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A Periodicity Theorem for Autonomous Functional Differential Equations*

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I. INTRODUCTION

Many periodicity problems in the theory of control, biological behavior, econometrics and other active areas of scientific research can be formulated in terms of differential equations in which the derivatives depend upon previous states of the systems. Such history dependent differential equations are called functional differential equations.

In this paper, we shall consider autonomous functional differential equations for which the derivatives at time t depend upon values of the solution on the finite time interval $[t - r, t]$, where r is a fixed, positive number. The main result is a periodicity theorem for such autonomous functional differential equations. The theorem will be applied to examples of first and second order differential difference equations.

The problem of determining conditions under which a functional differential equation will have a periodic solution has been undertaken by G. S. Jones in a series of papers, [*1*], [*2*], and [*3*]. He formulated certain asymptotic fixed point theorems which were applicable to operators arising in the study of functional differential equations of the type considered here. For a given equation, he defined an operator A with domain and range in an infinite dimensional Banach space B, such that a fixed point of A corresponded to a periodic solution of the equation. The problem thus is reduced to showing that the operator has a nontrivial fixed point in B. However, the equations considered by Jones (and here) have the origin of B as a critical point, and the operator A has the origin as a fixed point. This means that the origin of B must be removed from the set in which fixed points of A may lie. The approach used by Jones was to apply generalizations of standard fixed point theorems to the operator A. However, in order to

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apply Jones' method to a specific example, rather detailed information about the behavior of solutions near the origin must be obtained in order to ensure that his hypotheses are fulfilled.

The periodicity theorem proven in this paper presents a set of conditions under which a nonlinear autonomous functional differential equation has a periodic solution. The conditions of the theorem do not require a knowledge of solutions near the origin. The approach here is to use a saddle point property of the origin in B, and theorems about the eigenfunctions of an operator such as A. It is then shown that there is an eigenfunction of A with eigenvalue equal to unity. That eigenfunction is a nontrivial fixed point of A.

Application of the theorem requires that one identify the operator A and a cone K such that A maps K into itself. The mapping of the cone must be "compressive," that is cone elements sufficiently far from the origin must be mapped closer to the origin by A. In addition, A and K must satisfy certain boundedness conditions.

The theorem is applied to the equation, $\dot{x}(t) = \alpha x(t-1)[1 + x(t)]$, which was studied by Wright [*4*] and later Jones [*1*]. Wright showed that solutions of this equation are oscillatory if $\alpha > \pi/2$, and Jones proved the existence of periodic solutions. Here the existence of periodic solutions is proven with no more knowledge of the solutions than is given by Wright in [*4*]. In addition, a second order example, the van der Pol equation with retardation, $\ddot{x}(t) + \epsilon(x^2(t) - 1)\,\dot{x}(t) + x(t-r) = 0$, is shown to have a periodic solution for all $\epsilon > 0$, $r > 0$. It is well known that if $r = 0$, the equation has a unique, stable periodic solution for all $\epsilon > 0$. We prove only an existence theorem.

Before we state the main theorem of this paper, certain properties of autonomous functional differential equations with finite retardations must be presented. The properties are some of those reported by Hale in [*5*]. This section provides preliminary information and notation essential to later development.

The derivative in an autonomous functional differential equation depends on the past history of the system described by the equation. Because of this, it is necessary to prescribe data on an interval in order to begin a solution. It is natural to discuss such equations in the setting of a function space, rather than a vector space, for in the function space one point (function) is all that is needed to begin a solution. This is analogous to the ordinary autonomous differential equation, for which one point (vector) in a vector space is all that is needed to begin a solution.

Let r be a fixed positive number, and denote the set of functions continuous on the closed interval $[-r, 0]$, with range in the space of complex n-vectors E^n, by $C = C([-r, 0], E_n)$. The norm in E^n is $|\cdot|$, which may be any convenient vector space norm. The space just defined will be a Banach space under the topology generated by the usual sup norm $\|\cdot\|$, defined by, $\|g\| = \sup\{|g(\theta)| : -r < \theta < 0\}, g \in C$. This is the required function space. The origin of the space will be designated by $\{0\}$. If $\rho > 0$ is a given number, then the closed ball of radius ρ about $\{0\}$ is $B(\rho)$. The distance between two sets F and G is defined in the usual manner. The boundary of a set G in C will be denoted by ∂G.

The symbol x_t will be used throughout this paper and is defined as follows. Let x be any continuous function with domain $[-r, T)$, $T > 0$, and range in E^n. Then for each fixed $t, 0 \leqslant t < T$, the symbol x_t denotes the function with domain $[-r, 0]$ and whose graph coincides with the restriction of x to the interval $[t - r, t]$. That is x_t is an element of C and $x_t(\theta) \equiv x(t + \theta)$, $-r \leqslant \theta \leqslant 0$. Note that $x(t)$ denotes the value of the function x at the point t, hence $x(t) \in E^n$, and $x_t(\theta)$ denotes the value of the function x_t at the point θ, thus $x_t(\theta) \in E^n$.

Let X be an operator defined and continuous on a domain G which is in C and which has range in E^n. Then an autonomous functional differential equation is an equation of the form,

$$\dot{x}(t) = X(x_t), \tag{1.1}$$

where $\dot{x}(t)$ is a right-hand derivative. As we remarked, an initial function in C must be specified to begin a solution of (1.1), so there naturally arises an "initial value problem" for (1.1). A solution of this problem with initial value g at $t = 0$ is a continuous function $x(g)$ defined on a domain $[-r, T)$, $T > 0$, with range in E^n, and such that: (i) $x(g)_0 = g$, $x(g)_t \in G, 0 \leqslant t < T$; (ii) $\dot{x}(g)(t)$ exists, $0 \leqslant t < T$; (iii) $x(g)$ satisfies (1.1), $0 \leqslant t < T$. The operator X does not depend explicitly on t, so (1.1) is autonomous. The solution has the property of translation in time, just as do solutions of autonomous ordinary differential equations. Therefore, we shall assume that an initial function is specified at $t = 0$. There are theorems for functional differential equations, quite similar to those for ordinary differential equations, regarding existence and uniqueness of solutions, and dependence of the solution upon a parameter and initial data. Such theorems are given in Halanay [6], chapter 4.

The geometric concept of the path in E^n associated with the solution of an autonomous ordinary differential equation carries over to autonomous functional differential equations. If $x(g)$ is a solution of (1.1) on $[-r, T)$, then the path of $x(g)$ is the set $\{x(g)_t : 0 \leqslant t < T\}$. If g_0 is a constant function such that $X(g_0) = 0$, then the solution $x(g_0) \equiv g_0$ is constant and the path of $x(g_0)$ is the point g_0 in C. Such a point path will be called a critical point of (1.1). If (1.1) has a nonconstant ρ-periodic solution, then there is a g_0 in C such that $x(g_0)_t = x(g_0)_{t+\rho} = g_0$, and the path of $x(g_0)$ in C is closed.

Some characteristics of linear functional differential equations should be understood before investigating nonlinear functional differential equations. Accordingly, we consider the autonomous linear equation,

$$\dot{x}(t) = L(x_t) = \int_{-r}^{0} [d\beta(\theta)]\, x(t+\theta), \tag{1.2}$$

where $L(\cdot)$ is a continuous linear operator with domain C and range in E^n. The representation of $L(\cdot)$ by the Stieltjes integral is due to the Riesz representation theorem [7]. The $n \times n$ matrix, $\beta(\theta)$, has components which are functions of bounded variation defined on $[-r, 0]$.

Hale [5] showed that the operator $U(t)$ defined by $X(g)_t \overset{\text{def}}{=} U(t)\, g$, where $x(g)$ is the solution of (1.2) with initial value g at $t = 0$, is a bounded linear operator for all $t \geqslant 0$. It was through an investigation of $U(t)$ that Hale obtained the general properties of solutions of (1.2) given in [5].

