\}$.*

Proof. For each $R \in \mathcal{R}_{Y,Z}$, $\gcd(d_T, d_R) = 1$. By Lemma 2.2(vii), $|TR| = 1$ and we may define the following map.

$$\psi : \mathcal{R}_{Y,Z} \longrightarrow \mathcal{R}_{X,Z}$$
$$R \longmapsto S; \quad TR = \{S\}.$$

By Lemma 2.2(iii), ψ is surjective. Since $|\mathcal{R}_{Y,Z}| = |\mathcal{R}_{X,Z}|$, ψ must be a bijection. Consequently, $\sum_{R\in\mathcal{R}_{Y,Z}} d_R = \sum_{S\in\mathcal{R}_{X,Z}} d_S = \sum_{R\in\mathcal{R}_{Y,Z}} d_{TR}$. On the other hand, by Lemma 2.2(v), $d_R \leqslant d_{TR}$ for each $R \in \mathcal{R}_X$ and thus $d_R = d_{TR}$ for each $R \in \mathcal{R}_{Y,Z}$. Furthermore, by Lemma 2.2(v), $d_T \leqslant d_{TR} = d_R$ for each $R \in \mathcal{R}_{Y,Z}$. □

Proof of Theorem 1.3(ii). Let $\mathcal{C}$ be a reduced (m, n, r)-scheme and let $X \in \mathrm{Fib}(\mathcal{C})$ such that $\mathcal{C}_X$ is p-valanced. Clearly $m = \sum_{T\in\mathcal{R}_{X,Y}} d_T$ where $X, Y \in \mathrm{Fib}(\mathcal{C})$ with $X \neq Y$. Since $p \nmid m$, so there exists $T \in \mathcal{R}_{X,Y}$ such that $p \nmid d_T$. Since $\mathcal{C}_X$ is p-valenced, d_T is prime to $\prod_{R\in\mathcal{R}_X} d_R$. As $d_{\Delta_X} = 1$, it follows from Lemma 5.1 that $d_T \leqslant \min\{d_R \mid R \in \mathcal{R}_X\} = 1$, a contradiction. □

Lemma 5.2. *Let $\mathcal{C}$ be an $(m, n, 2)$-scheme and $R \in \mathcal{R}_{X,Y}$ where $X, Y \in \mathrm{Fib}(\mathcal{C})$. Then $d_R(d_R - 1) = \lambda(m - 1)$ for some non-negative integer λ.*

Proof. Let $\mathcal{C}$ be an $(m, n, 2)$-scheme and $X, Y \in \mathrm{Fib}(\mathcal{C})$. For each $R \in \mathcal{R}_{X,Y}$ we have by Lemma 2.2(i),

$$A_R A_{R^t} = \sum_{S\in\mathcal{R}_X} c_{RR^t}^{S} A_S = d_R I_X + c_{RR^t}^{\Delta_X^c}(J_X - I_X),$$

where $\Delta_X^c = (X \times X) \setminus \Delta_X$. It follows that $R \in \mathcal{R}_{X,Y}$ is regarded as the incident relation of a symmetric (m, d_R, λ)-design where $\lambda = c_{RR^t}^{\Delta_X^c}$. A basic property of symmetric deigns implies that $d_R(d_R - 1) = \lambda(m - 1)$. □

Remark 5.3. Let m and t be positive integers and q an odd prime power such that $m - 1 = 2^t q$. Then there are exactly four $d \in \{1, \dots, m-1\}$ such that $d(d-1) \equiv 0 \pmod{2^t q}$ by an elementary number theoretical argument. It follows that if $\mathcal{C}$ is a reduced $(m, n, 2)$-scheme, then $d_{X,Y}$ is uniquely determined for all $X, Y \in \mathrm{Fib}(\mathcal{C})$ with $X \neq Y$. Moreover, if m is also prime, then there is $\gamma \in \{2, \dots, m-2\}$ such that $\gcd(\gamma, m - \gamma) = 1$ and $d_{X,Y} = \{\gamma, m - \gamma\}$ for all $X, Y \in \mathrm{Fib}(\mathcal{C})$ with $X \neq Y$.

