Threshold and Stability Results for an Age-Structured SEIR Epidemic Model

XUE-ZHI LI
Department of Mathematics, Xinyang Teachers College
Henan 464000, P.R. China
and
Institute of Systems Science, Academia Sinica
Beijing 100080, P.R. China
xzli@iss06.iss.ac.cn

GENI GUPUR
Department of Mathematics, Xinjiang University
Urumqi 830046, P.R. China
and
Institute of Systems Science, Academia Sinica
Beijing 100080, P.R. China
geni@iss06.iss.ac.cn

GUANG-TIAN ZHU
Institute of Systems Science, Academia Sinica
Beijing 100080, P.R. China
zhught@iss06.iss.ac.cn

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Abstract—In this paper, an age-structured SEIR epidemic model is considered. The existence and uniqueness of a positive solution for this model is proved. Threshold results for the existence of endemic states are established under certain conditions. The local stability for the steady states is also examined. © 2001 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

Since the fact that the age structure of a population affects the dynamics of disease transmission was recognized [1–3], various age-structured epidemic models have been investigated by many authors, and a number of papers have been published on finding the threshold conditions for the disease to become endemic, describing the stability of steady-state solutions, and analyzing the global behavior of these age-structured epidemic models (see [4–17]). We may find that the epidemic models that most authors discussed mainly include S-I-S [4–8] and S-I-R [9–14], that...
is, the total population of a country or a district was subdivided into two or three compartments containing susceptibles, infectives, or immunes, it was assumed that there is no latent class, so a person who catches the disease becomes infectious instantaneously. In fact, many diseases, such as measles and mumps, have a latent period, a period during which individuals are exposed to the disease but are not yet infectious. Therefore, it is necessary and also it is of practical significance to consider the dynamic behavior about the age-structured epidemic model with latent period (i.e., the age-structured SEIR epidemic model). We note that the SEIR-type age-independent epidemic models have already been investigated by many authors (see [18–20]) and their threshold theorems are well obtained. However, the results about age-structured SEIR epidemic models are comparatively scarce. We also notice that most existing results in the literature (cf. [6–9]) are based on the assumption of a separable age-dependent transmission coefficient such that \( \beta(a,b) = \beta_1(a)\beta_2(b) \). Such a separable assumption has been proved to be equivalent to the assumption of a random mixing, which is less realistic. In this paper, there is assumed to be a nonseparable mixing function which is more general. This may cause some difficulties in the analysis of the model, especially in proving the stability results and the existence of an endemic steady state. To our knowledge, only a few works have been done with such mixing functions in the literature. The main purpose of this paper is to consider the age-structured SEIR epidemic model with a nonseparable age-dependent transmission coefficient in a constant-sized population. By using the semigroup theory, spectrum theory, and positive cone theory in functional analysis, we first establish the existence and uniqueness of nonnegative solutions to ensure that the model equations are well-posed. Second, we obtain threshold conditions in the sense that if the spectral radius of the operator \( T \), \( r(T) \), is less than 1, the zero solution is the only nonnegative solution, and if \( r(T) > 1 \), there exists a unique positive steady-state solution, by investigating the existence of such steady-state solutions. Finally, we complete the stability analysis in Theorem 5.8 so that \( r(T) \) is the reproductive number such that if \( r(T) < 1 \), the infection-free equilibrium is locally asymptotically stable, and if \( r(T) > 1 \), the infection-free equilibrium is unstable and a positive endemic solution appears, which is locally asymptotically stable under a certain condition given in Assumption 5.4. In a sense, we have examined that Greenhalgh’s conjectures [11] for an age-structured SIR-type epidemic model are still valid for the age-structured SEIR model.

2. THE MODEL

We subdivide a closed population into four compartments containing susceptible, exposed, infective, and recovered individuals. We assume that the population is in a stationary demographic state. Let \( N(a), 0 \leq a \leq r_m \) \((r_m \) denotes the highest age attained by the individuals in the population) be the density with respect to the age of the total number of individuals. Under our assumptions, \( N(a) \) satisfies

\[
N(a) = \mu^* N \exp \left( - \int_0^a \mu(\sigma) \, d\sigma \right),
\]

where \( \mu(a) \) denotes the instantaneous death rate at age \( a \) of the population, the constant \( N \) is the total size of the population, and \( \mu^* \) is the crude death rate. We assume that \( \mu(a) \) is nonnegative, locally integrable on \([0, r_m)\), and satisfies

\[
\int_0^{r_m} \mu(\sigma) \, d\sigma = +\infty.
\]

The crude death rate is determined such that

\[
\mu^* \int_0^{r_m} f(a) \, da = 1,
\]
where \( f(a) = \exp\left(- \int_0^a \mu(\sigma) \, d\sigma \right) \) is the survival function which is the proportion of individuals who survive to age \( a \). Then we have the relation 
\[
N(a) = \mu^* N f(a). 
\] (2.2)

Next let \( S(a, t) \), \( E(a, t) \), \( I(a, t) \), and \( R(a, t) \) be the age-densities of the susceptible, the latent, the infected, and the immune population, respectively, at time \( t \), so that 
\[
N(a) = S(a, t) + E(a, t) + I(a, t) + R(a, t). 
\] (2.3)

Let \( \alpha^{-1} \) and \( \gamma^{-1} \) be the average latent period and the average infectious period, respectively. Let \( \beta(a, b) \) be the age-dependent transmission coefficient, that is, the probability that a susceptible person of age \( a \) meets an infectious person of age \( b \) and becomes latent, per unit of time. Define the force of infectious \( \lambda(a, t) \) by 
\[
\lambda(a, t) = \int_0^r \beta(a, \sigma) I(\sigma, t) \, d\sigma. 
\] (2.4)

Then the transmission from the susceptible to the latent state is a Poisson process, i.e., the probability that a susceptible individual becomes latent during the small interval \((a, a + da)\) at time \( t \) is \( \lambda(a, t) \, da \). Moreover, we assume that the death rate of the population is not affected by the presence of the disease. Under the above assumption, the spread of the disease can be described by the system of partial differential equations:
\[
\begin{align*}
\frac{\partial S(a, t)}{\partial t} + \frac{\partial S(a, t)}{\partial a} &= -(\mu(a) + \lambda(a, t))S(a, t), \\
\frac{\partial E(a, t)}{\partial t} + \frac{\partial E(a, t)}{\partial a} &= \lambda(a, t)S(a, t) - (\mu(a) + \alpha)E(a, t), \\
\frac{\partial I(a, t)}{\partial t} + \frac{\partial I(a, t)}{\partial a} &= \alpha E(a, t) - (\mu(a) + \gamma)I(a, t), \\
\frac{\partial R(a, t)}{\partial t} + \frac{\partial R(a, t)}{\partial a} &= \gamma I(a, t) - \mu(a)R(a, t),
\end{align*}
\] (2.5a-d)

with boundary conditions
\[
S(0, t) = \mu^* N, \quad E(0, t) = 0, \quad I(0, t) = 0, \quad R(0, t) = 0. 
\] (2.6)

Consider the fractions of susceptible, latent, infectious, and immune population at age \( a \) and time \( t \):
\[
s(a, t) = \frac{S(a, t)}{N(a)}, \quad e(a, t) = \frac{E(a, t)}{N(a)}, \quad i(a, t) = \frac{I(a, t)}{N(a)}, \quad r(a, t) = \frac{R(a, t)}{N(a)}. 
\]

Then system (2.5a)-(2.5d) can be written in a simpler form
\[
\begin{align*}
\frac{\partial s(a, t)}{\partial t} + \frac{\partial s(a, t)}{\partial a} &= -\lambda(a, t)s(a, t), \\
\frac{\partial e(a, t)}{\partial t} + \frac{\partial e(a, t)}{\partial a} &= \lambda(a, t)s(a, t) - \alpha e(a, t), \\
\frac{\partial i(a, t)}{\partial t} + \frac{\partial i(a, t)}{\partial a} &= \alpha e(a, t) - \gamma i(a, t), \\
\frac{\partial r(a, t)}{\partial t} + \frac{\partial r(a, t)}{\partial a} &= \gamma i(a, t),
\end{align*}
\] (2.7a-d)

where
\[
\lambda(a, t) = \int_0^r \beta(a, \sigma) N(\sigma) i(\sigma, t) \, d\sigma, \quad N(a) = \mu^* N f(a), 
\] (2.9)

\[
s(a, t) + e(a, t) + i(a, t) + r(a, t) = 1. 
\] (2.10)

In the following, we mainly consider system (2.7a)-(2.7d) with the initial conditions
\[
s(a, 0) = s_0(a), \quad e(a, 0) = e_0(a), \quad i(a, 0) = i_0(a), \quad r(a, 0) = r_0(a). 
\] (2.11)
In this section, we shall show that the initial-boundary value problem (2.7a)-(2.7d), (2.8), and (2.11) has a unique solution. First we note that it suffices to consider the system in terms of only $s(a, t)$, $e(a, t)$, and $i(a, t)$ since, once these functions are known, $r(a, t)$ can be obtained by $r(a, t) = 1 - s(a, t) - e(a, t) - i(a, t)$.

First we introduce a new variable $\tilde{s}$ by $\tilde{s}(a, t) = s(a, t) - 1$. Then we obtain the new system for $\tilde{s}(a, t)$, $e(a, t)$, and $i(a, t)$,

\begin{align}
\frac{\partial \tilde{s}(a, t)}{\partial t} + \frac{\partial \tilde{s}(a, t)}{\partial a} &= -\lambda(a, t)(\tilde{s}(a, t) + 1), \\
\frac{\partial e(a, t)}{\partial t} + \frac{\partial e(a, t)}{\partial a} &= \lambda(a, t)(\tilde{s}(a, t) + 1) - \alpha e(a, t), \\
\frac{\partial i(a, t)}{\partial t} + \frac{\partial i(a, t)}{\partial a} &= \alpha e(a, t) - \gamma i(a, t),
\end{align}

(3.1a) (3.1b) (3.1c)

$\tilde{s}(0, t) = 0$, $e(0, t) = 0$, $i(0, t) = 0$.

Let us consider the initial-boundary value problem of the system composed of (3.1a)-(3.1c) as an abstract Cauchy problem on the Banach space $X = L^1(0, r_m) \times L^1(0, r_m) \times L^1(0, r_m)$ with the norm $\|\psi\| = \sum_{i=1}^{3} \|\phi_i\|_1$ for $\phi(a) = (\phi_1(a), \phi_2(a), \phi_3(a))^T \in X$, where $\|\cdot\|_1$ is the ordinary norm of $L^1(0, r_m)$. Let $A$ be a linear operator defined by

\begin{equation}
(A\phi)(a) = \begin{pmatrix}
- \frac{d\phi_1(a)}{da} \\
- \frac{d\phi_2(a)}{da} - \alpha \phi_2(a) \\
- \frac{d\phi_3(a)}{da} - \gamma \phi_3(a)
\end{pmatrix},
\end{equation}

(3.2)

$\phi(a) = (\phi_1(a), \phi_2(a), \phi_3(a))^T \in D(A)$,

where $p^T$ is the transpose of the vector $p$ and the domain $D(A)$ is given as

$D(A) = \{\phi \in X \mid \phi_i$ is absolutely continuous on $[0, r_m)$, $\phi(0) = (0, 0, 0)^T\}$.\)

Suppose that $\beta(a, b) \in L^\infty((0, r_m) \times (0, r_m))$. We define a nonlinear operator $F : X \to X$ by

\begin{equation}
(F\phi)(a) = \begin{pmatrix}
- (P\phi_3)(a)(1 + \phi_1(a)) \\
(P\phi_3)(a)(1 + \phi_1(a)) \\
\alpha \phi_2(a)
\end{pmatrix}, \quad \phi \in X,
\end{equation}

(3.3)

where $P$ is a bound linear operator on $L^1(0, r_m)$ given by

\begin{equation}
(Pf)(a) = \int_0^{r_m} \beta(a, \sigma)N(\sigma)f(\sigma) \, d\sigma.
\end{equation}

(3.4)

Note that $Pf \in L^\infty(0, r_m)$ for $f \in L^1(0, r_m)$, and hence, the nonlinear operator $F$ is defined on the whole space $X$. Let $u(t) = (\tilde{s}(\cdot, t), e(\cdot, t), i(\cdot, t))^T \in X$. Then we can rewrite the initial-boundary value problem (3.1a)-(3.1c) as the abstract semilinear initial value problem in $X$:

\begin{equation}
\frac{d}{dt}u(t) = Au(t) + F(u(t)), \quad u(0) = u_0 \in X,
\end{equation}

(3.5)
where \( u_0(a) = (\hat{s}_0(a), e_0(a), i_0(a))^T, \hat{s}_0(a) = s_0(a) - 1 \). It is easily seen that the operator \( A \) is the infinitesimal generator of \( C_0 \)-semigroup \( T(t), t \geq 0 \), and \( F \) is continuously Frechet differentiable on \( X \). Then for each \( u_0 \in X \), there exists a maximal interval of existence \([0,t_0)\), and a unique continuous mild solution \( t \mapsto u(t,u_0) \) from \([0,t_0)\) to \( X \) such that

\[
    u(t,u_0) = T(t)u_0 + \int_0^t T(t-s)F(u(s,u_0)) \, ds, \tag{3.6}
\]

for all \( t \in [0, t_0) \) and either \( t_0 = +\infty \) or \( \lim_{t \to t_0^-} \|u(t,u_0)\| = \infty \). Moreover, if \( u_0 \in D(A) \), then \( u(t,u_0) \in D(A) \) for \( 0 \leq t < t_0 \) and the function \( t \mapsto u(t,u_0) \) is continuously differentiable and satisfies (3.5) on \([0,t_0)\) (see [14, p. 194, Proposition 4.16]).