As in linear ordinary differential equations, there is a characteristic equation associated with (1.2). It is,

$$\det \Delta(\lambda) = 0, \qquad \Delta(\lambda) = I - \int_{-r}^{0} [d\beta(\theta)]\, e^{\lambda\theta} \tag{1.3}$$

where I is the $n \times n$ identity matrix. The roots of (1.3) are called the eigenvalues of L. They ι re real or occur in complex conjugate pairs; each root has finite multiplicity, and only a finite number of roots lie to the right of any given vertical line in the complex plane. If λ satisfies (1.3) and has multiplicity m, then there are exactly m functionally linearly independent solutions of (1.2) of the form $\phi_j(t) = p_j(t)\, e^{\lambda t}$, $(j = 1, 2,\ldots, m)$, where each $p_j(t)$ is a polynomial of degree less than m with coefficients in E^n. The set of functions $\{\phi_1(\theta), \phi_2(\theta),\ldots, \phi_m(\theta)\}$, $(-r \leqslant \theta \leqslant 0)$ forms a basis for a space $P(\lambda)$ in C, which is invariant under the operator $U(t)$. We shall call the space $P(\lambda)$ the generalized eigenspace associated with λ.

More generally, let $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_s\}$ be a set of eigenvalues of L where each λ_j has multiplicity m_j $(j = 1, 2,\ldots, s)$, and set $k = m_1 + m_2 + \cdots m_s$. Then there is a generalized eigenspace $P(\Lambda)$ associated with Λ which is spanned by a basis of k elements; in fact $P(\Lambda)$ is the union of all $P(\lambda_j)$. Properties of $P(\Lambda)$ are summed up in the following theorem.

THEOREM (Hale). *Let Λ and k be as defined above. Then there exists an $n \times k$ matrix function*

$$\Phi(\theta) = \{\phi_1(\theta), \phi_2(\theta),\ldots, \phi_k(\theta)\} \qquad (-r \leqslant \theta \leqslant 0) \tag{1.4}$$

whose columns are functionally linearly independent and form a basis for $P(\Lambda)$. Also there exists a real $k \times k$ matrix A whose eigenvalues are precisely the elements of Λ, and each λ_j has multiplicity m_j. The relation, $\Phi(\theta) = \Phi(0)\, e^{A\theta}$ holds, and $U(t)\,\Phi$ is a matrix solution of (1.2).

As corollaries to this Theorem we have:

COROLLARY (Hale). *For any ϕ in $P(\Lambda)$, there is a unique k vector b such that $\phi = \Phi b$. The solution of* (1.2) *starting from ϕ is $X(\phi)_t = \Phi e^{At} b$ for $t \geqslant 0$. This implies that the path of a solution starting from a point in $P(\Lambda)$ remains in $P(\Lambda)$ for $t \geqslant 0$.*

COROLLARY (Hale). *On the generalized eigenspace $P(\Lambda)$ the initial value problem for* (1.2) *with initial function $\phi = \Phi b$ reduces to the initial value problem for the k dimensional ordinary differential equation, $\dot{y}(t) = Ay(t)$, $y(0) = b$.*

There is a subspace $Q(\Lambda)$ in C which is complimentary to $P(\Lambda)$ and which is also invariant under the operator $U(t)$. To obtain a characterization of $Q(\Lambda)$, it is convenient to consider the adjoint equation,

$$\dot{v}(s) = -\int_{-r}^{0} [d\beta(\theta)]^T\, v(s-\theta), \qquad (s \leqslant 0) \tag{1.5}$$

where $[\cdot]^T$ denotes the transpose. Equation (1.5) is different from (1.2) in that to obtain a solution, an initial function g^* in $C([0, r], E^n) = C^*$ must be specified. Solutions of (1.5) are then obtained for decreasing s. The characteristic equation associated with (1.5), has roots exactly those of (1.3). Thus associated with Λ there is a generalized eigenspace $P^*(\Lambda)$ in C^* which has the same dimension as $P(\Lambda)$. In analogy to (1.4) there is an $n \times k$ matrix,

$$\Psi(\theta) = (\psi_1(\theta), \psi_2(\theta), \ldots, \psi_k(\theta)) \qquad (0 \leqslant \theta \leqslant r) \tag{1.6}$$

whose elements are functionally linearly independent and form a basis for $P^*(\Lambda)$. It is necessary to introduce the bilinear form,

$$(\psi, \phi) = \psi^T(0)\,\phi(0) - \int_{-r}^{0}\int_{0}^{\theta} \psi^T(\xi - \theta)[d\beta(\theta)]\,\phi(\xi)\, d\xi, \tag{1.7}$$

where ψ is in C^* and ϕ is in C, in order to state the next theorem. It gives some properties of the complimentary subspace $Q(\Lambda)$.

THEOREM (Hale). *There exists a subspace $Q(\Lambda)$ in C which has co-dimension k, and is characterized by, $Q(\Lambda) = \{g \in C : (\Psi, g) = 0\}$, where Ψ is the matrix* (1.6) *and (Ψ, g) is the k column vector obtained by substituting Ψ and g into*

(1.7). *Moreover, if* $g \in Q(\Lambda)$, *then* $U(t)g \in Q(\Lambda)$, $(t \geqslant 0)$. *Hence* $Q(\Lambda)$ *is invariant under* $U(t)$.

The subspaces $P(\Lambda)$ and $Q(\Lambda)$ form a coordinate system in C as is shown by the next theorem.

THEOREM (Hale). *Let* $P(\Lambda)$ *and* $Q(\Lambda)$ *be as above. The* $k \times k$ *matrix* $(\Psi, \Phi) = ((\psi_j, \phi_i))$ $(j, i = 1, 2,..., k)$ *is nonsingular. If* Ψ *and* Φ *have been chosen so that* $(\Psi, \Phi) = I$, *the* $k \times k$ *identity matrix, then any* g *in* C *has a unique decomposition*:

$$g = g^P + g^Q, \quad g^P = \Phi(\Psi, g) \in P(\Lambda), \qquad g^Q \in Q(\Lambda). \tag{1.8}$$

In the work to follow we shall take Λ_0 to be the special set:

$$\Lambda_0 = \{\lambda : \Delta(\lambda) = 0, \quad \mathrm{Re}(\lambda) > 0\}, \tag{1.9}$$

which will have finite number of elements.

II. THEOREM AND PROOF

We consider the system of autonomous functional differential equations with retardation $r > 0$,

$$\dot{x}(t) = L(x_t) + N(x_t), \tag{2.1}$$

where L is continuous on C, is linear, and at least one eigenvalue of L has positive real part. (The latter assures that Λ_0 of (1.9) is nonempty and that the generalized eigenspaces, $P(\Lambda_0) \stackrel{\text{def}}{=} P$ and $Q(\Lambda_0) \stackrel{\text{def}}{=} Q$, exist.) The operator N is continuous on C, $N(\{0\}) = \{0\}$, and given a $\delta > 0$, there is a continuous, nondecreasing function $\eta(\cdot)$, $\eta(0) = 0$, such that for any g', g'' in $B(\delta)$, $| N(g') - N(g'')| \leqslant \eta(\delta)\| g' - g'' \|$.

Equation (2.1) is specified to be nonlinear, since we are interested here in those systems in which the periodic solution is forced into existence by the presence of nonlinear elements.