Lemma 5.4. *Let $\mathcal{C}$ be a reduced $(m, n, 2)$-scheme. Suppose that $d_{X,Y} = \{a, b\}$ with $\gcd(a, b) = 1$ for all $X, Y \in \mathrm{Fib}(\mathcal{C})$. Then $n \leqslant 2$.*

Proof. Suppose that X, Y and Z are distinct fibers of $\mathcal{C}$ and let $\mathcal{R}_{X,Y} = \{R, R'\}$, $\mathcal{R}_{Y,Z} = \{S, S'\}$, $\mathcal{R}_{X,Z} = \{T, T'\}$ so that $d_R = d_S = d_T = a < b = d_{R'} = d_{S'} = d_{T'}$. By Lemma 2.2(i), (ii), (iii), $a^2 = d_R d_S = \alpha a + \beta b$ such that $a \mid b\beta$ and $\beta < a$. Since $\gcd(a, b) = 1$, it follows that $\beta = 0$ and $\alpha = a$. This implies that $c_{RS}^T = a = d_R$. It follows that

$$R_{out}(x) \subseteq S_{in}(z), \tag{12}$$

where $(x, z) \in T$. Now we take $y_1, y_2 \in R_{out}(x)$ so that $y_1 \neq y_2$. It follows from (12) that $T_{out}(x) \subseteq S_{out}(y_1) \cap S_{out}(y_2)$. This fact along with Lemma 2.2(iii) assert that $a = c_{SS^t}^{\Delta_Y^c}$ where $\Delta_Y^c = (Y \times Y) \setminus \Delta_Y$. Therefore, by Lemma 5.2, $a(a-1) = a(a+b-1)$. It follows that $ab = 0$, a contradiction. This completes the proof. □

Lemma 5.5. *Let $\mathcal{C}$ be a reduced $(m, n, 2)$-scheme. If $m - 1$ is a prime power, then $n = 1$.*

Proof. Let p be prime such that $m - 1 = p^t$ for some t. In this case, p does not divide m and we are done by Theorem 1.3(ii). □

Lemma 5.6. *Let $\mathcal{C}$ be a reduced (m, n, r)-scheme and $X, Y \in \mathrm{Fib}(\mathcal{C})$ with $X \neq Y$. If $m = 2r$, then the following hold:*

(i) *For each $T \in \mathcal{R}_{X,Y}$, $d_T = 2$.*
(ii) *For each $R \in \mathcal{R}_X$, $d_R \in \{1, 2, 4\}$ and*

$$\bigl|\{R \in \mathcal{R}_X \mid d_R = 1\}\bigr| = 2\bigl|\{R \in \mathcal{R}_X \mid d_R = 4\}\bigr|.$$

Proof. (i) Let $\mathcal{C}$ be a reduced (m, n, r)-scheme and $X, Y \in \mathrm{Fib}(\mathcal{C})$ with $X \neq Y$. Then as $d_T \geqslant 2$ for each $T \in \mathcal{R}_{X,Y}$, it follows from $m = \sum_{T \in \mathcal{R}_{X,Y}} d_T$ that $2r \leqslant m$ and the equality holds if and only if $d_T = 2$ for each $T \in \mathcal{R}_{X,Y}$.

(ii) Let $R \in \mathcal{R}_X$ and $T \in \mathcal{R}_{X,Y}$. Then by Lemma 2.2(i), (iii), there exist non-negative integers α and β such that $2d_R = d_R d_T = \alpha d_S + \beta d_{S'} = 2\alpha + 2\beta$ and $\alpha, \beta \leqslant 2$. This implies that $d_R \leqslant 4$. By Lemma 5.1, $d_R \in \{1, 2, 4\}$. We set $k_i := |\{R \in \mathcal{R}_X \mid d_R = i\}|$ for $i \in \{1, 2, 4\}$. Since $k_1 + k_2 + k_4 = |\mathcal{R}_X| = |\mathcal{R}_{X,Y}|$, it follows that $m = k_1 + 2k_2 + 4k_4 = 2(k_1 + k_2 + k_4)$. Therefore, $k_1 = 2k_4$. □

