Average distance and domination number

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Received 30 May 1996; received in revised form 10 February 1997

Abstract

We give a sharp upper bound on the average distance of a graph of given order and domination number and determine the extremal graphs.

1. Introduction

Let $G = (V, E)$ be a finite, simple and undirected graph with vertex set $V$ and edge set $E$. If $G$ is connected, the average distance $\mu(G)$ is defined to be the average of all distances in $G$.

$$
\mu(G) := \frac{1}{n(n-1)} \sum_{x, y \in V(G)} d_G(x, y),
$$

where $n = |V(G)|$ is the order of $G$ and $d_G(x, y)$ denotes the length of a shortest path joining the vertices $x$ and $y$. The average distance, or mean distance, has been investigated by several authors and under various names. The transmission $\sigma(G)$ of a graph $G$, defined as the sum of the distances between all ordered pairs of vertices and the Wiener index $W(G)$ of a graph $G$, defined as the sum of the distances between all unordered pairs of vertices, differ from the average distance only by the factor $n(n-1)$ and $\binom{n}{2}$, respectively.

The Wiener index, introduced in 1947 by the chemist Wiener [20], has numerous applications in physical chemistry (see e.g. [11]). It has been used in the characterization of many different types of chemical species, including alkanes, alkenes and arenes [15]. The index has been correlated with a large number of physiochemical properties in such species, e.g. the boiling point, refractive index, surface tension, viscosity, melting point and chromatographic retention time [14].

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PII S0166-218X(97)00067-X
Several upper and lower bounds on the average distance in terms of other graph parameters are known. Plesnič [13] essentially clarified the relation between the diameter, the radius, and the average distance. He constructed graphs with specified radius and diameter whose average distance is arbitrary close to any given real between 1 and the diameter. Based on the well-known Moore bound on the order of a graph with given diameter and maximum degree, a lower bound on the average distance of a connected, regular graph of given order and vertex degree was given by Cerf et al. [1]. The minimum and maximum average distance of a graph with given order and chromatic number have been determined by Tomescu and Melter [18]. Recently, Tomescu [17] has improved this result for 2-connected graphs. Algorithmic aspects of the average distance are investigated in [7, 3]. A generalization of the average distance, the average Steiner distance, was recently introduced by Dankelmann et al. [5, 6].

The computer program GRAFFITI [9] (1986) made the attractive conjecture

$$\mu(G) \leq \alpha(G),$$

where $\alpha(G)$ denotes the independence number of $G$. Chung [2] succeeded in proving the conjecture. She also established that equality holds only for the complete graph, i.e. for $\alpha = 1$. In [4] sharp upper and lower bounds for $\mu$, depending as well on the independence number as on the order, were given. It turned out that the extremal graphs attaining the upper bound consist of two complete graphs or stars, whose orders differ at most by one, connected by a path. It seems likely that graphs having a similar shape are good candidates for the solution of various extremal average distance problems. In fact, the unique graph of given order and matching number and maximum average distance has a similar structure (see [4]). In this paper we will show that, essentially, the same applies to extremal graphs of given order and domination number and maximal average distance.

The notation we will use is as follows. The neighbourhood $N_G(x)$ of a vertex $x \in V(G)$ is the set of all vertices adjacent to $x$. The closed neighbourhood $\bar{N}_G(x)$ of a vertex $x \in V$ contains $N_G(x)$ and the vertex $x$ itself; $d_G(x) := |N(x)|$ denotes the degree of the vertex $x$. We will drop the subscript if no confusion can occur. A vertex of degree one is called an end vertex. The domination number $\gamma(G)$ of a graph $G$ is the minimum cardinality of a subset $D$ of $V(G)$ such that each vertex of $G$ that is not contained in $D$ is adjacent to at least one vertex of $D$. A graph is called empty if its edge set is empty. By $K_n$ we denote the complete graph and by $P_n$ the path of order $n$, respectively; $nK_1$ is the empty graph of order $n$. A star is a tree containing one vertex that is adjacent to each other vertex. For disjoint graphs $G$ and $H$, the corona $G \circ H$ is obtained from $G$ and $|V(G)|$ disjoint copies of $H$, one for each vertex of $G$, by joining each vertex of $G$ to all vertices of its copy of $H$. A vertex of a graph $G$ is a cut vertex if its deletion increases the number of connected components of $G$. The diameter $d_{m}(G)$ of a connected graph $G$ is the maximum over all distances between vertices of $G$. The transmission $\sigma(x) = \sigma(x, G)$ of a vertex $x \in V$ is the sum of all distances between $x$ and each other vertex of $G$. 
The transmission $\sigma(G)$ of the graph $G$ is the sum of all transmissions of the vertices of $G$,

$$\sigma(x, G) := \sum_{y \in V} d(x, y),$$

$$\sigma(G) := \sum_{x \in V} \sigma(x) = \sum_{(x, y) \in V \times V} d(x, y).$$