A cone K in a Banach space C is a closed, convex set such that if $k \in K$, then $\beta k \in K$ for all $\beta \geqslant 0$; and at least one of g, $-g$ is not in K if $g \neq \{0\}$. Suppose now that the path of a solution of (2.1) starting from any element $k \neq \{0\}$ in a fixed cone K in C, leaves the cone, and returns to the cone at a time $\tau(k)$. The time $\tau(\cdot)$ must be continuous and such that for any $m > 0$ and all k in $B(m) \cap K$, then $r \leqslant \tau(k) \leqslant T(m)$ where $T(m)$ is finite. If $\tau(k)$ exists for all elements not equal to $\{0\}$ of a given cone K, then we shall say that system (2.1) maps K into itself. Moreover, if $x(k)$, $k \in K$, is a solution of (2.1), then we may define an operator A through

$$Ak \stackrel{\text{def}}{=} x(k)_{\tau(k)} . \tag{2.2}$$

The operator A is positive with respect to K, that is A maps K into itself. Moreover, A is continuous on K since $\tau(k)$ is continuous and the solution $x(k)$ is continuous with respect to initial conditions. That A is compact may be seen from the following. The Lipschitz constant associated with right hand side of (2.1) is a continuous, nondecreasing function $J(\delta)$, $\delta \geqslant 0$. Let $T(\delta) = T$ be an upper bound for $\tau(k)$, k in $K \cap B(\delta)$. Then it is known (Halanay [6], page 338) that the solution $x(k)$ satisfies the inequality,

$$\| x(k)_t \| \leqslant \| k \| \exp(J(\delta)\, t), \tag{2.3}$$

as long as $\| x(k)_t \| \leqslant \delta$. This implies that (2.3) is satisfied as long as $\| k \| \leqslant \delta \exp(-TJ(\delta)) = \delta'$. This shows that for any given $\delta > 0$, the functions $x(k)_{(\tau)k} = Ak$ are uniformly bounded if k is in $K \cap B(\delta')$. The derivative of $x(k)$ exists on $[\tau(k) - r, \tau(k)]$ and is given by (2.1); hence it is uniformly bounded. Also, Ak is a well defined function on $[-r, 0]$. By the Ascoli selection theorem, A of (2.2) is compact.

The main theorem of this paper is then:

THEOREM 2.1. *Suppose that the mapping operator A of* (2.2) *is defined for all $k \neq \{0\}$ in some cone K in C. Suppose also that A satisfies the conditions*:

I. *Let G be any open, bounded neighborhood of $\{0\}$, then* $\inf \| Ak \| > 0$ *if $k \in \partial G \cap K$.*

II. *There exists a finite $M > 0$ such that $\| Ak \| < \| k \|$ for all k in K satisfying $\| k \| \geqslant M$.*

Then if the cone K satisfies the condition,

III. $\inf \| k^P \| > 0$, *k in $\partial B(1) \cap K$,*

there exists at least one nontrivial periodic solution of system (2.1) *with period greater than r.*

Proof of Theorem 2.1. Suppose that the nonlinear operator A of (2.2) has a fixed point k^* in K which is not a critical point of (2.1). This means that the solution path starting from k^* returns to k^*, hence it is a closed path in C. The uniqueness of the solution assures that the closed path is a periodic solution.

An eigenfunction for a nonlinear operator A is defined in the same manner as if A were linear. That is, an element $\phi \neq \{0\}$ of the Banach space in which A operates is an eigenfunction of A if there exists a number μ (the eigenvalue) such that $A\phi = \mu\phi$. Thus if A is positive with respect to K and $\phi \in K$ is an eigenfunction, then $\mu > 0$.

The remainder of the proof is devoted to showing that A has an eigenfunction in K with eigenvalue equal to unity.

As in Krasnoselskii ([8], p. 248) we say that the eigenfunctions of an operator form a continuous branch of length R, if for every $R_1 < R$, the intersection of the set of eigenfunctions of the operator with the boundary of every open neighborhood of $\{0\}$ contained in $B(R_1)$ is nonempty. The continuous branch is of infinite length if there is no upper bound on R. The following lemma is due to Krasnoselskii ([8], p. 243).

LEMMA 2.1 (Krasnoselskii). *Let G be an open, bounded neighborhood of $\{0\}$ in a Banach space and let K be a cone in the space. Suppose that on $\partial G \cap K$ the operator A is positive with respect to K and is completely continuous (compact and continuous). If* $\inf \| Ak \| > 0$, $k \in \partial G \cap K$, *then the operator A has at least one eigenfunction in $\partial G \cap K$ with positive eigenvalue.*

Since the operator A of (2.2) is assumed to satisfy condition I of Theorem 2.1, it satisfies all the conditions of the above lemma. It therefore has a continuous branch of eigenfunctions of infinite length in K, since there is no restriction on the neighborhood G of the lemma. The next lemma shows that the eigenvalues have a certain continuity property.

LEMMA 2.2. *Let an operator A be positive with respect to a cone K, and be completely continuous. Assume A has a continuous branch of eigenfunctions of infinite length in the cone K. Let $F = \{\phi \in K : \phi$ an eigenfunction of A, $A\phi = \mu\phi\}$ and assume that the following conditions hold:*

IV. *There exist numbers $M < 0$, $\mu^* > 0$, such that for all ϕ in F with $\| \phi \| > M$, the associated eigenvalue $\mu < \mu^*$.*

V. *There exists an open bounded neighborhood of $\{0\}$, G, such that for all ϕ in $\partial G \cap K$, the associated eigenvalues $\mu > \mu^*$.*

Then μ^* is an eigenvalue of A and the corresponding eigenfunction ϕ^* is in F.

Note that the lemma is stated somewhat more generally than is necessary here. In the application of the lemma we shall take $\mu^* = 1$, and will prove that the operator A of (2.2) has a fixed point.

Proof of Lemma 2.2. Assume there is no such ϕ^* associated with μ^*. Let F_0 be the set of eigenfunctions of A in K located outside the neighborhood G. That is, $F_0 = \{\phi \in F : \phi \notin G\}$. Form the disjoints sets,

$$F_1 = \{\phi \in F_0 : \text{associated eigenvalue } \mu < \mu^*\}$$
$$F_2 = \{\phi \in F_0 : \text{associated eigenvalue } \mu > \mu^*\}$$

so that $F_1 \cup F_2$ contains all of F_0. Then there results:

(i) F_1 is bounded away from the origin. This is because $F_1 \subset F_0$ and F_0 is bounded away from the origin.

(ii) F_2 is bounded and also bounded away from the origin. The latter statement follows as in (i). If it were not bounded, then there would be a ϕ in F_2 with $\|\phi\| > M$, and by IV the corresponding eigenvalue $\mu < \mu^*$. This contradicts the definition of F_2.

(iii) The sets F_1 and F_2 are both closed. Let $\{\phi_n\}$ be a bounded sequence in F_1 converging to ϕ_0. Since F_1 is bounded away from the origin $\|\phi_0\| > 0$. By continuity of A, $\{A\phi_n\}$ converges to $A\phi_0$ and so the sequence $\{\mu_n\} = \{\|A\phi_n\|/\|\phi_n\|\}$ converges to $\|A\phi_0\|/\|\phi_0\| = \mu_0 \leqslant \mu^*$. Consequently the equation $A\phi_n = \mu_n\phi_n$ converges to, $A\phi_0 = \mu_0\phi_0$. If $\mu_0 = \mu^*$, there would be an eigenfunction ϕ^* for μ^* contrary to hypothesis. Thus $\mu_0 < \mu^*$ and $\phi_0 \in F_1$. Similarly F_2 is closed.

(iv) The sets F_1 and F_2 are separated by a finite distance. To show this, assume the contrary. Then there exist sequences, $\{\phi_n\}$ in F_1 and $\{\psi_n\}$ in F_2, $A\psi_n = \mu_n\psi_n$, such that

$$\lim \|\phi_n - \psi_n\| = 0 \quad \text{as} \quad n \to \infty. \tag{2.4}$$

By F_1 bounded away from the origin, F_2 bounded above, and (2.4) we may presume that the sequence $\{\psi_n\}$ is bounded above and away from the origin below. Since A is compact and $\inf \|\psi\| > 0$, $\psi \in F_2$, we have the estimate,

$$0 < \mu^* < \mu_n = \|A\psi_n\|/\|\psi_n\| \leqslant \sup \|A\psi_n\|/\inf \|\psi_n\| < \infty$$

where the sup and inf are over all ψ_n in $\{\psi_n\}$. Hence the sequence $\{\mu_n\}$ has a subsequence converging to a number $\mu_0 \geqslant \mu^*$. Moreover, compactness of A implies that $\{A\psi_n\}$ has a subsequence $\{A\psi_i\}$ which converges. Since $\sup \|A\psi_n\| > 0$, $\{A\psi_i\}$ converges to a nonzero element ψ^* which we may write as $\mu_0\psi_0$. In view of these convergences, the following estimate holds: for any $\epsilon > 0$, there exists an $N(\epsilon)$ such that for all $i > N(\epsilon)$,

$$\begin{aligned}\|\psi_i - \psi_0\| &= \|A\psi_i/\mu_i - \psi_0\| = \|(\mu_0/\mu_i - 1)\psi_0 + (A\psi_i - \mu_0\psi_0)/\mu_i\| \\ &\leqslant |\mu_0/\mu_i - 1|\,\|\psi_0\| + \|A\psi_i - \mu_0\psi_0\|/\mu^* \leqslant \epsilon/2 + \epsilon/2 = \epsilon\end{aligned}$$

Therefore $\{\psi_i\}$ converges to ψ_0, and ψ_0 is in F_2 by closure. However, by (2.4) the sequence $\{\phi_i\}$ also converges to ψ_0 and thus it is in F_1 by closure. This contradicts the fact that F_1 and F_2 are disjoint. Therefore, the distance d between F_1 and F_2 is positive, $d > 0$.