Lemma 5.7. *For each (m, n, r)-scheme, if m is prime, then $r - 1$ divides $m - 1$.*

Proof. Let $X \in \mathrm{Fib}(\mathcal{C})$ and consider the homogeneous component $(X, \mathcal{R}_X)$. Since $|X| = m$ is prime, by [6, Theorem 3.3] $d_R = d$ for all $R \in \mathcal{R}_X$ with $R \neq \Delta_X$. Then $m - 1 = \sum_{R \in \mathcal{R}_X, \Delta_X \neq R} d_R = (r-1)d$. □

Table 1

r \ m	4	5	6	7	8	9	10	11
2	1	1	1	$\leqslant 2$	1	1	1	$\leqslant 2$
3	1	1	1	1	$\leqslant 2$	1	1	1
4	1	$*$	1	1	$\leqslant 2$	1	1	$*$
5	$*$	1	$*$	$*$	1	1	$*$	$*$

Lemma 5.8. *Let $\mathcal{C} = (V, \mathcal{R})$ be an (m, n, r)-scheme. If m is odd, then each non-reflexive symmetric basis relation of $\mathcal{C}$ has even degree.*

Proof. Let $S \in \mathcal{R}_X \setminus \{\Delta_X\}$ be symmetric for some $X \in \mathrm{Fib}(\mathcal{C})$. Since $S \neq \Delta_X$, $|S|$ is even. By (2), $|S| = d_S m$ and thus d_S is even. □

Lemma 5.9. *(See [9, (3.2)].) Let $\mathcal{C}$ be a reduced $(m, n, 3)$-scheme. Then $\mathcal{C}_X$ is symmetric for each $X \in \mathrm{Fib}(\mathcal{C})$.*

Proof of Theorem 1.4. So far in this section we have been preparing some lemmas, which will be applied to enumerate reduced (m, n, r)-schemes for m up to 11. The enumeration process leads to Table 1 whose (r, m)'s entry characterizes n such that a reduced (m, n, r)-scheme can exist. The entries (r, m) such that $m < 2r$ are eliminated by Theorem 1.3(i) whereas $(2, m)$'s are eliminated by Lemma 5.5 except $(2, 7)$ and $(2, 11)$. If $\mathcal{C}$ is a reduced $(m, n, 2)$-scheme with $m \in \{7, 11\}$, then by Remark 5.3 and Lemma 5.4, $n \leqslant 2$. Thus we have eliminated the first row of Table 1.

Applying Lemma 5.6 for $(r, m) = (5, 10)$ we obtain that $d_X = \{1, 1, 2, 2, 4\}$. According to [11,5] there is no homogeneous scheme with $d_X = \{1, 1, 2, 2, 4\}$. Note that we can prove this fact in a theoretical way.

The entries $(4, 11)$ and $(5, 11)$ are eliminated by Lemma 5.7 whereas $(3, 7)$ and $(3, 11)$ are eliminated by Lemmas 5.8 and 5.9. An (r, m)-entry of Table 1 is denoted by $*$ if there exists no $(m, 1, r)$-scheme.

Table 2 shows the list of (r, m), $\sum_{i=1}^{r} a_i$ and $\sum_{i=1}^{r} b_i$ where $m = \sum_{i=1}^{r} a_i = \sum_{i=1}^{r} b_i$, $1 = a_1 \leqslant a_2 \leqslant \cdots \leqslant a_r$ and $2 \leqslant b_1 \leqslant b_2 \leqslant \cdots \leqslant b_r$ such that $d_X = \{a_1, \ldots, a_r\}$ and $d_{X,Y} = \{b_1, \ldots, b_r\}$ for some $(m, 1, r)$-scheme $(X, \mathcal{R}_X)$ not satisfying the assumption of Theorem 1.3 (see [5,11]). The remaining cases are processed by use of Table 2. This completes the elimination. □

Lemma 5.10. *If $\mathcal{C}$ is a reduced $(6, n, 3)$-scheme such that $d_X = \{1, 1, 4\}$ for some $X \in \mathrm{Fib}(\mathcal{C})$, then $d_{X,Y} \neq \{2, 2, 2\}$ for each $Y \in \mathrm{Fib}(\mathcal{C})$ with $Y \neq X$.*