**Lemma 3.1.** Let \( \Omega = \{(\hat{s}, e, i) \in X \mid \hat{s} \geq -1, e \geq 0, i \geq 0\} \) and let \( \hat{\Omega}_0 = \{(\hat{s}, e, i) \in X \mid -1 \leq \hat{s} \leq 0, 0 \leq e \leq 1, 0 \leq i \leq 1\} \). Then the mild solution \( u(t,u_0), u_0 \in \Omega \) of (3.5) enters \( \Omega_0 \) after a finite time and the set \( \Omega_0 \) is positively invariant.

**Proof.** From (2.7a), we have the representation

\[
    s(a,t) = \begin{cases} 
    \exp \left( -\int_0^a \lambda(\rho, t-a+\rho) \, d\rho \right), & t > a, \\
    s_0(a-t) \exp \left( -\int_0^a \lambda(a-t+\rho, \rho) \, d\rho \right), & a > t,
    \end{cases}
\]

which shows that \( \hat{s}(a, t) \geq -1 \) when \( s_0(a) \geq 0 \). From (3.1c), we get

\[
    i(a, t) = \begin{cases} 
    \alpha \int_0^a \rho e(\rho, t-a+\rho) \exp \left( -\gamma(\rho-a) \right) \, d\rho, & t > a, \\
    i_0(a-t) \exp \left( -\gamma t \right) + \alpha \int_0^t \rho e(a-t+\rho, \rho) \exp \left( -\gamma(t-\rho) \right) \, d\rho, & t < a.
    \end{cases}
\]

Substituting (3.8) into (2.9) and then defining

\[
    \lambda(a, t) = (Qe)(a, t),
\]

where \( Q \) is a transformation from \( e(a, t) \) to \( \lambda(a, t) \). Then, we can write (3.1b) as an abstract Cauchy problem

\[
    \frac{de(t)}{dt} = Be(t) + (Qe(t)) \left( 1 + \hat{s}(t) \right), \quad e(0) = e_0 \in L^1(0, r_m),
\]

where the operator \( B \) is defined by

\[
    B = -\frac{d}{d\alpha} - \alpha, \\
    D(B) = \{ \phi \in L^1(0, r_m) \mid \phi \text{ is absolutely continuous on } [0, r_m) \text{ and } \phi(0) = 0 \}. 
\]

Then we obtain

\[
    e(t) = S(t)e(0) + \int_0^t S(t-\tau)(Qe(\tau)) \left( 1 + \hat{s}(\tau) \right) \, d\tau, \tag{3.9}
\]

where \( S(t) = \exp(tB) \) is the positive \( C_0 \)-semigroup generated by the closed operator \( B \). If we assume that \( \hat{s}(t) \geq -1, e_0 \geq 0, \) and \( i_0 \geq 0 \), (3.9) shows that \( e(t) \) is also positive because \( S(t), t \geq 0, \) and \( Q \) are positive, and \( e(t) \) can be obtained by monotone iteration

\[
    e_0(t) = e_0, \\
    e_{n+1}(t) = S(t)e_0 + \int_0^t S(t-\tau)(QS(\tau)e_n(\tau)) \left( 1 + \hat{s}(\tau) \right) \, d\tau, \quad n = 0, 1, 2, \ldots.
\]
Consequently, thanks to the positivity of $e(t)$ and (3.8), we know that $i(t)$ is also positive. Hence, we obtain that $u(t, u_0) \in \Omega$ for all $t \geq 0$ when $u_0 \in \Omega$. Next, let $w(t) = \dot{s}(t) + e(t) + i(t)$. Then we have

\[
\frac{dw(t)}{dt} = Cw(t) - \gamma i(t), \quad w(0) = \dot{s}_0(a) + e_0(a) + i_0(a) \in L^1(0, r_m),
\]

where the operator $C$ is given by

\[
C = -\frac{d}{da},
\]

\[
D(C) = \{ \phi \in L^1(0, r_m) \mid \phi \text{ is absolutely continuous on } [0, r_m] \text{ and } \phi(0) = 0 \}.
\]

From (3.10), it follows that

\[
w(t) = U(t)w(0) - \int_0^t U(t - s) \gamma i(s) ds \leq U(t)w(0),
\]

where $U(t)$, $t \geq 0$ is the positive $C_0$-semigroup generated by the operator $C$. Since $U(t)$ is a nilpotent translation semigroup, we have $w(t)(a) \leq \dot{s}_0(a - t) + e_0(a - t) + i_0(a - t)$, $a > t$, and $w(t) \leq 0$ for $t \geq r_m$. Then it follows that the mild solution $u(t, u_0)$, $u_0 \in \Omega$ enters $\Omega_0$ for $t \geq \omega$, and if $u_0 \in \Omega_0$, then $u(t, u_0) \in \Omega_0$ for all $t \geq 0$. This completes the proof of Lemma 3.1.

By the above lemma, we know that the norm of the local solution $u(t, u_0)$, $u_0 \in D(A) \cap \Omega$, of (3.5) is finite as long as it is defined. Thus, we arrive at the following result.

**Theorem 3.2.** The abstract Cauchy problem (3.5) has a unique global classical solution on $X$ with respect to initial data $u_0 \in \Omega \cap D(A)$.

Therefore, it follows immediately that the initial-boundary value problem (2.7)-(2.9) has a unique positive global solution with respect to the positive initial data.

### 4. EXISTENCE OF STEADY STATES

Let $u^* = (s^*(a), e^*(a), i^*(a))$ be the steady-state solution for equation (2.7a)-(2.7d). Then it is easy to verify the following:

\[
s^*(a) = \exp \left( - \int_0^a \lambda^*(\sigma) d\sigma \right),
\]

\[
e^*(a) = \int_0^a \exp(-\alpha(a - \sigma)) \lambda^*(\sigma) \exp \left( - \int_0^\sigma \lambda^*(\eta) d\eta \right) d\sigma,
\]

\[
i^*(a) = \int_0^a \pi(a, \sigma) \lambda^*(\sigma) \exp \left( - \int_0^\sigma \lambda^*(\eta) d\eta \right) d\sigma,
\]

where

\[
\lambda^*(a) = \int_0^{r_m} \beta(a, \sigma) N(\sigma) i^*(\sigma) d\sigma,
\]

\[
\pi(a, \eta) = \alpha \int_\eta^a \exp(-\gamma(a - \sigma)) \exp(-\alpha(\sigma - \eta)) d\sigma.
\]

Substituting (4.1c) into (4.1d) and changing the order of integration, we obtain an equation for $\lambda^*(a)$.

\[
\lambda^*(a) = \int_0^{r_m} \phi(a, \sigma) \lambda^*(\sigma) \exp \left( - \int_0^\sigma \lambda^*(\eta) d\eta \right) d\sigma,
\]

where

\[
\phi(a, \sigma) = \int_\sigma^{r_m} \beta(a, \xi) N(\xi) \pi(\xi, \sigma) d\xi.
\]
From (4.1d), it follows that $|\lambda^*(a)| \leq \mu^* N \|\beta\|_\infty \|\xi^*\|_1$, where $\|\cdot\|_\infty$, $\|\cdot\|_1$ denote an $L^\infty$-norm and an $L^1$-norm, respectively. Then it follows from $\xi^* \in L^1(0, r_m)$ that $\lambda^* \in L^\infty(0, r_m)$. It is clear that one solution of (4.2) is $\lambda^*(a) \equiv 0$, which corresponds to the equilibrium state with no disease. In order to investigate a nontrivial solution for (4.2), we define a nonlinear operator $\Phi(x)$ in the Banach space $X = L^1(0, r_m)$ with the positive cone $X_+ = \{x \in X, x \geq 0, a.e.\}$ by

$$
(\Phi x)(a) = \int_0^{r_m} \phi(a, \sigma) x(\sigma) \exp \left(-\int_0^\sigma x(\eta) \, d\eta \right) \, d\sigma, \quad x \in X.
$$

(4.4)

Since the range of $\Phi$ is included in $L^\infty(0, r_m)$, the solutions of (4.2) correspond to fixed points of the operator $\Phi$. Observe that the operator $\Phi$ has a positive linear majorant $T$ defined by

$$
(T x)(a) = \int_0^{r_m} \phi(a, \sigma) x(\sigma) \, d\sigma, \quad x \in X.
$$

(4.5)

Here we summarize the Perron-Frobenius theory for positive operators in an ordered Banach space as long as it is needed for our purpose. Let $X$ be a real or complex Banach space and let $X^*$ be its dual, i.e., the space of all linear functionals on $X$. The value of $F \in X^*$ at $\psi \in X$ is denoted by $\langle F, \psi \rangle$. A close subset $X_+$ is called a cone if the following hold:

1. $x_+ + x_+ \subseteq x_+$,
2. $\lambda x_+ \subseteq x_+$ for $\lambda \geq 0$,
3. $x_+ \cap (-x_+) = \{0\}$,
4. $x_+ \neq \{0\}$.

We write $x \leq y$ if and only if $y - x \in X_+$ and write $x < y$ if $y - x \in X_+ \setminus \{0\}$. The cone $X_+$ is called total if the set $\{\psi - \phi \mid \psi, \phi \in X_+\}$ is dense in $X_+$. The dual cone $X_+^*$ is the subset of $X^*$ consisting of all positive linear functionals on $X$, i.e., $F \in X_+^*$ if and only if $F \in X^*$ and $\langle F, \psi \rangle \geq 0$ for all $\psi \in X_+$. $\psi \in X_+$ is called a nonsupporting point (or a quasi-interior point) if $\langle F, \psi \rangle > 0$ for all $F \in X_+^* \setminus \{0\}$. A positive linear functional $F \in X_+^*$ is called strictly positive if $\langle F, \psi \rangle > 0$ for all $\psi \in X_+ \setminus \{0\}$. Let $B(X)$ be the set of bounded linear operators of $X$ into $X$. $T \in B(X)$ is called positive with respect to the cone $X_+$ if $T(X_+) \subseteq X_+$. We say $T \geq S$ if $(T - S)X_+ \subseteq X_+$ for $S, T \in B(X)$. We denote the spectral radius of $T \in B(X)$ by $r(T)$.

**Definition 4.1.** A positive operator $T \in B(X)$ is called semi-nonsupporting if and only if for every pair $\psi \in X_+ \setminus \{0\}$, $F \in X_+^* \setminus \{0\}$, there exists a positive integer $p = p(\psi, F)$ such that $\langle F, T^p \psi \rangle > 0$. A positive operator $T \in B(X)$ is called nonsupporting if and only if for every pair $\psi \in X_+ \setminus \{0\}$, $F \in X_+^* \setminus \{0\}$, there exists an integer $p = p(\psi, F)$ such that $\langle F, T^n \psi \rangle > 0$ for all $n \geq p$.

The reader may refer to [21,22] for the proof of the following theorem.

**Theorem 4.2.** Let the cone $X_+$ be total, $T \in B(E)$ be seminonsupporting with respect to $X_+$, and let $r(T)$ be a pole of the resolvent $R(\lambda, T)$. Then the following hold:

1. $r(T) \in \rho(T) \setminus \{0\}$, $r(T)$ is a simple pole of the resolvent.
2. The eigenspace corresponding to $r(T)$ is one dimensional and the corresponding eigenvector $\psi \in X_+$ is a nonsupporting point. The relation $T \phi = \mu \phi$ with $\phi \in X_+$ implies that $\phi = c \psi$ for some constant $c$.
3. The eigenspace of $T^*$ corresponding to $r(T)$ is also a one-dimensional subspace of $X^*$ spanned by a strictly positive functional $F \in X_+^*$.
4. Assume that $X$ is a Banach lattice. If $T \in B(X)$ is nonsupporting, then the peripheral spectrum of $T$ consists only of $r(T)$, i.e., $|\lambda| < r(T)$ for $\lambda \in \sigma(T) \setminus \{r(T)\}$.

The following comparison theorem is due to [23].

**Theorem 4.3.** Suppose that $X$ is a Banach lattice. Let $S$ and $T$ be positive operators in $B(X)$.

1. If $S \leq T$, then $r(S) \leq r(T)$.
2. If $S$ and $T$ are semi-nonsupporting operators, then $S \leq T$, $S \neq T$ implies that $r(S) < r(T)$. 
After the above preparations, we first consider the nature of the majorant operator $T$ defined by (4.5). In the following, we shall make an assumption.

**Assumption 4.4.**

1. $\beta(a, \xi) \in L^\infty_X [(0, r_m) \times (0, r_m)]$.

2. \[ \lim_{h \to 0} \int_0^{r_m} |\beta(a + h, \xi) - \beta(a, \xi)| \, da = 0, \quad \text{uniformly for } \xi \in R. \] (4.6)

   where $\beta$ is extended by $\beta(a, \xi) = 0$ for $a, \xi \in (-\infty, 0) \cup (r_m, +\infty)$.