(v) The last result permits the final step of this proof. Let G be as in V, and construct the set G^* as follows,

$$G^* = G \cup G_2, \qquad G_2 = \{g \in C : \|\phi - g\| < d/2, \phi \in F_2\}.$$

Then G^* forms an open, bounded neighborhood of $\{0\}$. We proceed to show that there are no eigenfunctions of A in the intersection of K and the boundary of G^*. The set $\partial G^* \cap K$ is exterior to $G \cap K$ since the intersection of G and ∂G^* is empty. Therefore any eigenfunctions of A in $\partial G^* \cap K$ are in F_0, and $\partial G^* \cap F = \partial G^* \cap F_0$. Now, let k_0 be any element in $\partial G^* \cap K$. Then k_0 is not in F_2, since F_2 is in the interior of G^*. Moreover, k_0 is not in F_1, as is shown next. Certainly k_0 is in $\partial G \cap K$ or $\partial G_2 \cap K$, since $\partial G^* \cap K$ is contained in the union of these two sets. If k_0 were an eigenfunction in $\partial G \cap K$, then it could only be in F_2 by the definition of G. If k_0 were an eigenfunction in $\partial G_2 \cap K$, it could not be in F_1, since G_2 was constructed so that F_1 and ∂G_2 are disjoint. This shows that $\partial G^* \cap F$ is empty as was to be shown.

The last result contradicts the hypothesis that F is a continuous branch. Thus there is an eigenfunction ϕ^* in F corresponding to μ^*, and Lemma 2.2 is proven.

The problem has now been reduced to showing that the operator A of (2.2) satisfies the conditions of Lemma 2.2 with $\mu^* = 1$. It has already been shown that A of (2.2) is completely continuous, positive with respect to a cone K, and has a continuous branch of eigenfunctions of infinite length in K. Condition II of Theorem 2.1 assures that condition IV of Lemma 2.2 holds with $\mu^* = 1$. Hence we need only show that condition V holds. This shall be done by identifying a neighborhood G that has the required property. The next lemma, Theorem 3 of [9] gives a certain property of solutions of system (2.1) near the origin.

LEMMA 2.3 (Hale–Perello). *Assume that for system* (2.1) *the set* Λ_0 *of* (1.9) *is nonempty and that the generalized eigenspaces P and Q are established. Then given a constant s, $0 < s < 1$, there exists a $\delta = \delta(s) > 0$, $\delta(s) \to 0$ as $s \to 0$, and a function $V(y(f))$, which is a positive definite quadratic form in the components of the vector $y = y(f) = (\Psi, f)$, such that $\dot{V}(y) > 0$ if $\|f\| \leqslant \delta$ and $\|f^P\| \geqslant s\delta$. Here the function $\dot{V}(y)$ stands for the derivative of V along the paths defined in $B(\delta)$ by equation* (2.1). *In* [9] *Hale and Perello gave the following geometric interpretation of Lemma* 2.3. *In the ball $B(\delta)$ in which C with the set*

$$G = \{g \in C : \|g\| < \delta\} \cap \{g \in C : \|g^P\| < s\delta\} \tag{2.5}$$

deleted, all solution paths of equation (2.1) *are tending away from Q with increasing t in such a way that the P projection of the paths is increasing with t. Geometrically this situation is illustrated in Figure* 2.1, *a schematic representation of C. The P and Q spaces are shown as coordinate axes, and the cone K and set G are also represented.*

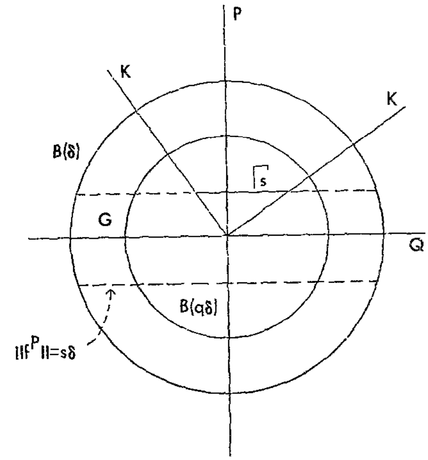

Fig. 2.1

FIG. 2.1. Schematic representation of C.

Lemma 2.3 may be applied to system (2.1) as follows. Suppose that the set G of (2.5) has the following two properties.

VI. The projection on P of any element in $\partial G \cap K$ is exactly $s\delta$. That is,

$$\partial G \cap K = \{k \in K : \| k \| \leqslant \delta, \qquad \| k^P \| = s\delta\} \stackrel{\text{def}}{=} \Gamma_s \tag{2.6}$$

VII. The solution path of (2.1) starting from any point k in Γ_s remains in $B(\delta)$ for $0 \leqslant t \leqslant \tau(k)$; $\tau(k)$ defined in (2.2).

The operator A of (2.2) will have an eigenfunction ϕ in Γ_s with eigenvalue $\mu > 0$, since A has a continuous branch of eigenfunctions of infinite length. If $x(\phi)$ is a solution of (2.1), then by Lemma 2.3, the P components of the solution path are bounded below by $s\delta$ during $[0, \tau(\phi)]$. That is, $\| x(\phi)_t{}^P \| \geqslant s\delta$ for $0 \leqslant t \leqslant \tau(\phi)$. In particular we have:

$$s\delta \leqslant \| x(\phi)^P_{\tau(\phi)} \| = \|(A\phi)^P \| = \mu \| \phi^P \| = \mu s\delta.$$

The above relation shows that $\mu \geqslant 1$. If $\mu = 1$, then ϕ is a fixed point A of and we are finished, hence we assume $\mu > 1$.

It is not difficult to show that the set G of (2.5) is a neighborhood of $\{0\}$. Hence we have shown that if conditions VI and VII above are satisfied, then the set G of (2.5) satisfies condition V of Lemma 2.2 with $\mu^* = 1$.

Property VII can be shown to hold by an argument similar to that used in proving that A of (2.2) is compact. If $\delta = \delta(s)$ is that of Lemma 2.3 then, since $\delta(s) \to 0$ as $s \to 0$, there is a $\delta' > \delta(s)$ for $0 \leqslant s < 1$. Let $J = J(\delta')$, $T = T(\delta')$, and $q = \exp(-TJ)$. Thus by (2.3), if $k \in K \cap B(q\delta)$, then $\| Ak \| \leqslant \delta$. Therefore if we require that

$$\Gamma_s \subset K \cap B(q\delta), \tag{2.7}$$

then property VII will hold.

It now remains to show that one can actually pick an s such that both (2.6) and (2.7) are true. The desired situation is shown schematically in Figure 2.1. We first need to know that there is a positive lower bound on the P components of elements in $\partial B(\delta) \cap K$. Were this not so, then elements in $\partial G \cap K$ could have a norm greater than $q\delta$, while the norm of the P component would be less than $s\delta$. Then neither (2.6) or (2.7) would hold.