In order to avoid large fractions, we will often deal with $\sigma(G)$ rather than with $\mu(G)$.

2. Results

First we derive upper bounds on the transmission of a vertex, depending on the order and domination number of the graph. They are essential for the proof of the main theorem.

The following useful observation is due to Walikar and Acharya [19].

Lemma 1 (Walikar and Acharya [19]). Let $H$ be a graph. The following two statements are equivalent.

(i) $\gamma(H - e) > \gamma(H)$ for each edge $e \in E(H)$.
(ii) $H$ is the union of vertex disjoint stars.

An immediate consequence of Lemma 1, or directly of the definition of the domination number, is the fact that every connected graph $G$ contains a spanning tree $T$ with the same domination number.

This implies that every extremal graph $G$ of given order and domination number and maximum average distance is a tree, since otherwise we could delete an edge of $G$ that does not belong to the spanning tree $T$. This would not change the domination number but decrease the average distance of $G$, in contradiction to $G$ having maximal average distance.

The following lemma is an extension of a result of Zelinka [21], which states that in a tree each vertex having maximum transmission is an end vertex. The proof is similar to the one given by Zelinka.

Lemma 2. (i) Let $G$ be a connected graph and $v \in V(G)$ a cut vertex. Then there is a vertex $w \in N(v)$ with $\sigma(w) > \sigma(v)$.

(ii) If $G$ is a tree and $v$ is neither an end vertex nor adjacent to an end vertex, then there is a vertex $w$, adjacent to an end vertex, with $\sigma(w) > \sigma(v)$.

Proof. (i) Let $G_1, G_2, \ldots, G_k$ be the components of $G - v$ and let $H_i$ be the induced subgraph of $G$ containing the vertices of $G_i$ and $v$. Without loss of generality, we can assume that $G_1$ is a component of least order.
Choose a vertex \( w \in N(v) \cap V(G_1) \). Using
\[
\sigma(w, H_1) \leq \sigma(w, G_1) + |V(G_1)|
\]
we obtain
\[
\sigma(w, G) = \sigma(w, G_1) + 1 + \sum_{i>1} \sum_{a \in V(G_i)} d_G(a, w)
\]
\[
\geq \sigma(v, H_1) + |V(G_1)| + 1 + \sum_{i>1} (\sigma(v, H_i) + |V(G_i)|)
\]
\[
= \sigma(v, G) + 1 - |V(G_1)| + \sum_{i>1} |V(G_i)| > \sigma(v, G).
\]
This proves (i). Statement (ii) follows by successively applying (i) to the neighbour in a smallest component. □

We now describe the extremal graphs for which the transmission of a single vertex in a graph of given order and domination number is maximum. As stated above, we have to consider only trees, but it is easy to prove that the inequality of Lemma 3 holds generally for connected graphs.

**Definition 1.** (i) For positive integers \( n, \gamma \) with \( 1 \leq \gamma \leq n/3 \) let \( H_{n,\gamma} \) be the graph consisting of a path \( P_{3\gamma-1} = (v_1, v_2, \ldots, v_{3\gamma-1}) \) and independent vertices \( w_1, \ldots, w_{n+1-3\gamma} \) that are all joined with \( v_{3\gamma-1} \) (Fig. 1).

(ii) For positive integers \( n, \gamma \) with \( n/3 < \gamma < n/2 \) let \( H_{n,\gamma} \) be the graph obtained from a path \( P_{2n-3\gamma+1} = (v_1, v_2, \ldots, v_{2n-3\gamma+1}) \) and independent vertices \( w_{3n-6\gamma+3}, \ldots, w_{2n-3\gamma+1} \) by joining \( v_i \) and \( w_i \) for \( 3n - 6\gamma + 3 \leq i \leq 2n - 3\gamma + 1 \) (Fig. 2).