3. There exist numbers $\kappa$ with $r_m > \kappa > 0$ and $\epsilon > 0$ such that

   \[ \beta(a, \xi) \geq \epsilon, \quad \text{for almost all } (a, \xi) \in (0, r_m) \times (r_m - \kappa, r_m). \] (4.7)

Then we can prove the following.

**Lemma 4.5.** Under Assumption 4.4, the operator $T : X \to X$ is nonsupporting and compact.

**Proof.** Define the positive linear functional $F \in X_+^*$ by

\[ \langle F, \psi \rangle = \int_0^{r_m} g(\sigma) \psi(\sigma) \, d\sigma, \quad \psi \in X, \] (4.8)

where $g(\sigma)$ is given by

\[ g(\sigma) = \int_\sigma^{r_m} s(\xi) N(\xi) \pi(\xi, \sigma) \, d\xi, \] (4.9)

where the function $s(\xi)$ is defined as $s(\xi) = 0$, $\xi \in (0, r_m - \kappa)$, $s(\xi) = \epsilon$, $\xi \in [r_m - \kappa, r_m)$. Hence, $\beta(a, \xi) \geq s(\xi)$ for almost all $(a, \xi) \in (0, r_m) \times (0, r_m)$. Since $g(\sigma) > 0$ for all $\sigma \in [0, r_m)$, the functional $F$ is strictly positive and

\[ \langle F, x \rangle e \leq T x, \quad e = 1 \in X_+, \quad x \in X_+. \]

Then for any integer $n$, we have

\[ T^{n+1} x \geq \langle F, x \rangle \langle F, e \rangle^n e. \]

Therefore, we obtain $\langle G, T^n x \rangle > 0$, $n \geq 1$ for every pair $x \in X_+ \setminus \{0\}$, $G \in X_+^* \setminus \{0\}$, that is, $T$ is nonsupporting. Next observe that

\[
\begin{align*}
\int_0^{r_m} |\phi(a + h, \sigma) - \phi(a, \sigma)| \, da \\
= \int_0^{r_m} \left| \int_\sigma^{r_m} \beta(a + h, \xi) N(\xi) \pi(\xi, \sigma) \, d\xi - \int_\sigma^{r_m} \beta(a, \xi) N(\xi) \pi(\xi, \sigma) \, d\xi \right| \, da \\
= \int_0^{r_m} \left| \int_\sigma^{r_m} \left[ \beta(a + h, \xi) - \beta(a, \xi) \right] N(\xi) \pi(\xi, \sigma) \, d\xi \right| \, da \\
\leq \mu^* \alpha N \sigma r_m \int_0^{r_m} \left| \int_\sigma^{r_m} \left[ \beta(a + h, \xi) - \beta(a, \xi) \right] \, d\xi \right| \, da.
\end{align*}
\] (4.10)

In order to prove the compactness of $T$, we identify the Banach space $X$ with the subspace of $L^1(R)$ such that $X = \{ \psi \in L^1(R) \mid \psi(a) = 0 \text{ for } a \in (-\infty, 0) \cup (r_m, \infty) \}$. Then we can interpret $T$ as an operator on $L^1(R)$ such that $X$ is its invariant subspace, so it is sufficient to
show that the operator $T$ is compact in $L^1(R)$. Let $K$ be a bounded subset of $L^1(R)$. Then it follows immediately that $T(K)$ is also a bounded subset. Observe that

$$\int_R \left| (Tx)(a + h) - (Tx)(a) \right| \, da \leq \int_R \int_R \left| \phi(a + h, \sigma) - \phi(a, \sigma) \right| \, |x(\sigma)| \, d\sigma \, da$$

$$\leq \|x\| \sup_{0 \leq \sigma \leq r_m} \int_R \left| \phi(a + h, \sigma) - \phi(a, \sigma) \right| \, da.$$  

Together with conditions (4.6) and (4.10), it follows that $T(K)$ is an equicontinuous family in $L^1$-norm. Moreover, it follows from $T(K) \subset X$ that

$$\int_{|\sigma| \geq r_m} |(Tx)(\sigma)| \, d\sigma = 0, \quad x \in K.$$

Thus, we can apply the compactness criterion by Frechet-Kolmogorov [24, p. 275], that is, $T(K)$ is relatively compact in $L^1(R)$. Thus, $T$ is a compact operator. This completes the proof.

From Theorem 4.2, it follows that the spectral radius $r(T)$ of operator $T$ is the only positive eigenvalue with a positive eigenvector and also an eigenvalue of the dual operator $T^*$ with a strictly positive eigenfunctional. Now we can prove the following.

**THEOREM 4.6. THRESHOLD RESULTS.** Let $r(T)$ be the spectral radius of the operator $T$ defined by (4.5). Then the following holds.

1. If $r(T) \leq 1$, the only nonnegative solution $x$ of the equation $x = \Phi(x)$ is the trivial solution $x \equiv 0$.
2. If $r(T) > 1$, the equation $x = \Phi(x)$ has at least one nonzero positive solution.

**PROOF.** Suppose that $r(T) \leq 1$. It is easily checked that $Tx - \Phi(x) \in X_+ \setminus \{0\}$ for $x \in X_+ \setminus \{0\}$. If there exists a $x_0 \in X_+ \setminus \{0\}$ being a solution of $x = \Phi(x)$, then $x_0 = \Phi(x_0) \leq T(x_0)$. Let $F^*_0 \in X^*_+ \setminus \{0\}$ be the adjoint eigenvector of $T$ corresponding to $r(T)$. Taking duality pairing, we find

$$\langle F^*_0, T(x_0) - x_0 \rangle = \langle (T^* - I^*) F^*_0, x_0 \rangle = (r(T) - 1) \langle F^*_0, x_0 \rangle > 0,$$

because $T(x_0) - x_0 \in X_+ \setminus \{0\}$ and $F^*_0$ is strictly positive. Then we have $r(T) > 1$, which is a contradiction. This shows that (1) holds.

Next we assume that $r(T) > 1$, under Assumption 4.4, in the same manner as the proof of Lemma 4.5, we can see that the operator $\Phi$ is a completely continuous operator in the Banach space $X$. Moreover, if we define the number $M_0$ by

$$M_0 = \sup_{0 \leq \sigma \leq r_m} \int_0^{r_m} \phi(a, \sigma) \, da,$$

the set $\Omega = \{x \in X \mid 0 \leq x, \|x\| \leq M_0\}$ is invariant (in fact, $\Phi(X_+ \subset \Omega)$ under the operator $\Phi$.

We define an operator $\Phi_r$ by

$$\Phi_r(x) = \begin{cases} \Phi(x), & \text{if } \|x\| \geq r, \quad x \in X_+, \\ \Phi(x) + (r - \|x\|)x_0, & \text{if } \|x\| \leq r, \quad x \in X_+, \end{cases}$$

where $x_0$ is the positive eigenvector of $T$ corresponding to $r(T) > 1$. Then $\Phi_r$ is also completely continuous and transmissions the set $\Omega_r = \{x \in X \mid 0 \leq x, \|x\| \leq M_0 + r\|x_0\|\}$ into itself. Since $\Omega_r$ is bounded, convex, and closed in $X$, $\Phi_r$ has a fixed point $x_r \in \Omega_r$ (Schauder's fixed-point theorem). Observe that the Frechet derivative of $\Phi(x)$ at $x = 0$ is the operator $T$ and $T$ does not have eigenvectors in $X_+$ corresponding to the eigenvalue one. Then we can apply the method of [25, Theorem 4.11], and it can be shown that the norms of these fixed points are greater than $r$ if $r$ is sufficiently small. That is, $\Phi$ has a positive fixed point. This completes the proof.
Subsequently, in order to investigate the uniqueness problem for nontrivial positive fixed points of the operator $\Phi$, we introduce the concept of concave operator (see [25]).

**DEFINITION 4.7.** Let $X_+$ be a cone in a real Banach space $X$ and $\leq$ be the partial ordering defined by $X_+$. A positive operator $A : X_+ \rightarrow X_+$ is called a concave operator if there exists a $u_0 \in X_+ \setminus \{0\}$ which satisfies the following.

1. For any $x \in X_+ \setminus \{0\}$, there exist $\alpha = \alpha(x) > 0$ and $\beta = \beta(x) > 0$ such that $\alpha u_0 \leq Ax \leq \beta u_0$, that is, $Ax$ is comparable with $u_0$.
2. $A(tx) \geq tAx$ for $0 \leq t \leq 1$ and for every $x \in X_+$ such that $\alpha(x)u_0 \leq x \leq \beta(x)u_0$, $(\alpha(x) > 0, \beta(x) > 0)$.

**LEMMA 4.8.** (See [25].) Suppose that the operator $A : X_+ \rightarrow X_+$ is monotone and concave. If for any $x \in X_+$ satisfying $\alpha_1 u_0 \leq x \leq \beta_1 u_0$ ($\alpha_1 = \alpha_1(x) > 0, \beta_1 = \beta_1(x) > 0$), and $0 < t < 1$, there exists $\eta = \eta(x, t) > 0$ such that

$$A(tx) \geq tAx + \eta u_0,$$

then $A$ has at most one positive fixed point.

Here, we make another assumption.

**ASSUMPTION 4.9.** For all $(a, \sigma) \in [0, r_m) \times [0, r_m)$, the inequality

$$\int_0^{r_m} \beta(a, \xi)N(\xi) \exp(-\gamma(\xi - \sigma)) \, d\xi - \phi(a, \sigma) \geq 0$$

holds.

Then we can prove the following.

**THEOREM 4.10.** Suppose that Assumption 4.9 holds. If $T > 1$, then $\Phi$ has only one positive fixed point.

**PROOF.** From Lemma 4.8 and Theorem 4.6, it is sufficient to show that under Assumption 4.9, the operator $\Phi$ is a monotone concave operator satisfying condition (4.10). From (4.4), (4.3), and (4.1e), it follows that

$$\Phi(x)(a) = \int_0^{r_m} \phi(a, \sigma) x(\sigma) \exp \left( - \int_0^\sigma x(\eta) \, d\eta \right) \, d\sigma$$

$$= \int_0^{r_m} \phi(a, \sigma) \left[ -\frac{d}{d\sigma} \left( \exp\left( - \int_0^\sigma x(\eta) \, d\eta \right) \right) \right] \, d\sigma$$

$$= -\phi(a, \sigma) \exp\left( - \int_0^\sigma x(\eta) \, d\eta \right)\left[ \int_0^{r_m} \beta(a, \xi)N(\xi) \frac{d}{d\sigma} \pi(\xi, \sigma) \, d\xi \right] \, d\sigma$$

$$= \phi(a, 0) - \alpha \int_0^{r_m} \exp\left( - \int_0^\sigma x(\eta) \, d\eta \right) \left[ \int_0^{r_m} \beta(a, \xi)N(\xi) \exp(-\gamma(\xi - \sigma)) \, d\xi \right] \, d\sigma$$

$$- \alpha \int_0^{r_m} \beta(a, \xi)N(\xi) \int_0^\xi \exp(-\gamma(\xi - \eta)) \cdot \exp(-\alpha(\eta - \sigma)) \, d\eta \, d\xi \, d\sigma$$

$$= \phi(a, 0) - \alpha \int_0^{r_m} \exp\left( - \int_0^\sigma x(\eta) \, d\eta \right) \left[ \int_0^{r_m} \beta(a, \xi)N(\xi) \exp(-\gamma(\xi - \sigma)) \, d\xi \right] \, d\sigma$$

$$- \int_0^{r_m} \beta(a, \xi)N(\xi) \exp(-\gamma(\xi - \sigma)) \, d\xi \int_0^{r_m} \beta(a, \xi)N(\xi) \pi(\xi, \sigma) \, d\xi \right] \, d\sigma$$

$$= \phi(a, 0) - \alpha \int_0^{r_m} \exp\left( - \int_0^\sigma x(\eta) \, d\eta \right) \left[ \int_0^{r_m} \beta(a, \xi)N(\xi) \exp(-\gamma(\xi - \sigma)) \, d\xi \right] \, d\sigma.$$
from which, together with Assumption 4.9, we know that $\Phi$ is a monotonic operator. Next from (4.4) and (4.3), we observe that

$$\alpha(x)u_0 \leq \Phi(x)(a) \leq \beta(x)u_0,$$

where $u_0 \equiv 1$ and

$$\alpha(x) = \int_0^{r_m} g(\sigma)x(\sigma)\exp\left(-\int_0^\sigma x(\eta)\,d\eta\right)\,d\sigma,$$

$$\beta(x) = M \int_0^{r_m} h(\sigma)x(\sigma)\exp\left(-\int_0^\sigma x(\eta)\,d\eta\right)\,d\sigma.$$

Here $M = \text{ess sup} \beta(a, b) < +\infty$, $g(\sigma)$ is given by (4.9) and $h(\sigma)$ is defined by

$$h(\sigma) = \int_\sigma^{r_m} N(\xi)\pi(\xi, \sigma)\,d\xi.$$

It follows that $\alpha(x) > 0$ and $\beta(x) > 0$ for $x \in X_+ \setminus \{0\}$. Moreover, we obtain

$$\Phi(tx)(a) - t\Phi(x)(a) = t \int_0^{r_m} \phi(a, \sigma)x(\sigma)\exp\left(-\int_0^\sigma x(\eta)\,d\eta\right)\left[\exp\left((1-t)\int_0^\sigma x(\eta)\,d\eta\right) - 1\right]\,d\sigma$$

$$\geq t \int_0^{r_m} g(\sigma)x(\sigma)\exp\left(-\int_0^\sigma x(\eta)\,d\eta\right)\left[\exp\left((1-t)\int_0^\sigma x(\eta)\,d\eta\right) - 1\right]\,d\sigma,$$

from which we conclude that $\Phi$ is a concave operator and condition (4.11) is satisfied by letting

$$u_0 = 1$$

and

$$\eta(x, t) = t \int_0^{r_m} g(\sigma)x(\sigma)\exp\left(-\int_0^\sigma x(\eta)\,d\eta\right)\left[\exp\left((1-t)\int_0^\sigma x(\eta)\,d\eta\right) - 1\right]\,d\sigma.$$

This completes the proof of Theorem 4.10.