LEMMA 2.4. *If condition* III *of Theorem* 2.1 *holds, then for all k in $\partial B(\delta) \cap K$, $\delta > 0$, there exists a minimal number $a(\delta) = \nu\delta$, $\nu > 0$ a constant, such that $\| k^P \| \geqslant a(\delta)$.*

Proof of Lemma 2.4. Fix δ and suppose there were no such minimal number. Then there must be a sequence $\{k_j\}$ in $\partial B(\delta) \cap K$ such that $\| k_j^P \|$ tends to 0 as j tends to ∞. However, $\{k_j/\delta\}$ is in $\partial B(1) \cap K$, which contradicts condition III. Hence there is a minimal number, $a(\delta) > 0$ for $\delta > 0$. Closure of $\partial B(\delta) \cap K$ ensures that there is at least one element k_0 in $\partial B(\delta) \cap K$ such that $\| k_0^P \| = a(\delta)$. Let $\beta > 0$ be fixed. Then by the cone property βk_0 is in $\partial B(\beta\delta) \cap K$ so that,

$$\|(\beta k_0)^P \| = \beta \| k_0^P \| = \beta a(\delta) \geqslant a(\beta\delta). \tag{2.8}$$

If in (2.8) we let $\beta = 1/\delta$, then $a(\delta) \geqslant \delta a(1)$; while if we replace δ by 1 and β by δ, then $\delta a(1) \geqslant a(\delta)$. This shows that

$$a(\delta) = \nu\delta, \qquad \nu = a(1) > 0, \tag{2.9}$$

which proves Lemma 2.4.

Finally the choice, $0 < s < 1$, $s < \nu q$, q as in (2.7) and ν as in (2.9) will ensure that (2.6) and (2.7) both hold.

To show (2.6) holds, suppose there were an element k_0 in $\partial G \cap K$ such that $\| k_0^P \| < s\delta$. Then, since

$$\partial G = \{g \in C : \| g \| = \delta, \| g^P \| \leqslant s\delta\} \cup \{g \in C : \| g \| \leqslant \delta, \| g^P \| = s\delta\},$$

the element k_0 has norm equal to δ. However, by Lemma 2.4 we have $\nu\delta \leqslant \| k_0^P \| < s\delta < s\delta/q$, which contradicts $s < q\nu$. (Note $0 < q < 1$.)

To show (2.7) holds, we note that for any k in Γ_s, $\| k^P \| = s\delta < \nu q\delta = a(q\delta)$. Now if $\| k \| \geqslant q\delta$, then $\| k^P \| \geqslant a(q\delta)$, which is not true. This proves $\Gamma_s \subset B(q\delta) \cap K$, and completes the proof of Theorem 2.1.

III. Applications

In this section, Theorem 2.1 will be used to show that two first order and one second order differential difference equations possess periodic solutions.

Example 1. The equation,

$$\dot{x}(t) = -\alpha x(t-1)[1 + x(t)], \tag{3.1}$$

has occurred in one form or another in several unrelated subjects. In [*4*] Wright studied (3.1) in detail, and showed that it had oscillatory solutions if $\alpha > \pi/2$. An alternate form of (3.1), the equation $\dot{x}(t) = ax(t)[M - x(t-r)]/M$ was suggested by Hutchinson [*10*] as a mathematical description of a population growing at a constant reproductive rate a toward a saturation value M. The term $[M - x(t)]/M$ represents a self-regulating mechanism which takes a time r to react to changes in the population level, $x(t)$. This equation was also studied by Kakutani and Markus [*11*], and by Cunningham [*12*]. The latter pointed out that it could be used to describe certain control systems, and that similar equations could arise in economic studies of the business cycle. It is interesting to note that Hutchinson states (no proof) that if $ar > \pi/2$, then sustained oscillations of solutions exist, while oscillations damp out or are non-existent if $ar < \pi/2$. Moreover, analogue computer results of Cunningham [*12*] show the same behavior of solutions. In the analysis to follow, we shall see why $ar = \alpha = \pi/2$ is a critical value.

Corollary 3.1. *If $\alpha > \pi/2$ then there exists a nontrivial periodic solution of* (3.1) *with period greater than* 1.

Proof. Clearly, the right hand side of (3.1) is of the class represented by (2.1). The characteristic equation for the linear part of (3.1) is $\lambda + \alpha \exp(-\lambda) = 0$. A result of Wright [*4*] assures us that Λ_0 of (1.9) is nonempty if $\alpha > \pi/2$. It is necessary to understand certain properties of the solutions of (3.1) before defining A and K. These properties are given in the following lemma due to Wright [*4*].

Lemma 3.1 (Wright). *Corresponding to any bounded, integrable function f*

on $[-1, 0]$ *there exists a unique solution to* (3.1) *for all* $t \geqslant 0$. *It has the following properties.*

(i) *If* $f(0) \gtreqless -1$, *then* $x(f)(t) \gtreqless -1$, $(t \geqslant 0)$.

(ii) *If* $f(0) > -1$, *then* $-1 < x(f)(t) < e^{\alpha} - 1$, $(t \geqslant 2)$.

(iii) *If* $\alpha > \pi/2$, *and* f *is positive* (*negative and* $f(0) > -1$) *on* $(-1, 0)$, *then* $x(f)$ *is oscillatory about zero and the oscillations do not tend to zero as* $t \to \infty$.

(iv) *For oscillatory* $x(f)$, *there is a first point*, $z_1(f) < 2$, *at which* $x(f)$ *is zero and* $\dot{x}(k)$ *is nonzero. Subsequent zeros are separated by at least a unit and are simple.*

A choice for the cone required by Theorem 2.1 is the set of nondecreasing, nonnegative functions,

$$K = \{k \in C([-1, 0], E^1) : 0 \leqslant k(\theta_1) \leqslant k(\theta_2), -1 \leqslant \theta_1 < \theta_2 \leqslant 0\}. \tag{3.2}$$

It is an easy exercise to show that condition (iii) of Lemma 3.1 holds for boundary elements as well as interior elements of K. Lemma 3.1 then assures that for any k in K there exists a first simple zero, $z_2(k)$, of $x(k)$ with the property that $\dot{x}(k)(z_2(k)) > 0$. Therefore, the operator A defined by,

$$Ak = x(k)_{\tau(k)}\,, \qquad \tau(k) = z_2(k) + 1, \tag{3.3}$$

defines a mapping of K into itself.

Showing that $\tau(k)$ is continuous on K amounts to showing that $z_2(k)$ is continuous on K. The proof of this fact is a direct result of the continuity of the solution $x(k)$ on K and item (iv) of Lemma 3.1. Hence the details are not reproduced. The boundedness of $z_2(k)$ is given by the next lemma.

LEMMA 3.2. *For each* $\delta > 0$, *there is a finite number* $T(\delta)$ *such that* $\tau(k) = z_2(k) + 1 \leqslant T(\delta)$ *for* $k \in B(\delta) \cap K$.

Proof. Define a mapping of K into C by the operator A^*, $A^*k = x(k)_1$, $(A^*k)(\theta) = x(k)(1 + \theta)$, $(-1 \leqslant \theta \leqslant 0)$. Then by the same arguments used in proving compactness of A, A^* is compact. Let the closure of the image of $B(\delta) \cap K$ under A^* be S, which is compact. Now consider $z_2(k)$ as the mapping of S into the real numbers, $z_2(k) = z_2(A^*k)$. Then $z_2(\cdot)$ is a continuous mapping of a compact set, and so has a finite, uniform bound, $T(\delta)$. This proves Lemma 3.2.

The next lemma shows that condition I of Theorem 2.1 holds.

LEMMA 3.3. *Let* G *be a bounded neighborhood of* $\{0\}$. *Then for* A *of* (3.3), $\inf \| Ak \| > 0$ *for* $k \in \partial G \cap K$.