Proof. Suppose by the contrary that $d_{X,Y} = \{2, 2, 2\}$ for some $Y \neq X$. By Lemma 5.6, $d_Y = \{1, 1, 4\}$. Taking $R, S \in \mathcal{R}_{X,Y}$ with $R \neq S$ we obtain from Lemma 2.2(vi) that $R^t R = S^t S = \{\Delta_Y, T\}$ where $T \in \mathcal{R}_Y$ with $T \neq \Delta_Y$ and $d_T = 1$. By Lemma 2.2(i), (ii), (iv), $4 = d_R d_{S^t} = \alpha + 4\beta$ for some non-negative integers $\alpha, \beta \leqslant 2$. This implies $\alpha = 0$ and $\beta = 1$, which contradicts Lemma 2.4. □

Lemma 5.11. *If $\mathcal{C}$ is a reduced $(8, n, 3)$-scheme such that $d_X = \{1, 1, 6\}$ for some $X \in \mathrm{Fib}(\mathcal{C})$, then we have the following*:

(i) *For each $Y \in \mathrm{Fib}(\mathcal{C})$ with $Y \neq X$, $d_{X,Y} \neq \{2, 2, 4\}$. Indeed, $d_{X,Y} = \{2, 3, 3\}$ for each $Y \in \mathrm{Fib}(\mathcal{C})$.*
(ii) *Let $\mathcal{R}_{X,Y} = \{R, S, S'\}$ such that $d_R = 2$ and $d_S = d_{S'} = 3$. Let $T \in \mathcal{R}_X$ with $T \neq \Delta_X$ and $d_T = 1$. Then $TR = \{R\}$, $TS = \{S'\}$ and $TS' = \{S\}$.*

Proof. (i) Suppose by the contrary that $d_{X,Y} = \{2, 2, 4\}$ for some $Y \in \mathrm{Fib}(\mathcal{C})$, and take $R \in \mathcal{R}_X$ and $S \in \mathcal{R}_{X,Y}$ so that $d_R = 6$ and $d_S = 2$. It follows from Lemma 2.2(i), (ii), (iii) that for some non-negative integers α, β, γ we have

$$12 = d_R d_S = 2\alpha + 2\beta + 4\gamma, \quad 6 \mid 2\alpha,\ 6 \mid 2\beta,\ 6 \mid 4\gamma,\ \alpha, \beta, \gamma \leqslant 2.$$

Table 2

(r,m)	$\sum_{R\in\mathcal{R}_X} d_R$	$\sum_{R\in\mathcal{R}_{X,Y}} d_R$	
(3, 6)	$1+1+4$	$2+2+2$	Not occur by Lemma 5.10
(3, 8)	$1+1+6$	$2+2+4$	Not occur by Lemma 5.11
		$2+3+3$	$n\leqslant 2$ by Lemma 5.12
	$1+3+4$	$2+2+4$	Not occur by Lemma 2.2(vi)
		$2+3+3$	Not occur by Lemma 2.2(vi)
(3, 9)	$1+2+6$	$2+2+5$	Not occur by Lemma 5.1
		$2+3+4$	Not occur by Lemma 5.13
		$3+3+3$	Not occur by Lemma 5.13
(3, 10)	$1+1+8$	$2+3+5$	Not occur by Lemma 5.1
		$3+3+4$	Not occur by Lemma 5.1
		$2+4+4$	Not occur by Lemma 5.14(i)
		$2+2+6$	Not occur by Lemma 5.14(i)
	$1+3+6$	$2+3+5$	Not occur by Lemma 5.1
		$3+3+4$	Not occur by Lemma 5.14(ii)
		$2+4+4$	Not occur by Lemma 5.1
		$2+2+6$	Not occur by Lemma 2.2(vi)
	$1+4+5$	$2+3+5$	Not occur by Lemma 5.1
		$3+3+4$	Not occur by Lemma 5.1
		$2+4+4$	Not occur by Lemma 2.2(vi)
		$2+2+6$	Not occur by Lemma 2.2(vi)
(4, 8)	$1+1+2+4$	$2+2+2+2$	$n\leqslant 2$ (see Lemma 5.16)
(4, 9)	$1+1+1+6$	$2+2+2+3$	Not occur by Lemma 2.2(vi) and Lemma 5.8
	$1+2+3+3$	$2+2+2+3$	Not occur by Lemma 5.17
(4, 10)	$1+2+2+5$	$2+2+2+4$	Not occur by Lemma 5.1
		$2+2+3+3$	Not occur by Lemma 5.1
	$1+1+4+4$	$2+2+3+3$	Not occur by Lemma 5.1
		$2+2+2+4$	Not occur by Lemma 5.18