We now determine what kind of conditions could guarantee Assumption 4.9. We say that if $\beta(a, \xi)N(\xi)$ is continuous and nonincreasing as a function of $\xi \in (0, r_m)$, then Assumption 4.9 holds. In the following, we will show this by considering two cases: $\gamma = \alpha$ and $\gamma \neq \alpha$.

**Case 1: $\gamma = \alpha$.** By (4.12), (4.3), (4.1.e), and the mean value theorem, we have

$$\int_\sigma^{r_m} \beta(a, \xi)N(\xi)\exp\left(-\gamma(\xi - \sigma)\right)\,d\xi - \phi(a, \sigma)$$

$$= \int_\sigma^{r_m} \beta(a, \xi)N(\xi)\left[\exp\left(-\gamma(\xi - \sigma)\right) - \pi(\xi, \sigma)\right]\,d\xi$$

$$= \int_\sigma^{r_m} \beta(a, \xi)N(\xi)\left[\exp\left(-\alpha(\xi - \sigma)\right) - \alpha\exp(-\alpha(\xi - \sigma))(\xi - \sigma)\right]\,d\xi$$

$$= \int_\sigma^{r_m+1/\alpha} \beta(a, \xi)N(\xi)\exp\left(-\alpha(\xi - \sigma)\right)[1 - \alpha(\xi - \sigma)]\,d\xi$$

$$+ \int_\sigma^{r_m} \beta(a, \xi)N(\xi)\exp\left(-\alpha(\xi - \sigma)\right)[1 - \alpha(\xi - \sigma)]\,d\xi$$

$$= \beta(a, \xi_1)N(\xi_1)\int_\sigma^{r_m} \exp\left(-\alpha(\xi - \sigma)\right)[1 - \alpha(\xi - \sigma)]\,d\xi$$

$$+ \beta(a, \xi_2)N(\xi_2)\int_{\sigma+1/\alpha}^{r_m} \exp\left(-\alpha(\xi - \sigma)\right)[1 - \alpha(\xi - \sigma)]\,d\xi$$

$$= \beta(a, \xi_1)N(\xi_1)\cdot \frac{1}{\alpha e} + \beta(a, \xi_2)N(\xi_2)\left[(r_m - \sigma)\exp(-\alpha(r_m - \sigma)) - \frac{1}{\alpha e}\right]$$

$$\geq \beta(a, \xi_2)N(\xi_2)\left[\frac{1}{\alpha e} + (r_m - \sigma)\exp(-\alpha(r_m - \sigma)) - \frac{1}{\alpha e}\right]$$

$$= \beta(a, \xi_2)N(\xi_2)(r_m - \sigma)\exp(-\alpha(r_m - \sigma)) \geq 0.$$
In the above calculation, we applied the continuous and nonincreasing of \(\beta(a, \xi)N(\xi)\) on \(\xi\), i.e., \(\xi_1 \leq \xi_2\) implies \(\beta(a, \xi_1)N(\xi_1) \geq \beta(a, \xi_2)N(\xi_2)\) for \(\forall a \in (0, r_m)\).

**CASE 2:** \(\gamma \neq \alpha\). Without loss of generality, we assume that \(\gamma > \alpha\), the discussion for \(\alpha > \gamma\) is similar to \(\gamma > \alpha\). Note that

\[
\int_{\sigma}^{r_m} \beta(a, \xi)N(\xi) \exp\left(-\alpha(\xi - \sigma)\right) d\xi = \int_{\sigma}^{r_m} \beta(a, \xi)N(\xi) \left[\frac{\gamma}{\gamma - \alpha} \exp\left(-\gamma(\xi - \sigma)\right) - \frac{\alpha}{\gamma - \alpha} \exp\left(-\alpha(\xi - \sigma)\right)\right] d\xi.
\]

It is easy to check that the function

\[
f(\xi) = \frac{\gamma}{\gamma - \alpha} \exp\left(-\gamma(\xi - \sigma)\right) - \frac{\alpha}{\gamma - \alpha} \exp\left(-\alpha(\xi - \sigma)\right)
\]

has the following properties.

1. There exists only one zero point

\[
x_0 = \frac{\ln \gamma - \ln \alpha}{\gamma - \alpha} + \sigma, \quad \text{on } (\sigma, r_m).
\]

2. \(f(\xi)\) has only one minimal value point

\[
x_{\text{min}} = \sigma + \frac{2(\ln \gamma - \ln \alpha)}{\gamma - \alpha}, \quad \text{on } (\sigma, r_m),
\]

and its minimal value is

\[
f(x_{\text{min}}) = \frac{1}{\gamma - \alpha} \gamma^{\alpha/(\gamma - \alpha)} \cdot \alpha^{\gamma/(\gamma - \alpha)} \left[\left(\frac{\alpha}{\gamma}\right)^{\gamma/(\gamma - \alpha)} - \left(\frac{\alpha}{\gamma}\right)^{\alpha/(\gamma - \alpha)}\right] < 0.
\]

3. \(f(\xi)\) is monotone decreasing on \((\sigma, x_{\text{min}})\) and is monotone increasing on \((x_{\text{min}}, r_m)\).

From the above discussion of \(f(\xi)\) and the mean value theorem, we have

\[
\int_{\sigma}^{r_m} \beta(a, \xi)N(\xi) \left[\frac{\gamma}{\gamma - \alpha} \exp\left(-\gamma(\xi - \sigma)\right) - \frac{\alpha}{\gamma - \alpha} \exp\left(-\alpha(\xi - \sigma)\right)\right] d\xi
\]

\[
= \int_{\sigma}^{x_0} \beta(a, \xi)N(\xi) \left[\frac{\gamma}{\gamma - \alpha} \exp\left(-\gamma(\xi - \sigma)\right) - \frac{\alpha}{\gamma - \alpha} \exp\left(-\alpha(\xi - \sigma)\right)\right] d\xi
\]

\[
+ \int_{x_0}^{r_m} \beta(a, \xi)N(\xi) \left[\frac{\gamma}{\gamma - \alpha} \exp\left(-\gamma(\xi - \sigma)\right) - \frac{\alpha}{\gamma - \alpha} \exp\left(-\alpha(\xi - \sigma)\right)\right] d\xi
\]

\[
= \beta(a, \xi_1)N(\xi_1) \int_{\sigma}^{x_0} \left[\frac{\gamma}{\gamma - \alpha} \exp\left(-\gamma(\xi - \sigma)\right) - \frac{\alpha}{\gamma - \alpha} \exp\left(-\alpha(\xi - \sigma)\right)\right] d\xi
\]

\[
+ \beta(a, \xi_2)N(\xi_2) \int_{x_0}^{r_m} \left[\frac{\gamma}{\gamma - \alpha} \exp\left(-\gamma(\xi - \sigma)\right) - \frac{\alpha}{\gamma - \alpha} \exp\left(-\alpha(\xi - \sigma)\right)\right] d\xi
\]

\[
\geq \beta(a, \xi_2)N(\xi_2) \left[\frac{1}{\gamma - \alpha} \exp\left(-\alpha(r_m - \sigma)\right) - \frac{1}{\gamma - \alpha} \exp\left(-\gamma(r_m - \sigma)\right)\right]
\]

\[
\geq 0.
\]

In particular, Assumption 4.9 holds if \(\beta\) is independent of age of infectives \(\sigma\), because \(N(\sigma)\) is a decreasing function.

The assumption that \(\beta(a, \xi)N(\xi)\) is nonincreasing on \(\xi\) implies that the number of age \(\alpha\) infected by younger individuals is always greater than the number of those infected by older individuals. This assumption may not be realistic for some diseases. Here we only use it to explain the fact that there are many functions which satisfy Assumption 4.9.
5. STABILITY ANALYSIS FOR EQUILIBRIUM SOLUTIONS

In order to investigate the local stability of the equilibrium solutions \((s^*(a), e^*(a), \eta^*(a))\) of (2.1a)-(2.7c), we first rewrite (2.7a)-(2.7c) into equations for small perturbations. Let

\[
\begin{align*}
  s(a, t) &= s^*(a) + \zeta(a, t), \\
  e(a, t) &= e^*(a) + \eta(a, t), \\
  i(a, t) &= i^*(a) + \delta(a, t).
\end{align*}
\]

From (2.7a)-(2.7c), we have

\[
\begin{align*}
  \frac{\partial \zeta(a, t)}{\partial t} + \frac{\partial \zeta(a, t)}{\partial a} &= -\lambda(a, t) [\zeta(a, t) + s^*(a)] - \lambda^*(a) \zeta(a, t), \\
  \frac{\partial \eta(a, t)}{\partial t} + \frac{\partial \eta(a, t)}{\partial a} &= \lambda(a, t) [\zeta(a, t) + s^*(a)] + \lambda^*(a) \zeta(a, t) - \alpha \eta(a, t), \\
  \frac{\partial \delta(a, t)}{\partial t} + \frac{\partial \delta(a, t)}{\partial a} &= \alpha \eta(a, t) - \gamma \delta(a, t),
\end{align*}
\]

where

\[
\begin{align*}
  \lambda(a, t) &= \int_0^{r_m} \beta(a, \sigma) N(a) \delta(\sigma, t) d\sigma, \\
  \lambda^*(a) &= \int_0^{r_m} \beta(a, \sigma) N(a) i^*(\sigma) d\sigma, \\
  \zeta(0, t) &= 0, \\
  \eta(0, t) &= 0, \\
  \delta(0, t) &= 0.
\end{align*}
\]

Therefore, we can formulate (5.1a)-(5.1c) as an abstract semilinear problem on the Banach space \(X\).

\[
\frac{du(t)}{dt} = Au(t) + G(u(t)), \quad u(t) = (\zeta(t), \eta(t), \delta(t))^T \in X,
\]

where the generator \(A\) is defined by

\[
(A\phi)(a) = \left(-\frac{d\phi_1(a)}{da}, -\frac{d\phi_2(a)}{da} - \alpha \phi_2(a), -\frac{d\phi_3(a)}{da} - \gamma \phi_3(a)\right)^T,
\]

with the domain \(D(A) = \{\phi \in X \mid \phi_i \text{ is absolutely continuous on } [0, r_m], \ i = 1, 2, 3, \phi(0) = 0\}\).

The nonlinear term \(G\) is defined as

\[
G(u) = (- (Pu_3) (u_1 + s^*) - \lambda^* u_1, (Pu_3) (u_1 + s^*) + \lambda^* u_1, \alpha u_2)^T,
\]

for \(u = (u_1, u_2, u_3)^T \in X\), where the operator \(P\) is defined by (3.4). The linearized equation around \(u = 0\) is given by

\[
\frac{d}{dt} u(t) = (A + C) u(t),
\]

where the bounded linear operator \(C\) is the Frechet derivative of \(G(u)\) at \(u = 0\) and given by

\[
Cu = (- (Pu_3) s^* - \lambda^* u_1, (Pu_3) s^* + \lambda^* u_1, \alpha u_2)^T.
\]

Now let us consider the resolvent equation for \(A + C\)

\[
(\lambda I - A - C) \phi = \psi, \quad \phi \in D(A), \quad \psi \in X, \quad \lambda \in C.
\]

Then we have

\[
\begin{align*}
  \frac{d\phi_1(a)}{da} + (\lambda + \lambda^*(a)) \phi_1(a) &= \psi_1(a) - (P \phi_3)(a) s^*(a), \\
  \frac{d\phi_2(a)}{da} + (\lambda + \alpha) \phi_2(a) &= \psi_2(a) + (P \phi_3)(a) s^*(a) + \lambda^*(a) \phi_1(a), \\
  \frac{d\phi_3(a)}{da} + (\lambda + \gamma) \phi_3(a) &= \psi_3(a) + \alpha \phi_2(a),
\end{align*}
\]

\[
(5.1a), (5.1b), (5.1c)
\]
From (5.7a), we obtain
\[ \phi_1(\sigma) = \exp(-\lambda a) \Pi(a) \int_0^a [\psi_1(\sigma) - (P \phi_3)(\sigma) \Pi(\sigma)] \exp(\lambda \sigma) \Pi^{-1}(\sigma) \, d\sigma, \quad (5.8) \]
where \( \Pi(a) \) is defined by
\[ \Pi(a) = \exp \left( - \int_0^a \lambda^*(\sigma) \, d\sigma \right) = s^*(a). \]