Proof. Note that there must be a $\delta_0 > 0$ such that

$$\inf \| k \| = \inf | k(0)| \geqslant \delta_0 > 0, \qquad k \in \partial G \cap K, \tag{3.4}$$

since $\partial G \cap K$ is bounded away from $\{0\}$. Assume that $\inf \| Ak \| = 0$ for $k \in \partial G \cap K$. Since $\| Ak \| = x(k)(z_2(k) + 1)$ our assumption implies that, $\inf x(k)(t) = 0$ for $k \in \partial G \cap K$ and $z_2(k) \leqslant t \leqslant z_2(k) + 1$. That is, there exists a sequence of solutions with initial values in $\partial G \cap K$ which tend to zero on a unit interval. The derivatives of these solutions also tend to zero because Ak is monotone increasing. By (3.1) this implies that these solutions tend to zero on the preceding unit interval and, $\inf | x(k)(t)| = 0$, for $z_2(k) - 1 \leqslant t \leqslant z_2(k)$. Since G is bounded, $z_2(k)$ is uniformly bounded for all k in $\partial G \cap K$. Thus one can continue the process, and in a finite number of steps conclude that $\inf | x(k)(z_1(k) + 1)| = 0$ for $k \in \partial G \cap K$. Note that $x(k)(z_1(k) + 1)$ is the minimal value of $x(k)$ on $[z_1, z_2]$. By a similar process, one can work back from $z_1(k) + 1$ and in a finite number of steps conclude that, $\inf | x(k)(0)| = \inf | k(0)| = 0$ for $k \in \partial G \cap K$. The latter contradicts (3.4) and Lemma 3.3 is proven.

Condition II of Theorem 2.1 will be fulfilled if we take $M = e^\alpha - 1$. The result of Wright, (ii) of Lemma 3.1, shows that $\| Ak \| < \| k \|$ if $\| k \| \geqslant e^\alpha - 1$.

Finally, we show that condition III of Theorem 2.1 is fulfilled to complete the proof of Corollary 3.1.

LEMMA 3.4. $\operatorname{Inf} \| k^P \| > 0$, $k \in \partial B(1) \cap K$, *where K is the cone of* (3.2).

Proof. A result of Wright [4] states that if $\alpha > \pi/2$, then there is a complex eigenvalue of the linear part of (3.1), $\lambda = \sigma + i\gamma$, such that $\sigma > 0$ and $0 < \gamma < \pi$. Corresponding to $\Lambda_0 = (\lambda, \bar{\lambda})$ are the matrices $\Phi = (\phi, \bar{\phi})$ of (1.4) and $\Psi = (\psi, \bar{\psi})$ of (1.6) where $\phi = \exp(\lambda\theta)/(1 + \lambda)$, $-1 \leqslant \theta \leqslant 0$, and $\psi = \exp(-\lambda\delta)$, $0 \leqslant \delta \leqslant 1$. The eigenfunctions ϕ and ψ have been chosen so that $(\Phi, \Psi) = I$.

Now assume that there exists a sequence $\{k_j\}$ in $\partial B(1) \cap K$ such that $\inf \| k_j^P \| = 0$. Since the elements of Φ are functionally linearly independent, our assumption reduces to $\inf |(\Psi, k_j)| = 0$. This implies, $\inf | R(k_j)| = 0$, and $\inf | I(k_j)| = 0$, where $R(\cdot)$ and $I(\cdot)$ are respectively the real and imaginary parts of (ψ, k_j). A computation according to (1.7) yields the results,

$$R(k_j) = k_j(0) - \alpha \int_{-1}^{0} k_j(s)\, e^{-\sigma(s+1)} \cos \gamma(s + 1)\, ds$$

$$I(k_j) = \alpha \int_{-1}^{0} k_j(s)\, e^{-\sigma(s+1)} \sin \gamma(s + 1)\, ds \tag{3.5}$$

Now, $\alpha \exp(-\sigma(s+1)) \sin \gamma(s+1)$ is positive on $(-1, 0]$ since $0 < \gamma < \pi$. Therefore since $\inf |I(k_j)| = 0$, k_j nondecreasing and nonnegative on $[-1, 0]$ implies that $k_j(s) \to 0$ for $-1 \leqslant s < 0$. Note that $\| k_j \| = k_j(0) = 1$ since $k_j \in \partial B(1) \cap K$. The latter two statements and (3.5) imply that $\inf | R(k_j)| = 1$, a contradiction. This proves Lemma 3.4.

Example 2. The equation,

$$\dot{x}(t) = -\alpha x(t-1)[1 - x^2(t)], \tag{3.6}$$

was discussed by Jones [*2*], who not only showed that it had a periodic solution $\alpha > \pi/2$, but also, in [*3*], obtained an explicit solution in terms of the Jacoby elliptic functions. The method of Theorem 2.1 can also be used to prove,

COROLLARY 3.2. *If* $\alpha > \pi/2$ *there exists a nontrivial solution of equation* (3.6) *with period greater than* 1.

Corollary 3.2 will not be proven in detail. The proof is similar to that of Corollary 3.1, however a different operator A than that used in Corollary 3.1 is necessary. As before, certain properties of the solutions, similar to those in Lemma 3.1, can be ascertained. These required properties are given by Jones in [*2*]. We specifically note that if $-1 < f(0) < 1$, then for all $t \geqslant z_1 + 1$, the solution $x(f)$ of (3.6) is bounded above and below by $\beta = (e^{2\alpha} - 1)/(e^{2\alpha} + 1) < 1$.

It is not possible to use the cone K of (3.2) with the operator A of (3.3) for equation (3.6), for if k in K has $k(0) > 1$, then there is a finite escape time for $x(k)$. Hence A is not defined for all of K. Moreover, the element $k_1(\theta) = 1$ $(-1 \leqslant \theta \leqslant 0)$, is in K, and is a fixed point of A. If the cone K of (3.2) is still to be used, the operator A must be modified.

Note that the operator A of (3.3) is defined on $K \cap B(\beta)$. If the operator A^* is defined by, A^*k when $\| k \| \leqslant \beta$ and $A(\beta k/\| k \|)$ if $\| k \| > \beta$, then A^* is positive with respect to K and satisfies the requirements of the positive operator in (2.2) and I of Theorem 2.1. It satisfies II as well, with $M = \beta$ for then $\| A^*k \| < \| k \|$ if $\| k \| \geqslant \beta$. Notice that this implies that any fixed point of A^* must have norm less than β, hence is a fixed point of A, since $A^* = A$ then. That K fulfills condition III has already been shown. Thus there is a fixed point k^* of A in K.

Example 3. The van der Pol equation with retardation in the "spring" term is:

$$\ddot{x}(t) + \epsilon(x^2(t) - 1)\,\dot{x}(t) + x(t-r) = 0, \qquad \epsilon > 0, \quad r > 0. \tag{3.7}$$

If $r = 0$, then (3.7) is the well known van der Pol equation which has a stable, unique periodic solution for all $\epsilon > 0$. We shall extend this result to show that (3.7) has a nontrivial periodic solution for all $\epsilon > 0$ and all $r > 0$. To do this, we put equation (3.7) in the form

$$\dot{w}(t) = \begin{cases} \dot{x}(t) = y(t) - F(x(t)), & F(x) = \epsilon(x^3/3 - x) \\ \dot{y}(t) = -x(t-r). \end{cases} \tag{3.8}$$

Let $g = (f, a)^T$, (T designates transpose), be a vector where f is a function in $C([-r, 0], E^1)$ and a is in E^1. Then the initial value problem for (3.8) consists of finding a solution $w(g) = (x(g), y(g))^T$ such that $x(g)(t) = f(t)$ for t in $[-r, 0]$, and $y(g)(0) = a$. Accordingly, we let $w(g)_t = [x(g)_t, y(g)(t)]^T$ where $x(g)_0 = f$, $y(0) = a$. We do not need to use the Banach space $C = C([-r, 0], E^2)$ which was used in proving Theorem 2.1, since an initial function in C contains more information than is necessary to solve the initial value problem for (3.8). The theorem will still be valid if we use the Banach space,

$$C_0 = \{g = (f, a)^T, f \in C([-r, 0], E^1), a \in E^1\}.$$

We will now prove:

COROLLARY 3.3. *For all values ϵ, r such that $\epsilon > 0$, $r > 0$, there exists at least one nontrivial periodic solution of equation* (3.8) *with period greater than* $2r$.