This implies that $\gamma=0$ and $12=2\alpha+2\beta\leqslant 8$, a contradiction.

(ii) As $d_T=1$, Lemma 2.2(v), (vii) asserts that $d_{TR}=2$ and $TR=\{R\}$, since R is the unique basis relation in $\mathcal{R}_{X,Y}$ of degree 2. By the same observation $d_{TS}=3$ and $TS\in\mathcal{R}_{X,Y}$. If $TS=\{S\}$, then by Lemma 2.2(i), $c_{TS}^S=1$ and Lemma 2.2(ii) implies that $c_{SS^t}^T=3$. Therefore, applying Lemma 2.2(i), (iv) we have $9=d_Sd_{S^t}=3+3+c_{SS^t}^{T'}6$ where $T'\in\mathcal{R}_X$ with $d_{T'}=6$, a contradiction. □

Lemma 5.12. *If $\mathcal{C}$ is a reduced $(8,n,3)$-scheme such that $d_X=\{1,1,6\}$ for some $X\in\mathrm{Fib}(\mathcal{C})$, then $n\leqslant 2$.*

Proof. Suppose by the contrary that X,Y and Z are distinct fibers of $\mathcal{C}$. Then by Lemma 5.11, $d_{X,Y}=d_{Y,Z}=d_{X,Z}=\{2,3,3\}$. Let $R\in\mathcal{R}_{X,Y}$ and $S\in\mathcal{R}_{Y,Z}$ with $d_R=2$ and $d_S=3$. It follows from Lemma 2.2(i), (ii), (iii), there exist non-negative integers α,β,γ such that

$$6=d_Rd_S=2\alpha+3\beta+3\gamma,\quad 6\mid 2\alpha,\ \alpha\leqslant 2.$$

This implies $\alpha=0$ and $RS=\{S'\}$ where $S'\in\mathcal{R}_{Y,Z}$ with $d_{S'}=3$. Since $c_{RS}^{S'}=2$, by Lemma 2.4, $R^tR\cap SS^t=\{\Delta_Y,T\}$ for some $T\in\mathcal{R}_Y$ with $d_T=1$. Thus $9=d_Sd_{S'}=3+3+6\alpha$. It follows that $3=6\alpha$, a contradiction. □

Lemma 5.13. *Let $\mathcal{C}$ be a reduced $(9,n,3)$-scheme and $d_X=\{1,2,6\}$ for some fiber X. Then $d_{X,Y}\notin\{\{2,3,4\},\{3,3,3\}\}$ for each $Y\in\mathrm{Fib}(\mathcal{C})$.*

Proof. Suppose that $d_{X,Y}=\{2,3,4\}$ and take the basis relations $R\in\mathcal{R}_X$ and $S\in\mathcal{R}_{X,Y}$ so that $d_R=6$ and $d_S=2$. It follows from Lemma 2.2(i), (ii), (iii), there exist non-negative integers α,β,γ such that

$$12=d_Rd_S=2\alpha+3\beta+4\gamma,\quad 6\mid 2\alpha,\ 6\mid 3\beta,\ 6\mid 4\gamma,\ \alpha,\beta,\gamma\leqslant 2.$$

This implies that $\gamma=0$ and $12=2\alpha+3\beta\leqslant 10$, a contradiction.