By (5.7b) and (5.7c), we get
\[ \phi_2(\sigma) = \int_0^a \exp(- (\lambda + \alpha)(a - \sigma)) [\psi_2(\sigma) + (P \phi_3)(\sigma) \Pi(\sigma) + \lambda^*(\sigma) \phi_1(\sigma)] \, d\sigma, \quad (5.9) \]
\[ \phi_3(\sigma) = \int_0^a \exp(- (\lambda + \gamma)(a - \sigma)) [\psi_3(\sigma) + \alpha \phi_2(\sigma)] \, d\sigma. \quad (5.10) \]

Substituting (5.8) and (5.9) into (5.10), we obtain
\[ \phi_3(\sigma) = \int_0^a \exp(- (\lambda + \gamma)(a - \sigma)) \psi_3(\sigma) \, d\sigma \]
\[ + \alpha \int_0^a \exp(- (\lambda + \gamma)(a - \sigma)) \int_0^\sigma \exp(-(\lambda + \alpha)(\sigma - \eta)) \psi_2(\eta) \, d\eta \, d\sigma \]
\[ + \alpha \int_0^a \exp(- (\lambda + \gamma)(a - \sigma)) \int_0^\sigma \exp(-(\lambda + \alpha)(\sigma - \eta)) (P \phi_3)(\eta) \Pi(\eta) \, d\eta \, d\sigma \]
\[ + \alpha \int_0^a \exp(- (\lambda + \gamma)(a - \sigma)) \int_0^\sigma \exp(-(\lambda + \alpha)(\sigma - \eta)) \exp(-\lambda \eta) \Pi(\eta) \lambda^*(\eta) \]
\[ \times \int_0^\eta \psi_1(\xi) \exp(\lambda \xi) \Pi^{-1}(\xi) \, d\xi \, d\eta \, d\sigma \]
\[ - \alpha \int_0^a \exp(- (\lambda + \gamma)(a - \sigma)) \int_0^\sigma \exp(-(\lambda + \alpha)(\sigma - \eta)) \exp(-\lambda \eta) \Pi(\eta) \lambda^*(\eta) \]
\[ \times \int_0^\eta \exp(\lambda \xi) (P \phi_3)(\xi) \, d\xi \, d\eta \, d\sigma. \quad (5.11) \]

Using (5.11) and (3.4), we obtain
\[ (P \phi_3)(\sigma) = J(\sigma) + K(\sigma) + L(\sigma) + X(\sigma) + Y(\sigma), \quad (5.12) \]
where
\[ J(\sigma) = \int_0^a \beta(\sigma, \sigma) N(\sigma) \int_0^\sigma \exp(-(\lambda + \gamma)(\sigma - \eta)) \psi_3(\eta) \, d\eta \, d\sigma, \]
\[ K(\sigma) = \alpha \int_0^a \beta(\sigma, \sigma) N(\sigma) \int_0^\sigma \exp(-(\lambda + \gamma)(\sigma - \eta)) \]
\[ \times \int_0^\eta \exp(-(\lambda + \alpha)(\eta - \xi)) \psi_2(\xi) \, d\xi \, d\eta \, d\sigma, \]
\[ L(\sigma) = \alpha \int_0^a \beta(\sigma, \sigma) N(\sigma) \int_0^\sigma \exp(-(\lambda + \gamma)(\sigma - \eta)) \]
\[ \times \int_0^\eta \exp(-(\lambda + \alpha)(\eta - \xi)) (P \phi_3)(\xi) \Pi(\xi) \, d\xi \, d\eta \, d\sigma, \]
\[ X(\sigma) = \alpha \int_0^a \beta(\sigma, \sigma) N(\sigma) \int_0^\sigma \exp(-(\lambda + \gamma)(\sigma - \eta)) \int_0^\eta \exp(-(\lambda + \alpha)(\eta - \xi)) \]
\[ \times \exp(-\lambda \xi) \Pi(\xi) \lambda^*(\xi) \int_0^\xi \psi_1(\rho) \exp(\lambda \rho) \Pi^{-1}(\rho) \, d\rho \, d\xi \, d\eta \, d\sigma, \]
\[ Y(\sigma) = -\alpha \int_0^a \beta(\sigma, \sigma) N(\sigma) \int_0^\sigma \exp(-(\lambda + \gamma)(\sigma - \eta)) \int_0^\eta \exp(-(\lambda + \alpha)(\eta - \xi)) \]
\[ \times \exp(-\lambda \xi) \Pi(\xi) \lambda^*(\xi) \int_0^\xi \exp(\lambda \rho) (P \phi_3)(\rho) \, d\rho \, d\xi \, d\eta \, d\sigma. \]
Define

\[ \pi_\lambda(\sigma, \xi) = \alpha \int_\xi^\sigma \exp(- (\lambda + \gamma)(\sigma - \eta)) \exp(- (\lambda + \alpha)(\eta - \xi)) \, d\eta, \]

\[ \phi_\lambda(a, \xi) = \int_\xi^a \beta(a, \sigma) N(\sigma) \pi_\lambda(\sigma, \xi) \, d\sigma, \]

\[ \theta_\lambda(a, \eta) = \int_\eta^a \beta(a, \sigma) N(\sigma) \exp(- (\lambda + \gamma)(\sigma - \eta)) \, d\sigma. \]

Then we can rewrite the above representations for \( J(a), K(a), L(a), X(a), \) and \( Y(a) \) as

\[ J(a) = \int_0^a \int_\eta^a \beta(a, \sigma) N(\sigma) \text{exp}(- (\lambda + \gamma)(\sigma - \eta)) \, d\sigma \psi_3(\eta) \, d\eta, \]

\[ = \int_0^a \theta_\lambda(a, \eta) \psi_3(\eta) \, d\eta, \]

\[ K(a) = \alpha \int_0^a \beta(a, \sigma) N(\sigma) \int_0^\sigma \text{exp}(- (\lambda + \gamma)(\sigma - \eta)) \times \text{exp}(- (\lambda + \alpha)(\eta - \xi)) \, d\eta \psi_2(\xi) \, d\xi \, d\sigma \]

\[ = \int_0^a \beta(a, \sigma) N(\sigma) \int_0^\sigma \pi_\lambda(\sigma, \xi) \psi_2(\xi) \, d\xi \, d\sigma \]

\[ = \int_0^a \int_\xi^a \beta(a, \sigma) N(\sigma) \pi_\lambda(\sigma, \xi) \, d\sigma \psi_2(\xi) \, d\xi \]

\[ = \int_0^a \phi_\lambda(a, \xi) \psi_2(\xi) \, d\xi, \]

\[ L(a) = \alpha \int_0^a \beta(a, \sigma) N(\sigma) \int_0^\sigma \text{exp}(- (\lambda + \gamma)(\sigma - \eta)) \times \text{exp}(- (\lambda + \alpha)(\eta - \xi)) \, d\eta \, d\xi \, d\sigma \]

\[ = \int_0^a \beta(a, \sigma) N(\sigma) \int_0^\sigma \pi_\lambda(\sigma, \xi) (P\phi_3)(\xi) \, d\xi \, d\sigma \]

\[ = \int_0^a \int_\xi^a \beta(a, \sigma) N(\sigma) \pi_\lambda(\sigma, \xi) \, d\sigma (P\phi_3)(\xi) \, d\xi \]

\[ = \int_0^a \phi_\lambda(a, \xi) (P\phi_3)(\xi) \, d\xi, \]

\[ X(a) = \alpha \int_0^a \beta(a, \sigma) N(\sigma) \int_0^\sigma \text{exp}(- (\lambda + \gamma)(\sigma - \eta)) \text{exp}(- (\lambda + \alpha)(\eta - \xi)) \, d\eta \]

\[ \times \text{exp}(- (\lambda \xi)\Pi(\xi)\lambda^*(\xi)) \int_0^{\xi} \phi_1(\rho) \text{exp}(\lambda\rho)\Pi^{-1}(\rho) \, d\rho \, d\xi \, d\sigma \]

\[ = \int_0^a \beta(a, \sigma) N(\sigma) \int_0^{\sigma} \pi_\lambda(\sigma, \xi) \text{exp}(- (\lambda \xi)\Pi(\xi)\lambda^*(\xi)) \int_0^{\xi} \phi_1(\rho) \text{exp}(\lambda\rho)\Pi^{-1}(\rho) \, d\rho \, d\xi \, d\sigma \]

\[ = \int_0^a \int_\xi^a \beta(a, \sigma) N(\sigma) \pi_\lambda(\sigma, \xi) \, d\sigma \text{exp}(- (\lambda \xi)\Pi(\xi)\lambda^*(\xi)) \int_0^{\xi} \phi_1(\rho) \text{exp}(\lambda\rho)\Pi^{-1}(\rho) \, d\rho \, d\xi \]

\[ = \int_0^a \phi_\lambda(a, \xi) \text{exp}(- (\lambda \xi)\Pi(\xi)\lambda^*(\xi)) \int_0^{\xi} \phi_1(\rho) \text{exp}(\lambda\rho)\Pi^{-1}(\rho) \, d\rho \, d\xi, \]

\[ Y(a) = -\alpha \int_0^a \beta(a, \sigma) N(\sigma) \int_0^\sigma \text{exp}(- (\lambda + \gamma)(\sigma - \eta)) \text{exp}(- (\lambda + \alpha)(\eta - \xi)) \, d\eta \]

\[ \times \text{exp}(- (\lambda \xi)\Pi(\xi)\lambda^*(\xi)) \int_0^{\xi} \text{exp}(\lambda\rho) (P\phi_3)(\rho) \, d\rho \, d\xi \, d\sigma \]

\[ = -\int_0^a \beta(a, \sigma) N(\sigma) \int_0^{\sigma} \pi_\lambda(\sigma, \xi) \text{exp}(- (\lambda \xi)\Pi(\xi)\lambda^*(\xi)) \int_0^{\xi} \text{exp}(\lambda\rho) (P\phi_3)(\rho) \, d\rho \, d\xi \, d\sigma \]
\[
- \int_0^\infty \int_0^\infty \beta(a, \sigma)N(\sigma)\pi_A(a, \xi) \, d\sigma \exp(-\lambda \xi) \Pi(\xi) \lambda^*(\xi) \int_0^\xi \exp(\lambda \rho) (P\phi_3)(\rho) \, d\xi
\]
\[
- \int_0^\infty \phi_A(a, \xi) \exp(-\lambda \xi) \Pi(\xi) \lambda^*(\xi) \int_0^\xi \exp(\lambda \rho) (P\phi_3)(\rho) \, d\rho \, d\xi.
\]

If we define linear operators on the Banach space \(L^1(0, r_m)\) by

\[
(S_\lambda \psi)(a) = \int_0^{r_m} \theta_A(a, \eta)\psi(\eta) \, d\eta,
\]
\[
(T_\lambda \psi)(a) = \int_0^{r_m} \phi_A(a, \xi)\Pi(\xi)\psi(\xi) \, d\xi,
\]
\[
(U_\lambda \psi)(a) = \int_0^{r_m} \phi_A(a, \xi) \exp(-\lambda \xi)\Pi(\xi)\lambda^*(\xi) \int_0^\xi \psi(\rho) \exp(\lambda \rho) \, d\rho \, d\xi,
\]
\[
(V_\lambda \psi)(a) = (T_\lambda \psi)(a) - (U_\lambda \psi)(a),
\]
then the following expression holds:

\[
(V_\lambda \psi)(a) = \int_0^{r_m} \chi_\lambda(a, \rho)\psi(\rho) \, d\rho,
\]

where

\[
\chi_\lambda(a, \rho) = \alpha \int_\rho^{r_m} \Pi(\xi) \exp(-\lambda(\xi - \rho)) \int_\xi^{r_m} \beta(a, \sigma)N(\sigma) \times \left[ \exp(-(\lambda + \gamma)(\sigma - \xi)) - \pi_A(a, \xi) \right] \, d\sigma \, d\xi.
\]