Proof of Corollary 3.3. It is not difficult to show that (3.8) is an equation of the type considered in Theorem 2.1. However, we need information about the roots of the characteristic equation for the linear part of (3.8), which is,

$$\lambda^2 - \epsilon\lambda + e^{-r\lambda} = 0. \tag{3.9}$$

LEMMA 3.6. *For all $\epsilon \geq 0, r > 0$ there exists at least one complex conjugate pair $\lambda_0 = \sigma_0 + i\gamma_0$, $\bar{\lambda}_0$, of roots of* (3.9) *such that $\sigma_0 > 0$ and $0 < \gamma_0 < \pi/r$.*

Proof. Only the idea of the proof is given, since a verification of the details is a lengthy, but straightforward task. Let $\lambda = \sigma + i\gamma$, and represent the real and imaginary parts of the function in (3.9) by $R(\sigma, \gamma)$ and $I(\sigma, \gamma)$. Then these functions form continuous surfaces in $(\sigma, \gamma, \cdot)$ space. It can then be shown that the surfaces cut the $(\sigma, \gamma, 0)$ plane in continuous curves, and that the two curves intersect each other but once in the region $\sigma > 0$, $0 < \gamma < \pi/r$. The point of intersection (σ_0, γ_0) is simultaneously a root of $R(\cdot, \cdot)$ and $I(\cdot, \cdot)$. Hence $\lambda_0 = \sigma_0 + i\gamma_0$ is a root of (3.9).

We next turn our attention to finding a cone and an operator as required by Theorem 2.1. It turns out that a satisfactory cone is,

$$K_0 = \{k = (k_1(\theta), k_2)^T : 0 = k_1(-r) \\ \leqslant k_1(\theta_1) \leqslant k_1(\theta_2), -r \leqslant \theta_1 < \theta_2 \leqslant 0; k_2 \geqslant 0\}.$$

Notice that the function k_1 has the monotonicity properties of the elements of the cone K of (3.2) in the previous examples. It has the additional property that the function is zero at the left hand end. Before we can define an operator, we need the following properties of the solution of (3.8).

LEMMA 3.7. *For any k in K_0, the solution $w(k)$ of* (3.8) *exists for all $t \geqslant 0$ and has the following properties.*

(i) *The solution $w(k)$ is oscillatory, that is, both $x(k)$ and $y(k)$ change sign for arbitrarily large t.*

(ii) *At $t = 0$, $w(k)_0$ is in K_0. The solution path then leaves K_0 and within a finite time greater than r enters the cone $-K_0 = \{-k : k \in K_0\}$.*

(iii) *If the solution is in $-K_0$, it leaves that cone and within a finite time greater than r enters the cone K_0.*

Proof. Equations (3.8) are symmetrical in the sense that if $w(k)$ is a solution, then $-w(k) = w(-k)$ is also a solution. Clearly if (ii) and (iii) hold, then $w(k)$ exists for all $t \geqslant 0$. Moreover, if (ii) holds then, by the symmetry of (3.8), (iii) holds. These together prove (i). Accordingly, we shall only prove that item (ii) is true.

Let the projection of a point $g = (f, a)^T$ in C_0 into E^2 be the pair $(f(0), a)$. If the solution $w(g)$ of (3.8) is defined for t in $[0, a)$, then the projection in E^2 of the set of functions $\{w(g)_t : 0 \leqslant t < a\}$ is the set

$$\{[x(g)_t(0), y(g)(t)] : 0 \leqslant t < a\},$$

which shall be called the projection of $w(g)$.

To show that (ii) holds, we will analyze the behavior of the projections of $w(k)$ where k is in K_0. Figure 3.1 shows the (x, y) plane and the projection of a solution path for the case where the projection of the initial function k is located above the curve $y = F(x)$. We shall next show that the projection of solutions have the aspect of the curve $0 - 1 - 2 - 3 - 4$ in Figure 3.1. The point 0 corresponds to the projection of k. Let t_j be the value of t at the points 1, 2, 3 and 4 of Figure 3.1: $t_1 =$ the time at which the projection first crosses the curve $y = F(x)$, $t_2 =$ the first time $y(k)(t) = 0$, $t_3 =$ the first time $x(k)(t) = 0$, $t_4 = t_3 + r$. Time t increases in the direction of the arrow. We thus have $0 < t_1 < t_2 < t_3 < t_4$ for the projection shown. If the projection crosses the x axis at a point x in $(0, \sqrt{3}]$, then $t_2 \leqslant t_1 < t_3$. We shall assume k is fixed and write $x(t)$ in lieu of $x(k)(t)$, etc.

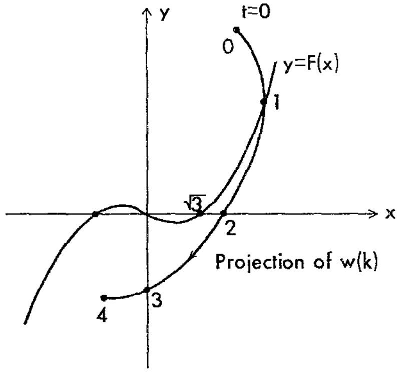

FIG. 3.1. Projection of a solution to (3.8).

Notice that the projection of the cone K_0 into E^2 is in the quadrant $x \geqslant 0$, $y \geqslant 0$. We shall assume that the projection $(k_1(0), k_2)$ of the initial function k is above the curve $y = F(x)$. If $(k_1(0), k_2)$ is located on or below the curve $y = F(x)$ then the times t_2, t_3 and t_4 are still well defined, as will be evident from the analysis.

Between the points 0 and 1, we have for t in $[0, t_1)$,

$$\dot{x}(t) = y(t) - F(x(t)) > 0,$$

and $\dot{y}(t) = -x(t-r) \leqslant 0$. Now, t_1 is finite. If it were not, then we would have $\dot{x}(t) > 0$ and $\dot{y}(t) < -x(r) < 0$ for all $t \geqslant 2r$. This means that the projection must cross the curve $y = F(x)$ at a finite time, t_1.

Between the points 1 and 4 we have $\dot{x}(t) = y(t) - F(x(t)) < 0$, and $\dot{y}(t) = -x(t-r) < 0$, for $t_1 < t < t_3 + r = t_4$. Once the projection crosses the curve $y = F(x)$, it cannot intersect it again until after t_3. Suppose it did at a time t_0, $t_1 < t_0 \leqslant t_3$. Then the intersection must be a crossing because at time t_0 the slope of the projection dy/dx is $-\infty$. Since the crossing is from below to above, $\dot{y}$ must be positive at t_0. This is a contradiction, since $x(t)$ is nonnegative for t in $[-r, t_0]$. Because of the above behavior of the projections, $\dot{x}(t)$ is bounded away from zero for all t in $[t_2, t_3]$. This implies that the projection crosses the vertical line $x = 0$ at a finite time, t_3. Now the point 4 is under the curve $y = F(x)$ and has y coordinate negative. If the projection were to cross the curve at some time t^*, $t_3 < t^* < t_4$ then $\dot{y}(t^*) > 0$ because the projection must cross the curve from underneath. However, on $t_3 \leqslant t < t_4$, $\dot{y}(t) < 0$. The projection therefore has the aspect shown between the points 1 and 4 of Figure 3.1. It is now evident that $w(k)_{t_4}$ is in $-K_0$. This proves Lemma 3.7.

The following corollary to Lemma 3.7 gives additional necessary information concerning the behavior of the projections of solutions.

COROLLARY. *For* $k \in K_0$, *assume* $\| k \| \geqslant m > \sqrt{3}$. *Then there exists an* $a > 0$ *such that* $\inf | \dot{x}(k)(t)| \geqslant a > 0$ *for* $t \in [t_2, t_3]$.