Suppose that $d_{X,Y}=\{3,3,3\}$ for some $Y\in\mathrm{Fib}(\mathcal{C})$ and take distinct $R,S\in\mathcal{R}_{X,Y}$. By Lemma 2.2(i), (iv), for some non-negative integers α,β we have $9=d_Rd_{S^t}=2\alpha+6\beta=2(\alpha+3\beta)$, a contradiction. □

Let $X,Y\in\mathrm{Fib}(\mathcal{C})$ with $X\neq Y$ and $R,S,S'\in\mathcal{R}_{X,Y}$. Then $R^tR\cap S^tS'\neq\emptyset$ if and only if $RS^t\cap RS'^t\neq\emptyset$. We use this fact in the proof of the following lemma.

Lemma 5.14. *Let $\mathcal{C}$ be a reduced $(10,n,3)$-scheme. Then the following hold*:

(i) *If $d_X=\{1,1,8\}$ for some $X\in\mathrm{Fib}(\mathcal{C})$, then $d_{X,Y}\notin\{\{2,2,6\},\{2,4,4\}\}$ for each $Y\in\mathrm{Fib}(\mathcal{C})$.*
(ii) *If $d_X=\{1,3,6\}$ for some $X\in\mathrm{Fib}(\mathcal{C})$, then $d_{X,Y}\neq\{3,3,4\}$ for each $Y\in\mathrm{Fib}(\mathcal{C})$.*

Proof. (i) Suppose that $d_{X,Y}=\{2,2,6\}$ for some $Y\in\mathrm{Fib}(\mathcal{C})$. Take $R,S\in\mathcal{R}_{X,Y}$ with $R\neq S$ and $d_R=d_S=2$. By Lemma 2.2(i), (iv), (iii), $4=d_Rd_{S^t}=\alpha+8\beta$ for some non-negative integers α,β with $\alpha,\beta\leqslant 2$. It follows that $\alpha=0$ and $4=8\beta$, a contradiction.

Suppose that $\mathcal{R}_{X,Y}=\{R,S,S'\}$ such that $d_R=2$ and $d_S=d_{S'}=4$. By Lemma 2.2(i), (iv), $8=d_Rd_{S^t}=\alpha+8\beta$ for some non-negative integers α,β with $4\mid\alpha\leqslant 2$. This implies that $\alpha=0$ and then $\beta=1$. Therefore, $RS^t=\{T'\}$ where $T'\in\mathcal{R}_X$ with $d_{T'}=8$. By the same observation, $RS'^t=\{T'\}$. Therefore, $T\in S^tS'\cap R^tR$ where $T\neq\Delta_Y$. On the other hand, by Lemma 2.2(vi), $d_T=1$. It follows from Lemma 2.2(i), (ii), (iv), (iii) that for some non-negative integers α,β we have $16=d_{S^t}d_{S'}=\alpha+8\beta$ with $4\mid\alpha$ and $0<\alpha\leqslant 4$. This implies that $\alpha=4$ and thus $12=8\beta$, a contradiction.

(ii) Take $R\in\mathcal{R}_X$ and $S\in\mathcal{R}_{X,Y}$ with $d_R=3$ and $d_S=4$. By Lemma 2.2(i), (ii), (iii), for some non-negative integers α,β,γ we have $12=d_Rd_S=3\alpha+3\beta+4\gamma$ with $12\mid\alpha$ and $12\mid\beta$ and $\alpha,\beta,\gamma\leqslant 3$. This implies that $\alpha=\beta=0$ and $\gamma=3$. Hence $RS=\{S\}$. By Lemma 2.3, $d_{L_S}\mid\gcd(10,4)=2$ which is a contradiction, since $d_{L_S}>d_R=3$. □

Lemma 5.15. *Let $\mathcal{C}$ be a reduced $(8,n,4)$-scheme such that $d_X=\{1,1,2,4\}$ for some $X\in\mathrm{Fib}(\mathcal{C})$. Then for all $X,Y\in\mathrm{Fib}(\mathcal{C})$ with $X\neq Y$, there exists $R\in\mathcal{R}_{X,Y}$ such that $RR^t=\{\Delta_X,S\}$ (resp. $R^tR=\{\Delta_Y,S'\}$) where S is the unique basis relation in $\mathcal{R}_X$ with $d_S=2$ (resp. S' is the unique basis relation in $\mathcal{R}_Y$ with $d_{S'}=2$).*