It is not difficult to verify the above expression if we note that

\[
(U_\lambda \psi)(a) = \int_0^{r_m} \phi_A(a, \xi) \exp(-\lambda \xi)\Pi(\xi)\lambda^*(\xi) \int_0^\xi \psi(\rho) \exp(\lambda \rho) \, d\rho \, d\xi
\]
\[
= \int_0^{r_m} \left( \frac{d\Pi(\xi)}{d\xi} \right) \left[ \phi_A(a, \xi) \exp(-\lambda \xi) \int_0^\xi \psi(\rho) \exp(\lambda \rho) \, d\rho \right] \, d\xi
\]
\[
= -\Pi(\xi)\phi_A(a, \xi) \exp(-\lambda \xi) \int_0^\xi \psi(\rho) \exp(\lambda \rho) \, d\rho \bigg|_0^{r_m}
\]
\[
+ \int_0^{r_m} \Pi(\xi) \frac{\partial}{\partial \xi} \left[ \phi_A(a, \xi) \exp(-\lambda \xi) \int_0^\xi \psi(\rho) \exp(\lambda \rho) \, d\rho \right] \, d\xi
\]
\[
= \int_0^{r_m} \left[ \Pi(\xi) \frac{\partial \phi_A}{\partial \xi}(a, \xi) \exp(-\lambda \xi) \int_0^\xi \psi(\rho) \exp(\lambda \rho) \, d\rho \right. 
\]
\[
- \lambda \Pi(\xi)\phi_A(a, \xi) \exp(-\lambda \xi) \int_0^\xi \psi(\rho) \exp(\lambda \rho) \, d\rho + \Pi(\xi)\phi_A(a, \xi)\psi(\xi) \bigg] \, d\xi
\]
\[
= (T_\lambda \psi)(a) + \int_0^{r_m} \Pi(\xi) \frac{\partial \phi_A}{\partial \xi}(a, \xi) \exp(-\lambda \xi) \int_0^\xi \psi(\rho) \exp(\lambda \rho) \, d\rho \, d\xi
\]
\[
- \lambda \int_0^{r_m} \Pi(\xi)\phi_A(a, \xi) \exp(-\lambda \xi) \int_0^\xi \psi(\rho) \exp(\lambda \rho) \, d\rho \, d\xi
\]
\[
= (T_\lambda \psi)(a) + \int_0^{r_m} \Pi(\xi) \left[ \int_\xi^{r_m} \beta(a, \sigma)N(\sigma) \left[ -\alpha \exp(-(\lambda + \gamma)(\sigma - \xi)) 
\right.ight. 
\]
\[
+ (\lambda + \alpha)\pi_A(a, \xi) \right] \, d\sigma \exp(-\lambda \xi) \int_0^\xi \psi(\rho) \exp(\lambda \rho) \, d\rho \, d\xi
\]
\[
- \lambda \int_0^{r_m} \Pi(\xi)\phi_A(a, \xi) \exp(-\lambda \xi) \int_0^\xi \psi(\rho) \exp(\lambda \rho) \, d\rho \, d\xi
\]
\[
= (T_\lambda \psi)(a) + \int_0^{r_m} \Pi(\xi) \left[ \int_\xi^{r_m} \beta(a, \sigma)N(\sigma) \right. 
\]
\[
- \alpha \exp(-(\lambda + \gamma)(\sigma - \xi)) \right] \, d\sigma \exp(-\lambda \xi) \int_0^\xi \psi(\rho) \exp(\lambda \rho) \, d\rho \, d\xi
\]
\[
- \lambda \int_0^{r_m} \Pi(\xi)\phi_A(a, \xi) \exp(-\lambda \xi) \int_0^\xi \psi(\rho) \exp(\lambda \rho) \, d\rho \, d\xi
\]
\[
= (T_\lambda \psi)(a) + \int_0^{r_m} \Pi(\xi) \left[ \int_\xi^{r_m} \beta(a, \sigma)N(\sigma) \right. 
\]
\[
- \alpha \exp(-(\lambda + \gamma)(\sigma - \xi)) \right] \, d\sigma \exp(-\lambda \xi) \int_0^\xi \psi(\rho) \exp(\lambda \rho) \, d\rho \, d\xi
\]
\[
- \lambda \int_0^{r_m} \Pi(\xi)\phi_A(a, \xi) \exp(-\lambda \xi) \int_0^\xi \psi(\rho) \exp(\lambda \rho) \, d\rho \, d\xi
\]
\[
= (T_\lambda \psi)(a) + \int_0^{r_m} \Pi(\xi) \left[ \int_\xi^{r_m} \beta(a, \sigma)N(\sigma) \right. 
\]
\[
- \alpha \exp(-(\lambda + \gamma)(\sigma - \xi)) \right] \, d\sigma \exp(-\lambda \xi) \int_0^\xi \psi(\rho) \exp(\lambda \rho) \, d\rho \, d\xi
\]
\[
- \lambda \int_0^{r_m} \Pi(\xi)\phi_A(a, \xi) \exp(-\lambda \xi) \int_0^\xi \psi(\rho) \exp(\lambda \rho) \, d\rho \, d\xi
\]
\[
= (T_\lambda \psi)(a) + \int_0^{r_m} \Pi(\xi) \left[ \int_\xi^{r_m} \beta(a, \sigma)N(\sigma) \right. 
\]
\[
- \alpha \exp(-(\lambda + \gamma)(\sigma - \xi)) \right] \, d\sigma \exp(-\lambda \xi) \int_0^\xi \psi(\rho) \exp(\lambda \rho) \, d\rho \, d\xi
\]
\[
- \lambda \int_0^{r_m} \Pi(\xi)\phi_A(a, \xi) \exp(-\lambda \xi) \int_0^\xi \psi(\rho) \exp(\lambda \rho) \, d\rho \, d\xi
\]
\[
= (T_\lambda \psi)(a) + \int_0^{r_m} \Pi(\xi) \left[ \int_\xi^{r_m} \beta(a, \sigma)N(\sigma) \right. 
\]
\[
- \alpha \exp(-(\lambda + \gamma)(\sigma - \xi)) \right] \, d\sigma \exp(-\lambda \xi) \int_0^\xi \psi(\rho) \exp(\lambda \rho) \, d\rho \, d\xi
\]
\[
- \lambda \int_0^{r_m} \Pi(\xi)\phi_A(a, \xi) \exp(-\lambda \xi) \int_0^\xi \psi(\rho) \exp(\lambda \rho) \, d\rho \, d\xi
\]
\[ (T_\lambda \psi)(a) - \alpha \int_0^\tau \Pi(\xi) \int_\xi^\tau \beta(a, \sigma) N(\sigma) \exp(-\lambda + \gamma)(\sigma - \xi)) d\sigma \exp(-\lambda \xi) \]
\[ \times \int_0^\xi \psi(\rho) \exp(\lambda \rho) d\rho d\xi + \alpha \int_0^\tau \Pi(\xi) \phi_\lambda(a, \xi) \exp(-\lambda \xi) \int_0^\xi \psi(\rho) \exp(\lambda \rho) d\rho d\xi \]
\[ = (T_\lambda \psi)(a) - \alpha \int_0^\tau \int_\xi^\tau \Pi(\xi) \exp(-\lambda(\xi - \rho)) \]
\[ \times \int_\xi^\tau \beta(a, \sigma) N(\sigma) \{ \exp(-\lambda + \gamma)(\sigma - \xi)) - \Pi(\sigma, \xi) \} d\sigma d\xi \psi(\rho) d\rho. \]

From the above definitions and (5.12), it follows that
\[ (P\phi_3)(a) = (S_\lambda \psi_3)(a) + (T_\lambda \psi_2 \Pi^{-1})(a) + (U_\lambda \psi_1 \Pi^{-1})(a) - (U_\lambda P\phi_3)(a). \]

Hence, we have
\[ (P\phi_3)(a) = (I - T_\lambda + U_\lambda)^{-1} [(S_\lambda \psi_3)(a) + (T_\lambda \psi_2 \Pi^{-1})(a) + (U_\lambda \psi_1 \Pi^{-1})(a)] \]
\[ = (I - V_\lambda)^{-1} [(S_\lambda \psi_3)(a) + (T_\lambda \psi_2 \Pi^{-1})(a) + (U_\lambda \psi_1 \Pi^{-1})(a)]. \]

From (5.8), (5.9), (5.11), and (5.16), we can conclude the following.

**Lemma 5.1.** The perturbed operator \( A + C \) has a compact resolvent and
\[ \sigma(A + C) = P_\sigma(A + C) = \{ \lambda \in C \mid 1 \in P_\sigma(V_\lambda) \}, \]
\[ \sigma(A + C) = P_\sigma(A + C). \]

where \( \sigma(A) \) and \( P_\sigma(A) \) denote the spectrum of \( A \) and the point spectrum of \( A \), respectively.

**Proof.** From (5.8) and (5.16), we obtain the expression for \( \phi_1 \),
\[ \phi_1(a) = \int_0^a \exp(-\lambda a) \Pi(a) \exp(\lambda \sigma) \Pi^{-1}(a) \psi(a) \, d\sigma \]
\[ - \int_0^a \exp(-\lambda a) \Pi(a) \exp(\lambda \sigma) \Pi^{-1}(a) \Pi(\sigma) (P\phi_3)(\sigma) \, d\sigma \]
\[ = (H \psi_1)(a) - W(\psi_1, \psi_2, \psi_3)(a), \]
where the operators \( H \) and \( W \) are defined by
\[ (H \psi_1)(a) = \int_0^a G(a, \sigma) \psi_1(a) \, d\sigma, \]
\[ W(\psi_1, \psi_2, \psi_3)(a) = \int_0^a G(a, \sigma) \Pi(\sigma) (I - V_\lambda)^{-1} [(S_\lambda \psi_3)(\sigma) \]
\[ + (T_\lambda \psi_2 \Pi^{-1})(\sigma) + (U_\lambda \psi_1 \Pi^{-1})(\sigma)] \, d\sigma, \]
where
\[ G(a, \sigma) = \exp(-\lambda a) \exp(\lambda \sigma) \Pi(\sigma) \Pi^{-1}(\sigma). \]

Since \( H \) is a Volterra operator with a continuous kernel, it is a compact operator on \( L^1(0, r_m) \). On the other hand, in the same manner as the proof of Lemma 4.5, we can prove that \( T_\lambda \) and \( U_\lambda \) are compact for all \( \lambda \in C \). Let \( \Lambda = \{ \lambda \in C \mid 1 \in \sigma(V_\lambda) \} \). Then it follows that when \( \lambda \in C \setminus \Lambda \), the operator \( W \) is a compact operator from \( X \) to \( L^1(0, r_m) \). In the same way, we can prove that \( \phi_2(a) \) and \( \phi_3(a) \) can be represented by compact operators from \( X \) to \( L^1(0, r_m) \). Consequently, we know that \( A + C \) has a compact resolvent. So we get that \( \sigma(A + C) = P_\sigma(A + C) \) (see [26, p. 187]).

From the above argument, it follows that \( C \setminus \Lambda \subset \rho(A + C) \) (\( \rho(A + C) \) denotes the resolvent set of \( A + C \)), that is, \( \Lambda \supset \sigma(A + C) = P_\sigma(A + C) \). Since \( V_\lambda \) is a compact operator, we know that
\( \sigma(V_{\lambda}) \setminus \{0\} = P_\sigma(V_{\lambda}) \setminus \{0\} \) and if \( \lambda \in \Lambda \), there exists an eigenfunction \( \psi_{\lambda} \) such that \( V_{\lambda} \psi_{\lambda} = \psi_{\lambda} \). Then it is easily seen that if we define the following functions:

\[
\begin{align*}
\phi_1(a) &= -\exp(-\lambda a) \Pi(a) \int_0^a \exp(\lambda \sigma) \psi_{\lambda}(\sigma) \, d\sigma, \\
\phi_2(a) &= \int_0^a \exp(-(-\lambda + \alpha)(a - \sigma)) [\psi_{\lambda}(\sigma) \Pi(\sigma) - \lambda^*(\sigma) \phi_1(\sigma)] \, d\sigma, \\
\phi_3(a) &= \alpha \int_0^a \exp(-(-\lambda + \gamma)(a - \sigma)) \phi_2(\sigma) \, d\sigma,
\end{align*}
\]

\((\phi_1(a), \phi_2(a), \phi_3(a))^T\) gives an eigenvector of \( A + C \) corresponding to the eigenvalue \( \lambda \). Then \( \Lambda \subset P_\sigma(A + C) \) and we conclude that (5.17) holds.

**Lemma 5.2.** Let \( T(t), t \geq 0 \) be the \( C_0 \)-semigroup generated by the perturbed operator \( A + C \). Then \( T(t), t \geq 0 \) is eventually norm continuous and

\[
\omega_0(A + C) = s(A + C) = \sup \{ \text{Re} \mu \mid \mu \in \sigma(A + C) \},
\]

where \( \omega_0(A + C) \) denotes the growth bound of the semigroup \( T(t), t \geq 0 \), and \( s(A + C) \) is the spectral bound of the generator \( A + C \).

**Proof.** We define bounded operators \( C_1 \) and \( C_2 \) by

\[
C_1 \phi = (-\lambda^* \phi_1, \lambda^* \phi_1, \alpha \phi_2)^T, \quad C_2 \phi = (-s^*(P\phi_3), s^*(P\phi_3), 0)^T, \quad \phi \in X.
\]

Then \( C = C_1 + C_2 \) and \( A + C_1 \) generates a \( C_0 \)-semigroup \( S(t), t \geq 0 \). Since \( S(t) \) is a nilpotent semigroup, it is eventually norm continuous. Using Assumption 4.4 and similar proof to Lemma 4.5, we can prove that \( C_2 \) is a compact operator in \( X \). Therefore, from Theorem 1.30 in [27, p. 44], \( T(t) \) is also eventually norm continuous. Since the spectral mapping theorem holds for the eventually norm continuous semigroup [27, p. 87], we obtain (5.18).

If \( \omega_0(A + C) < 0 \), the equilibrium \( u = 0 \) of system (5.2) is locally exponentially asymptotically stable in the sense that there exist \( \epsilon > 0 \), \( M \geq 1 \), and \( \gamma > 0 \) such that if \( x \in X \) and \( ||x|| \leq \epsilon \), then the solution \( u(t, x) \) of (5.2) exists globally and \( ||u(t, x)|| \leq M \exp(\gamma t) ||x|| \) for all \( t \geq 0 \). This is the main part of the principle of linearized stability (see [14]). Therefore, in order to study the stability of the equilibrium states, we have to know the structure of the set of singular points \( \Lambda = \{ \lambda \in C \mid 1 \in P_\sigma(V_{\lambda}) \} \). Since \( ||V_{\lambda}|| \to 0 \), if \( \text{Re} \lambda \to +\infty \), \( I - V_{\lambda} \) is invertible for large values of \( \text{Re} \lambda \). By the theorem of Steinberg [28], the function \( \lambda \to (I - V_{\lambda})^{-1} \) is meromorphic in the complex domain, and hence, the set \( \Lambda \) is a discrete set whose elements are poles of \( (I - V_{\lambda})^{-1} \) of finite order.