Proof. By Lemma 3.7, the projections of solutions have the aspect shown in Figure 3.1. Assume that there is a sequence $\{k_j\}$ in K_0 with $\| k_j \| \geqslant m$, such that $\inf | \dot{x}(k_j)(t)| = \inf | y(k_j)(t) - F(x(k_j)(t))| = 0$ where the infimum is over all j and all t in $[t_2, t_3]$. In view of Lemma 3.7, the only possible way the projections could do this is for the sequence of projections to approach the curve $y = F(x)$ tangentially, but not cross it. In that case, the slope of the projection must be between 0 and 2ϵ, the range of slopes of the curve $y = F(x)$ where the projections could approach it. However, by (3.8), the projection slope, dy/dx, must approach ∞ as a projection curve approaches $y = F(x)$. This proves the corollary.

The above lemma suggests that we define an operator on K_0 as follows,

$$Ak = w(k)_{\tau(k)}, \qquad \tau(k) \overset{\text{def}}{=} t_3(k) + r, \qquad k \in K_0, \tag{3.10}$$

where $t_3(k)$ is the first point at which $x(k) = 0$, $\dot{x}(k) < 0$, and $y(k) < 0$. Lemma 3.7 shows that $t_3(k)$ exists for all $k \neq \{0\}$ in K_0, and thus that the operator A defines a mapping of K_0 into $-K_0$. The symmetry of (3.8) suggests that we consider the operator $(-A)$ defined by $(-A)\,k = -Ak$. Clearly $(-A)$ maps K_0 into K_0. Moreover, if ϕ is a fixed point of $(-A)$ we have: $(-A)\,\phi = \phi$, $A\phi = -\phi$, and $A^2\phi = A(-\phi) = -A\phi = \phi$; so that ϕ is also a fixed point of A^2. However, A^2 is a mapping of K_0 into itself by equation (3.8). Therefore we will investigate the operator A and show that $(-A)$ has a fixed point in K_0.

That $\tau(k)$ is continuous on K_0 and uniformly bounded for all k in $K_0 \cap B(\delta)$ in this case may be shown by a method similar to that used in the first example. Hence details are not reproduced here.

The method of showing that condition I of Theorem 2.1 is satisfied is quite similar to that used in proving Lemma 3.3. Thus the details of showing that $\inf \|(-A)\,k\| > 0$ for $k \in \partial G \cap K_0$ are not given.

We will not show that condition II of Theorem 2.1 is fulfilled for $(-A)$, but will show directly that condition IV in Lemma 2.2 holds. Recall that condition II was only used to show that condition IV was true.

LEMMA 3.8. *If ϕ is an eigenfunction of $(-A)$, then there exists an $M < \infty$ such that if $\|\phi\| > M$, the associated eigenvalue μ satisfies $0 < \mu < 1$.*

Proof. Suppose $\phi \in K_0$ is an eigenfunction of $(-A)$ and the projection of ϕ is on or below the curve $y = F(x)$. The projection of $A\phi$ lies below the curve $y = F(x)$ in the quadrant $x < 0$, $y < 0$; thus the projection of $(-A)\phi$ will lie above the curve $y = F(x)$ in the quadrant $x > 0$, $y > 0$, by the symmetry of (3.8). Since the projections of ϕ and $(-A)\phi = \mu\phi$ both lie on a straight line from the origin of E^2 and are contained in the quadrant $x > 0, y > 0$, the projection of $\mu\phi$ is between 0 and the projection of ϕ. If the vector norm is the usual Euclidean norm, then the projection of the set in K of elements with the same norm is a circle in E^2. Therefore $0 < \mu < 1$ and the lemma is true for all eigenfunctions which have projections lying on or below the curve $y = F(x)$.

Let t_j, $j = 1, 2, 3, 4$, be the times defined in the proof of Lemma 3.7. Assume ϕ is an eigenfunction of $(-A)$ and the projection of ϕ is above the curve $y = F(x)$. Then by considering only the scaler component of $(-A)\phi = \mu\phi$, namely $-y(\phi)(t_4) = \mu\phi_2$, it can be shown that $0 < \mu < 1$. Let $m > \sqrt{3}$ and a be as in the Corollary to Lemma 3.7 and $\|\phi\| \geqslant m$. We obtain the following estimates from the equations for $\dot{x}$ and $\dot{y}$. (The argument ϕ is omitted.)

$$t_3 - t_2 = \int_{t_2}^{t_3} dt = \int_{x(t_2)}^{0} \frac{dx}{y - F(x)} < x(t_2)/a$$

$$|y(t_4)| = \int_{t_2}^{t_4} x(t - r)\,dt < x(t_2)(t_3 - t_2 + r).$$

Let $b = r + 1/a$, and note that $x(t_2) > \sqrt{3}$, then $|y(t_4)| < b[x(t_2)]^2$. Now, the inverse of $F(x)$, $F^{-1}(y)$, is a well defined, monotone increasing function as long as its range is restricted to $[\sqrt{3}, \infty)$; which will be our case. There is a number $c > 0$ such that $b[F^{-1}(y)]^2 < y$ for all $y > c$. We may assume that m is chosen so that $\|\phi\| > m$ assures that the second component of ϕ, $\phi_2 > c$. Since $x(t_2) > \sqrt{3}$, there is a number $y_2 > 0$ such that $x(t_2) = F^{-1}(y_2)$. Because the projections have the aspect shown in Figure 3.1, $y_2 < \phi_2$. These considerations permit us to write the inequality:

$$|y(t_4)| < b[x(t_2)]^2 = b[F^{-1}(y_2)]^2 < b[F^{-1}(\phi_2)]^2 < \phi_2,$$

which was needed to complete the proof of Lemma 3.8.

That condition III of Theorem 2.1 is fulfilled by K_0 can be shown by a method similar to that used to prove Lemma 3.4, since the condition on the eigenvalue λ_0 of Lemma 3.6, $0 < \gamma_0 < \pi/r$, holds. Hence details are not presented.

It has been shown that Theorem 2.1 applied to the operator $(-A)$ assures that there is a fixed point of $(-A)$ in K_0. The fixed point is also a fixed point of A^2, which implies there is a periodic solution of (3.8). Since the projection of $w(k)$ takes at least time r to go from K_0 to $-K_0$, the period of the periodic solution is greater than $2r$. This proves Corollary 3.3.

A companion example to equation (3.7) is the equation

$$\ddot{y}(t) + (y^2(t) - \epsilon)\,\dot{y}(t) + y(t - r) = 0, \tag{3.11}$$

which was obtained from (3.7) by the change of variable $x = y/\sqrt{\epsilon}$. The methods used to prove that (3.7) has a periodic solution for all $r > 0$ and $\epsilon > 0$ carry over without change to (3.11); and the analysis holds for $r > 0$, $\epsilon \geqslant 0$. Thus (3.11) has a nontrivial periodic solution, $y^*(\epsilon)$, for all $\epsilon \geqslant 0$. Therefore, if we fix $r > 0$ and let $\| y^*(\epsilon)\|$ be the maximum value of the amplitude of $y^*(\epsilon)$, then there exists a number $a > 0$ such that $\| y^*(\epsilon)\| \geqslant a > 0$ for $0 \leqslant \epsilon \leqslant 1$. Thus the periodic solution $x^*(\epsilon)$ of (3.7) has the property that for fixed $r > 0$, $\| x^*(\epsilon)\| = \| y^*(\epsilon)\|/(\epsilon)^{1/2} \geqslant a/(\epsilon)^{1/2}$, $0 < \epsilon \leqslant 1$. Therefore the amplitude of $x^*(\epsilon)$ tends to ∞ as ϵ tends to 0, providing r is fixed.

This behavior is of course quite at variance with the behavior of the van der Pol equation, (3.7) with $r = 0$. In that equation the amplitude of the periodic solution is close to two for small ϵ. The reason for this behavior is that in (3.7) as $\epsilon \to 0$ with $r > 0$ fixed, the equation tends to the linear equation,

$$\ddot{x}(t) + x(t - r) = 0, \tag{3.12}$$

which by Lemma 3.6 has at least one pair of eigenvalues with positive real part. These give rise to a solution of (3.12) which is an expontially increasing oscillation. The nonlinear term, $\epsilon(x^2 - 1)\,\dot{x}$, which is added to (3.12) to form (3.7) must overcome the effect of the exponentially increasing oscillations. It forces the oscillation back upon itself to form a periodic solution.

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