Proof. Let $\mathcal{C}$ be a reduced $(8,n,4)$-scheme such that $d_X=\{1,1,2,4\}$ for some $X\in\mathrm{Fib}(\mathcal{C})$. Then $d_{X,Y}=\{2,2,2,2\}$ for all $X,Y\in\mathrm{Fib}(\mathcal{C})$ with $X\neq Y$. Let $T\in\mathcal{R}_X$ with $T\neq\Delta_X$ and $d_T=1$. Then $d_{TR}=2$ and $|TR|=1$. Suppose that $TR=\{R\}$ for each $R\in\mathcal{R}_{X,Y}$. Then $T\notin RS^t$ for all $R,S\in\mathcal{R}_{X,Y}$ with $R\neq S$. Thus by Lemma 2.2(i), (iii), $4=d_Rd_{S^t}=2\alpha+4\beta$ for some non-negative integers α,β with $\alpha\leqslant 2$. By Lemma 2.4, $\beta=0$ and $\alpha=2$. This implies that $R\mathcal{R}_{Y,X}\subsetneq\mathcal{R}_X$, which contradicts Lemma 2.2(iii). Thus there exists $R\in\mathcal{R}_{X,Y}$ such that $TR\neq\{R\}$. Equivalently, $T\notin RR^t$. It follows from Lemma 2.2(vi) that $RR^t=\{\Delta_X,S\}$ where S is the unique basis relation in $\mathcal{R}_X$ with $d_S=2$ □

Lemma 5.16. *If $\mathcal{C}$ is a reduced $(8,n,4)$-scheme such that $d_X=\{1,1,2,4\}$ for some $X\in\mathrm{Fib}(\mathcal{C})$, then $n\leqslant 2$.*

Proof. Suppose by the contrary that X,Y and Z are distinct fibers of $\mathcal{C}$. Then by Lemma 5.15, there exist $R\in\mathcal{R}_{X,Y}$ and $T\in\mathcal{R}_{Y,Z}$ such that $R^tR=TT^t=\{\Delta_X,S\}$ where S is the unique basis relation in $\mathcal{R}_Y$ with $d_S=2$. It follows from Lemma 2.2(i) that $c^S_{TT^t}=c^S_{R^tR}=1$ (see Fig. 2). Let $(y,y')\in S$. Then there exists $(x,z)\in X\times Z$ such that $R_{in}(y)\cap R_{in}(y')=\{x\}$ and $T_{out}(y)\cap T_{out}(y')=\{z\}$. As $d_T=2$, we may assume that $T_{out}(y)=\{z,z_1\}$ and $T_{out}(y')=\{z,z_2\}$. Note that $z_1\neq z_2$, otherwise $c^S_{TT^t}\geqslant 2$, a contradiction. This means that $(R\circ T)_{out}(x)=\{z,z_1,z_2\}$ and thus $d_{RT}=d_{R\circ T}=3$, which is a contradiction, since d_{RT} must be a sum of degrees in $d_{X,Z}=\{2,2,2,2\}$. □

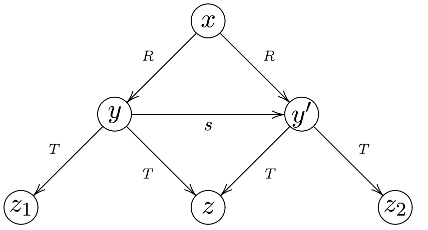

Fig. 2.