Now we shall make use of positive operator theory once more. Our main purpose here is to determine the dominant singular point, i.e., the element of the set \( \Lambda \) with the largest real part. From (5.17) and (5.18), the dominant singular point gives the growth bound of the semigroup \( T(t) \) generated by \( A + C \). First we show the following.

**Lemma 5.3.** Suppose that the following assumption holds.

**Assumption 5.4.**

\[
e^*(r_m) < \exp(-\alpha r_m).
\]

Then the operator \( V_{\lambda}, \lambda \in R \) is nonsupporting with respect to \( X_+ \) and the following holds:

\[
\lim_{\lambda \to +\infty} r(V_{\lambda}) = +\infty, \quad \lim_{\lambda \to +\infty} r(V_{\lambda}) = 0.
\]
PROOF. From (5.15), it follows that

\[
\chi_{\lambda}(a, \rho) = \alpha \int_{\rho}^{m} \Pi(\xi) \exp(-\lambda(\xi - \rho)) \int_{\xi}^{m} \beta(a, \sigma)N(\sigma) \times \left[ \exp(- (\lambda + \gamma)(\sigma - \xi)) - \pi_{\lambda}(\sigma, \xi) \right] d\sigma d\xi
\]

\[
\begin{align*}
&= \alpha \int_{\rho}^{m} \int_{\rho}^{\sigma} \Pi(\xi) \exp(-\lambda(\xi - \rho)) \left[ \exp(- (\lambda + \gamma)(\sigma - \xi)) 
- \alpha \int_{\xi}^{\sigma} \exp(- (\lambda + \gamma)(\sigma - \eta)) \cdot \exp(- (\lambda + \alpha)(\eta - \xi)) \eta \right] d\xi \beta(a, \sigma)N(\sigma) d\sigma \\
&= \alpha \int_{\rho}^{m} \int_{\rho}^{\sigma} \Pi(\xi) \exp(-\lambda(\xi - \rho)) \exp(- (\lambda + \gamma)(\sigma - \xi)) \\
&\quad - \alpha \exp((\lambda + \gamma)\xi) \int_{\xi}^{\sigma} \exp((\gamma - \alpha)\eta) \eta \right] d\xi \beta(a, \sigma)N(\sigma) d\sigma \\
&= \alpha \int_{\rho}^{m} \int_{\rho}^{\sigma} \Pi(\xi) \left[ \exp((\gamma - \alpha)\eta) \eta \right] d\xi \beta(a, \sigma)N(\sigma) d\sigma
\end{align*}
\]

(5.21)

In the following, we will show that Assumption 5.4 guarantees that the operator \( V_{\lambda} \) is strictly positive. In fact, note that

\[
\int_{\rho}^{\sigma} \Pi(\xi) \left[ \exp((\gamma - \alpha)\eta) \eta \right] d\xi \beta(a, \sigma)N(\sigma) d\sigma
\]

\[
\begin{align*}
&= \Pi(\rho) \exp(\alpha\rho) \int_{\rho}^{\sigma} \exp((\gamma - \alpha)\eta) \eta \right] d\xi \beta(a, \sigma)N(\sigma) d\sigma \\
&= \Pi(\rho) \exp(\alpha\rho) \int_{\rho}^{\sigma} \exp((\gamma - \alpha)\eta) \eta \right] d\xi \beta(a, \sigma)N(\sigma) d\sigma
\end{align*}
\]

(5.22)

Since \( \Pi(\rho) \exp(\alpha\rho) \int_{\rho}^{\sigma} e^{\alpha\xi} \lambda^{*}(\xi) \Pi(\xi) d\xi \) is an increasing function of \( \rho \), we have

\[
\int_{\rho}^{\sigma} \left[ \Pi(\rho) \exp(\alpha\rho) - \Pi(0) - \int_{0}^{\sigma} \exp(\alpha\xi)\lambda^{*}(\xi) \Pi(\xi) d\xi \right] \exp((\gamma - \alpha)\eta) \eta \right] d\xi \beta(a, \sigma)N(\sigma) d\sigma
\]

(5.23)
From (5.21)-(5.23), it follows that

\[
\chi_\lambda(a, \rho) \geq \alpha \int_{\rho}^{r_m} \left[ \exp(-\alpha r_m) - e^*(r_m) \right] \beta(a, \sigma) N(\sigma) \exp(-\lambda + \gamma) \exp(\lambda \rho) \int_{\rho}^{\sigma} \exp(\gamma \eta) \, d\eta \, d\sigma
\]

\[
\geq \alpha \int_{\rho}^{r_m} \left[ \exp(-\alpha r_m) - e^*(r_m) \right] \beta(a, \sigma) N(\sigma) \exp(-\lambda + \gamma) \exp(\lambda + \alpha) \int_{\rho}^{\sigma} \exp(\gamma \eta) \, d\eta \, d\sigma
\]

\[
= \int_{\rho}^{r_m} \left[ \exp(-\alpha r_m) - e^*(r_m) \right] \beta(a, \sigma) N(\sigma) \pi(\lambda, \sigma, \rho) \, d\sigma
\]

\[
= G_0(r_m) \int_{\rho}^{r_m} \beta(a, \sigma) N(\sigma) \pi_\lambda(\sigma, \rho) \, d\sigma,
\]

where

\[
G_0(r_m) = \exp(-\alpha r_m) - e^*(r_m), \quad \pi_\lambda(\sigma, \rho) = \alpha \int_{\rho}^{\sigma} \exp(-\lambda + \gamma)(\sigma - \eta)) \exp(-\lambda + \alpha)(\eta - \rho)) \, d\eta.
\]

From the above discussion, we know that if Assumption 5.4 holds, then the operator \( V_\lambda, \lambda \in R \) is positive. By expression (5.24), we have for \( \lambda \in R \),

\[
\chi_\lambda(a, \rho) \geq G_0(r_m) \phi_\lambda(a, \rho).
\]

Therefore, in order to show the nonsupporting property of \( V_\lambda, \lambda \in R \), it suffices to prove that the integral operator \( \hat{T}_\lambda \) defined by

\[
\left( \hat{T}_\lambda \psi \right)(a) = \int_{0}^{r_m} \phi_\lambda(a, \rho) \psi(\rho) \, d\rho, \quad \psi \in X,
\]

is nonsupporting. It is easy to verify the inequality

\[
\hat{T}_\lambda \psi \geq (f_\lambda, \psi)e, \quad e = 1 \in X_+, \quad \psi \in X_+,
\]

where the linear function \( f_\lambda \) is defined by

\[
(f_\lambda, \psi) = \int_{0}^{r_m} \left[ \int_{\rho}^{r_m} s(x) N(x) \pi_\lambda(x, \rho) \, dx \right] \psi(\rho) \, d\rho.
\]

Then it follows that for all integers \( n \),

\[
\hat{T}_\lambda^n \psi \geq (f_\lambda, \psi)^n e.
\]

Since \( f_\lambda \) is strictly positive and the constant function \( e = 1 \) is a quasi-interior point of \( L^1(0, r_m) \), it follows that \( (F, \hat{T}_\lambda \psi) > 0 \) for every pair \( \psi \in X_+ \setminus \{0\}, \ F \in X_+ \setminus \{0\} \). Then \( \hat{T}_\lambda, \lambda \in R \) is nonsupporting. Next we show (5.20).

From (5.25) and (5.27), we obtain

\[
V_\lambda \psi \geq G_0(r_m) \hat{T}_\lambda \psi \geq G_0(r_m) (f_\lambda, \psi)e, \quad \lambda \in R, \quad \psi \in X_+.
\]
Taking duality pairing with the eigenfunctional \( F_\lambda \) of \( V_\lambda \) that corresponds to \( r(V_\lambda) \), we have

\[
\text{SEIR Epidemic Model} \\
\tau(V_\lambda) \langle F_\lambda, \psi \rangle \geq G_0 (r_m) \langle F_\lambda, e \rangle (f_\lambda, \psi).
\]

If we let \( \psi = e \), we arrive at the inequality

\[
\tau(V_\lambda) \geq G_0 (r_m) \langle f_\lambda, e \rangle,
\]

where

\[
\langle f_\lambda, e \rangle = \int_{r_m}^{r_m} \int_{r_m}^{r_m} s(x) N(x) \pi_\lambda(x, \rho) \, dx \, d\rho = \int_{r_m}^{r_m} \int_{r_m}^{r_m} s(x) N(x) \pi_\lambda(x, \rho) \, d\rho \, dx = \int_{r_m}^{r_m} s(x) N(x) \int_{r_m}^{r_m} \pi_\lambda(x, \rho) \, d\rho \, dx.
\]

If \( \alpha = \gamma \), then \( \pi_\lambda(x, \rho) = \alpha e^{-(\lambda+\alpha)(x-\rho)}(x-\rho) \), and hence,

\[
0 \leq \int_{r_m}^{r_m} \pi_\lambda(x, \rho) \, d\rho = \alpha \int_{r_m}^{r_m} e^{-(\lambda+\alpha)(x-\rho)}(x-\rho) \, d\rho = -\frac{\alpha}{\lambda + \alpha} e^{-(\lambda+\alpha)x} \left[ x + \frac{1}{\lambda + \alpha} - \frac{1}{\lambda + \alpha} e^{(\lambda+\alpha)x} \right].
\]

From (5.28) and (5.29), it follows that

\[
\langle f_\lambda, e \rangle \geq e \int_{r_m}^{r_m} N(x) \left( -\frac{\alpha}{\lambda + \alpha} e^{-(\lambda+\alpha)x} \left[ x + \frac{1}{\lambda + \alpha} - \frac{1}{\lambda + \alpha} e^{(\lambda+\alpha)x} \right] \right) \, dx,
\]

since \( N(x) > 0 \) for \( x \in [r_m - \alpha, r_m) \), we know that

\[
\lim_{\lambda \to -\infty} r(V_\lambda) = +\infty.
\]

If \( \alpha \neq \gamma \), without loss of generality, we assume that \( \gamma > \alpha \) (the discussion for \( \alpha > \gamma \) is similar to \( \gamma > \alpha \)), then

\[
\pi_\lambda(x, \rho) = \alpha \int_{\rho}^{\infty} e^{-(\lambda+\gamma)(x-\eta)} \cdot e^{-(\lambda+\alpha)(\eta-\rho)} \, d\eta
\]

\[
= \alpha e^{-(\lambda+\gamma)x} \cdot e^{(\lambda+\alpha)\rho} \int_{\rho}^{\infty} e^{(\gamma-\alpha)\eta} \, d\eta
\]

\[
= \frac{\alpha}{\gamma - \alpha} e^{-(\lambda+\gamma)x} \cdot e^{(\lambda+\alpha)\rho} \left[ e^{(\gamma-\alpha)x} - e^{(\gamma-\alpha)\rho} \right]
\]

\[
= \frac{\alpha}{\gamma - \alpha} \left[ e^{-(\lambda+\gamma)(x-\rho)} - e^{-(\lambda+\gamma)(x-\rho)} \right],
\]

\[
0 \leq \int_{\rho}^{\infty} \pi_\lambda(x, \rho) \, d\rho = \frac{\alpha}{\gamma - \alpha} \int_{\rho}^{\infty} \left[ e^{-(\lambda+\gamma)(x-\rho)} - e^{-(\lambda+\gamma)(x-\rho)} \right] \, d\rho
\]

\[
= \frac{\alpha}{\gamma - \alpha} \left( \frac{1}{\lambda + \alpha} - \frac{1}{\lambda + \gamma} + \frac{1}{\lambda + \gamma} e^{-(\lambda+\gamma)x} - \frac{1}{\lambda + \alpha} e^{-(\lambda+\alpha)x} \right)
\]

\[
= \frac{\alpha}{\gamma - \alpha} \left[ \left( \frac{1}{\lambda + \alpha} - \frac{1}{\lambda + \gamma} \right) - \frac{1}{\lambda} e^{-(\lambda+\gamma)x} \left( \frac{e^{-\alpha x}}{1 + \alpha/\lambda} - \frac{e^{-\gamma x}}{1 + \gamma/\lambda} \right) \right].
\]
Since $N(x) > 0$ for $x \in [r_m - \kappa, r_m)$, from (5.28) and (5.29), we get

$$
(f, \varphi) \geq \varepsilon \int_{r_m - \kappa}^{r_m} N(x) \frac{\alpha}{\gamma - \alpha} \left[ \left( 1 + \frac{1}{\lambda + \alpha} - \frac{1}{\lambda + \gamma} \right) - \frac{1}{\lambda} e^{-\lambda x} \left( \frac{e^{-\alpha x}}{1 + \alpha/\lambda} - \frac{e^{-\gamma x}}{1 + \gamma/\lambda} \right) \right] \, dx,
$$

therefore,

$$
\lim_{\lambda \to -\infty} r(V_\lambda) = +\infty.
$$

On the other hand, we obtain

$$
V_\lambda \varphi \leq T_\lambda \varphi \leq \tilde{T}_\lambda \varphi \leq (g_\lambda, \varphi) \varepsilon, \quad \lambda \in R, \quad \varphi \in X_+,
$$

where the positive functional $g_\lambda$ is defined by

$$
(g_\lambda, \varphi) = M \int_0^{r_m} \int_\rho^{r_m} N(x) \pi_\lambda(x, \rho) \, dx \, \varphi(\rho) \, d\rho,
$$

where $M = \text{ess sup} \beta(a, \xi)$. Then we obtain the estimate

$$
r(V_\lambda) \leq (g_\lambda, \varepsilon) = \begin{cases}
M \int_0^{r_m} N(x) \left[ \frac{1}{\lambda + \alpha} e^{-(\lambda + \alpha)x} \left( z + \frac{1}{\lambda + \alpha} - \frac{1}{\lambda + \gamma} e^{(\lambda + \alpha)x} \right) \right] \, dx, & \text{if } \gamma = \alpha, \\
M \int_0^{r_m} N(x) \left[ \frac{1}{\lambda + \alpha} - \frac{1}{\lambda + \gamma} \right] - \frac{1}{\lambda} e^{-\lambda x} \left( \frac{e^{-\alpha x}}{1 + \alpha/\lambda} - \frac{e^{-\gamma x}}{1 + \gamma/\lambda} \right) \, dx, & \text{if } \gamma \neq \alpha.
\end{cases}
$$

From which we can conclude that

$$
\lim_{\lambda \to -\infty} r(V_\lambda) = 0.
$$

This completes the proof.