Example 5.1. The association scheme `as16 No. 122` as in [5] induces the thin residue fission (see [10, Proposition 3.1]), which is a reduced $(8,2,4)$-scheme whose relational matrix is

$$\left(\begin{array}{cccccccc|cccccccc}
0&1&2&2&3&3&3&3&4&4&5&5&6&7&6&7\\
1&0&2&2&3&3&3&3&4&4&5&5&7&6&7&6\\
2&2&0&1&3&3&3&3&5&5&4&4&6&7&7&6\\
2&2&1&0&3&3&3&3&5&5&4&4&7&6&6&7\\
3&3&3&3&0&1&2&2&6&7&6&7&4&4&5&5\\
3&3&3&3&1&0&2&2&7&6&7&6&4&4&5&5\\
3&3&3&3&2&2&0&1&6&7&7&6&5&5&4&4\\
3&3&3&3&2&2&1&0&7&6&6&7&5&5&4&4\\
\hline
4'&4'&5'&5'&6'&7'&6'&7'&0'&1'&2'&2'&3'&3'&3'&3'\\
4'&4'&5'&5'&7'&6'&7'&6'&1'&0'&2'&2'&3'&3'&3'&3'\\
5'&5'&4'&4'&6'&7'&7'&6'&2'&2'&0'&1'&3'&3'&3'&3'\\
5'&5'&4'&4'&7'&6'&6'&7'&2'&2'&1'&0'&3'&3'&3'&3'\\
6'&7'&6'&7'&4'&4'&5'&5'&3'&3'&3'&3'&0'&1'&2'&2'\\
7'&6'&7'&6'&4'&4'&5'&5'&3'&3'&3'&3'&1'&0'&2'&2'\\
6'&7'&7'&6'&5'&5'&4'&4'&3'&3'&3'&3'&2'&2'&0'&1'\\
7''&6'&6'&7''&5'&5'&4'&4'&3'&3'&3'&3'&2'&2'&1'&0'
\end{array}\right).$$

Also the thin residue fission of the association scheme `as16 No. 51` as in [5], is a reduced $(8,2,3)$-scheme.

Let $R, S, T \in \mathcal{R}$ such that $RS = T$. If $d_T \leqslant d_R$, then it is known that $R = TS^t$. We use this fact in the proof of the following lemma.

Lemma 5.17. *Let $\mathcal{C}$ be a reduced $(9,n,4)$-scheme such that $d_X = \{1,2,3,3\}$ for some $X \in \mathrm{Fib}(\mathcal{C})$. Then $d_{X,Y} \neq \{2,2,2,3\}$ for each $Y \in \mathrm{Fib}(\mathcal{C})$.*

Proof. Suppose by the contrary that $R_1, R_2, R_3 \in \mathcal{R}_{X,Y}$ with $d_{R_i} = 2$ for $i \in \{1,2,3\}$. For all $i, j \in \{1,2,3\}$ with $i \neq j$, by Lemma 2.2(i), (iv) we have $4 = d_{R_i} d_{R_j^t} = 2\alpha + 3\beta + 3\gamma$. This implies that $\beta = \gamma = 0$ and $\alpha = 2$. Hence, for all $i, j \in \{1,2,3\}$ with $i \neq j$, $R_i R_j^t = \{T\}$ where $T \in \mathcal{R}_X$ with $d_T = 2$. It follows that $\{R_1\} = TR_2 = \{R_3\}$, a contradiction. □

Lemma 5.18. *Let $\mathcal{C}$ be a reduced $(10,n,4)$-scheme such that $d_X = \{1,1,4,4\}$ for some $X \in \mathrm{Fib}(\mathcal{C})$. Then $d_{X,Y} \neq \{2,2,2,4\}$ for each $Y \in \mathrm{Fib}(\mathcal{C})$.*

Proof. Suppose by the contrary that $d_{X,Y} = \{2,2,2,4\}$ for some $Y \in \mathrm{Fib}(\mathcal{C})$ with $Y \neq X$. According to [11,5], $d_Y \in \{\{1,1,4,4\},\{1,2,2,5\}\}$. It follows from Lemma 5.1 that $d_Y = \{1,1,4,4\}$. Take $R, S \in \mathcal{R}_{X,Y}$

with $R \neq S$ and $d_R = d_S = 2$. By Lemma 2.2(i), (iv), (iii), there exist non-negative integers α, β, γ such that

$$4 = d_R d_{S^t} = \alpha + 4\beta + 4\gamma, \quad \alpha, \beta, \gamma \leqslant 2.$$

This implies that $4 \mid \alpha$. Hence, $\alpha = 0$ and $\beta + \gamma = 1$ which contradicts Lemma 2.4. □

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