From Assumption 5.4 and expression (5.21), the kernel $\chi_\lambda(a, \rho)$ is strictly decreasing as a function of $\lambda \in R$. Using Proposition 4.3, we know that the function $\lambda \to r(V_\lambda)$ is strictly decreasing for $\lambda \in R$. Moreover, if there exists $\lambda \in R$ such that $r(V_\lambda) = 1$, then $\lambda \in \Lambda$, because $r(V_\lambda) \in P_\lambda(V_\lambda)$. From the monotonicity of $r(V_\lambda)$ and (5.20), it is easy to see that the following holds.

**LEMMA 5.5.** Under Assumption 5.4, there exists a unique $\lambda_0 \in R \cap \Lambda$ such that $r(V_{\lambda_0}) = 1$, and $\lambda_0 > 0$ if $r(V_0) > 1$; $\lambda_0 = 0$ if $r(V_0) = 1$; $\lambda_0 < 0$ if $r(V_0) < 1$.

Next, by using the similar argument as Theorem 6.13 in [29], we can prove that $\lambda_0$ is the dominant singular point.

**LEMMA 5.6.** Suppose that Assumption 5.4 holds. If there exists a $\lambda \in \Lambda, \lambda \neq \lambda_0$, then $\text{Re} \lambda < \lambda_0$.

**Proof.** Suppose that $\lambda \in \Lambda$ and $V_\lambda \varphi = \varphi$, then $|V_\lambda \varphi| = |\varphi|$, where $|\varphi(a)| = |\varphi(a)|$. From expression (5.21), it follows that $V_{\text{Re} \lambda} \varphi \geq |\varphi|$. Taking duality pairing with $F_{\text{Re} \lambda} \in X_+^*$ on both sides, we have $r(V_{\text{Re} \lambda}) \langle F_{\text{Re} \lambda}, |\varphi| \rangle \geq \langle F_{\text{Re} \lambda}, |\varphi| \rangle$, from which we conclude that $r(V_{\text{Re} \lambda}) \geq 1$, because $F_{\text{Re} \lambda}$ is strictly positive. Since $r(V_\lambda), \lambda \in R$ is a decreasing function, we obtain that $\text{Re} \lambda \leq \lambda_0$. If $\text{Re} \lambda = \lambda_0$, then $V_{\lambda_0} |\varphi| = |\varphi|$. In fact, if we suppose that $V_{\lambda_0} |\varphi| > |\varphi|$, taking duality pairing with the eigenfunctional $F_0$ corresponding to $r(V_{\lambda_0}) = 1$ on both sides yields $\langle F_0, |\varphi| \rangle > \langle F_0, |\varphi| \rangle$, which is a contradiction. Then we can write that $|\varphi| = c_{\psi_0} \varphi$ for some constant $c$ which we may assume to be one, where $\psi_0$ is the eigenfunction corresponding to $r(V_{\lambda_0}) = 1$. Hence, $\varphi(a) = \psi_0(a) \exp(i\nu(a))$ for some real-valued function $\nu(a)$. If we substitute this relation into $V_{\lambda_0} \psi_0 = |V_\lambda \psi_0|$, we obtain

$$
\alpha \int_0^{r_m} \int_\rho^{r_m} \Pi(\xi) \left[ e^{\gamma \xi} - \alpha e^{\alpha \xi} \int_\xi^{r_m} e^{(\gamma - \alpha) \eta} \, d\eta \right] \, d\xi
\times \beta(a, \sigma) N(\sigma) e^{-\lambda_0 (\sigma - \rho)} e^{-\gamma \sigma} \, d\sigma \psi_0(\rho) \, d\rho
\quad = \alpha \int_0^{r_m} \int_\rho^{r_m} \Pi(\xi) \left[ e^{\gamma \xi} - \alpha e^{\alpha \xi} \int_\xi^{r_m} e^{(\gamma - \alpha) \eta} \, d\eta \right] \, d\xi
\times \beta(a, \sigma) N(\sigma) e^{-(\lambda_0 + \lambda \nu)} e^{(\gamma \sigma - \rho)} \, d\sigma \psi_0(\rho) \, d\rho.
$$
From [29, Lemma 6.12], it follows that \(-\Im\lambda(\sigma) + \nu(\rho) = \theta\) for some constant \(\theta\). Using the relation \(V_0 \psi = \psi\), we obtain that \(e^{i\theta} V_0 \psi_0 = \psi_0 e^{i\nu(\rho)}\), so \(\theta = \nu(\rho)\), which implies that \(\Im\lambda = 0\). This completes the proof of Lemma 5.6.

**Theorem 5.7.** Under Assumption 5.4, the equilibrium state
\[
(s^*(a), e^*(a), i^*(a))',
\]
for (2.7a)-(2.7c) is locally asymptotically stable if \(r(V_0) < 1\) and locally unstable if \(r(V_0) > 1\).

**Proof.** From Lemmas 5.5 and 5.6, we conclude that \(\sup\{\Re \lambda : \lambda \in P(V_0)\} = \lambda_0\). Hence, it follows that \(s(A + C) = \sup\{\Re \lambda : \lambda \in P(V_0)\} < 0\) if \(r(V_0) < 1\), and \(s(A + C) > 0\) if \(r(V_0) > 1\). This completes the proof.

Now we can state the local stability results for our SEIR epidemic model.

**Theorem 5.8. Local Stability Results.** Let \(r(T)\) be the spectral radius of the operator \(T\) defined by (4.5). Then the following hold.

1. If \(r(T) < 1\), the trivial equilibrium point of (2.7a)-(2.7c) is locally asymptotically stable.
2. If \(r(T) > 1\), the trivial equilibrium point of (2.7a)-(2.7c) is unstable.
3. If \(r(T) > 1\) and Assumption 5.4 holds for an endemic steady state, it is locally asymptotically stable.

**Proof.** By our definition (4.5) and (5.26), note that \(T = \hat{T}_0\). Since Assumption 5.4 is satisfied for the trivial steady state, it is sufficient to consider only the case that Assumption 5.4 holds. From (5.13) and (5.24), we know that \(U, V\) are positive operators for \(\lambda \in \mathbb{R}\) under Assumption 5.4, and it follows that
\[
V_\lambda \leq T_\lambda \leq \hat{T}_\lambda, \quad \text{for} \quad \lambda \in \mathbb{R},
\]
which implies that \(r(V_0) \leq r(\hat{T}_0) = r(T)\), where the equality holds if and only if \(\lambda^*(a) = 0\), which corresponds to the trivial equilibrium state. Hence, for the trivial equilibrium state, Theorem 5.7 says that if \(r(T) = r(V_0) < 1\), it is locally asymptotically stable and it is unstable if \(r(T) = r(V_0) > 1\). Next we show the result (3). By Theorem 5.7, it suffices to show that \(r(V_0) < 1\) for the endemic equilibrium state. From (5.13), we obtain the inequality \(r(V_\lambda) < r(T_\lambda), \lambda \in \mathbb{R}\), since \(T_\lambda\) is nonsupporting for \(\lambda \in \mathbb{R}\) and \(V_\lambda \neq T_\lambda\) when \(\lambda^*(a) \neq 0\). In particular, the nonsupporting operator \(T_0\) has an expression
\[
(T_0 \psi)(a) = \int_0^\infty \phi(a, \sigma) \exp \left( - \int_0^\sigma \lambda^* \eta \right) \psi(\sigma) d\sigma.
\]
Since \(\lambda^*(a)\) is a nontrivial positive solution of \(x = \Phi(x)\), it follows that \(T_0\) has a positive eigenfunction \(\lambda^*(a)\) corresponding to eigenvalue one. Since a nonsupporting operator has only one positive eigenfunction corresponding to its spectral radius, we conclude that \(r(T_0) = 1\), and hence, \(r(V_0) < 1\). This shows that the endemic equilibrium state satisfying Assumption 5.4 is locally asymptotically stable. The proof of Theorem 5.8 is complete.

We have not determined what kind of conditions could guarantee Assumption 5.4, since it would be difficult to answer the question if we consider it under most general conditions. Let us consider a simple case in the following example.

**Example 5.9.** Suppose that the transmission coefficient \(\beta\) is constant and \(\alpha \neq \gamma\). In this case, the steady state is given by
\[
s^*(a) = e^{-\lambda^* a},
\]
\[
e^*(a) = \lambda^* \int_0^\infty e^{-\alpha(a-\sigma)} e^{-\lambda^* \sigma} d\sigma = \frac{\lambda^*}{\alpha - \lambda^*} \left( e^{-\lambda^* a} - e^{-\alpha a} \right),
\]
\[
i^*(a) = \int_0^\infty \pi(a, \eta) \lambda^* e^{-\lambda^* \eta} d\eta
\]
\[
= \frac{\alpha \lambda^*}{\gamma - \alpha} \cdot \frac{1}{\alpha - \lambda^*} \left[ e^{-\lambda^* a} - e^{-\alpha a} \right] - \frac{\alpha \lambda^*}{\gamma - \lambda^*} \cdot \frac{1}{\gamma - \lambda^*} \left[ e^{-\lambda^* a} - e^{-\gamma a} \right].
\]
where the constant force of infection $\lambda^*$ at the steady state is given by

$$\lambda^* = \beta \int_0^{r_m} N(a) \alpha(a) da. \quad (5.33)$$

Define a function $f(\lambda)$ as follows:

$$f(\lambda) = \frac{\alpha \beta}{\gamma - \alpha} \int_0^{r_m} \left[ \frac{1}{\alpha - \lambda^*} (e^{-\lambda a} - e^{-\alpha a}) - \frac{1}{\gamma - \lambda} (e^{-\lambda a} - e^{-\gamma a}) \right] N(a) da. \quad (5.34)$$

Then (5.33) yields the characteristic equation $\lambda^*(1 - f(\lambda^*)) = 0$. In particular, it follows that $f(0) = \tau(T)$. Since $f(\lambda)$ has an expression

$$f(\lambda) = \frac{\alpha \beta}{\gamma - \alpha} \int_0^{r_m} \int_0^\alpha \left( e^{-\alpha(a-x)} - e^{-\gamma(a-x)} \right) e^{-\lambda x} dx N(a) da,$$

then $f(\lambda)$ is strictly decreasing for $\lambda \in R$. If $f(0) \leq 1$, $\lambda^* = 0$ is the only nonnegative solution of the characteristic equation and if $f(0) > 1$, there exists another possible solution which is given as a unique positive solution of the equation $f(\lambda) = 1$. If we define the critical value of the transmission rate by

$$\beta^* = \frac{\gamma - \alpha}{\alpha} \left( \int_0^{r_m} \left[ \frac{1}{\alpha} (1 - e^{-\alpha a}) - \frac{1}{\gamma} (1 - e^{-\gamma a}) \right] N(a) da \right)^{-1}, \quad (5.35)$$

then $f(0) = \beta/\beta^*$, and the equation $f(\lambda) = 1$ has only positive root if and only if $\beta > \beta^*$ and it has a zero solution if $\beta = \beta^*$. On the other hand, we obtain

$$G_0(r_m) = e^{-\alpha r_m} - e^* (r_m) = e^{-\alpha r_m} - \frac{\lambda^*}{\alpha - \lambda^*} \left( e^{-\lambda^* r_m} - e^{-\alpha r_m} \right) = \frac{\alpha}{\alpha - \lambda^*} e^{-\alpha r_m} - \frac{\lambda^*}{\alpha - \lambda^*} e^{-\lambda^* r_m}. \quad (5.36)$$

From (5.36), we know that a necessary condition to show that $G_0(r_m) > 0$ is $\lambda^* < r_m^{-1} < \alpha$ (from the physical meaning, it is always that $r_m^{-1} < \alpha$). Moreover, if $\lambda^*$ (the total infectivity) is sufficiently small, then $G_0(r_m) > 0$. That is, inequality (5.19) holds. This situation is possible if $\beta$ is sufficiently near to the critical value $\beta^*$, because in this case, the positive root of $\lambda^*$ of $f(\lambda) = 1$ is small enough.

**REFERENCES**