We note that a result of Ore [12] states that every graph of order \( n \) without isolated vertices has domination number at most \( n/2 \). Fink et al. [10] proved that equality holds only for \( C_4 \) and for graphs of the form \( H \circ K_1 \) for some \( H \).

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**Fig. 1.** The extremal graph \( H_{n,\gamma} \) from Definition 1(i) and Lemma 3 for \( \gamma \leq n/3 \).
The reason for the different shapes of the extremal graphs for $\gamma \leq n/3$ and $\gamma > n/3$ is the fact that the path $P_n$, which is the unique graph of order $n$ maximizing the transmission of a vertex, has domination number $[n/3]$.

**Lemma 3.** Let $G$ be a tree of order $n$ and domination number $\gamma$. Then, for each vertex $v \in V(G)$,

$$
\sigma(v, G) \leq \begin{cases} 
(3\gamma - 1)(n - \frac{3}{2}\gamma) & \text{if } \gamma \leq \frac{n}{3}, \\
(2n - 3\gamma + 1)^2 - \frac{1}{2}(3n - 6\gamma + 3)(3n - 6\gamma + 2) & \text{if } \gamma > \frac{n}{3}.
\end{cases}
$$

Equality holds if and only if $G = H_{n, \gamma}$ and $v = v_1$.

**Proof.** (i) We first consider the case $\gamma \leq n/3$. Let $W$ be a shortest path between two diametral vertices and let $D$ be a minimum dominating set of $G$. Since every vertex of $D$ dominates at most 3 vertices of $W$, we have $dm(G) \leq \frac{3\gamma}{2} - 1$, and thus

$$
\sigma(v, G) \leq 1 + 2 + \cdots + (3\gamma - 2) + (3\gamma - 1)(n - 3\gamma + 1) = (3\gamma - 1)(n - \frac{3}{2}\gamma).
$$

The uniqueness of the extremal graph is obvious.

(ii) Let $\gamma > n/3$. We will first prove the following bound on the diameter of a tree $H$ of order $n$ and domination number at least $\gamma \geq (n + 2)/3$,

$$
dm(H) \leq 2n - 3\gamma + 1.
$$

The proof of (2) is by induction on $n$.

It is easy to verify that the statement holds for $n \leq 7$ or for $\gamma = [n/3]$. In the latter case the extremal graph is a path. So we can assume $n \leq 3\gamma - 3$. Let $H$ have maximum diameter among all trees of order $n$ and domination number at least $\gamma$. Let $a, z$ be a diametral pair of vertices. Then $a$ is an end vertex with a unique neighbour $b$. By our choice of $a$ and $z$, each neighbour of $b$ except one is an end vertex. If $b$ were adjacent to another end vertex $a' \neq a$, then the graph $H' = H - a'b + a'a$ would have diameter $dm(H') = dm(H) + 1$ and $\gamma(H') \geq \gamma(H)$, contradicting the choice of $H$. Hence $b$ is adjacent to exactly one end vertex and has degree $d_H(b) = 2$.

Let $c \neq a$ be the other neighbour of $b$. 

![Fig. 2. The extremal graph $H_{n, \gamma}$ from Definition 1(ii) and Lemma 3 for $\gamma > n/3$.](image-url)
Case 1: \(d_H(c) = 2\). Then \(H - a - b - c\) is connected and has diameter at least \(dm(H) = 3\). Induction yields

\[ dm(H) \leq dm(H - a - b - c) + 3 \leq 2(n - 3) - 3(\gamma - 1) + 1 + 3 = 2n - 3\gamma + 1. \]

Case 2: \(d_H(c) \geq 3\). Let \(d\) be a neighbour of \(c\) that is not on the \(c, z\)-path. Then \(d\) is an end vertex since otherwise, by the choice of \(a\) and \(z\), the vertex \(d\) is adjacent only to \(c\) and to some end vertices, implying that \(H' = H - cd + bd\) has domination number \(\gamma(H') = \gamma(H)\) and diameter \(dm(H) + 1\), which contradicts the choice of \(H\). Consequently, we have \(\gamma(H - a - b) = \gamma(H) - 1\) and thus, by induction,

\[ dm(H) \leq dm(H - a - b) + 1 \leq 2(n - 2) - 3(\gamma - 1) + 1 + 1 = 2n - 3\gamma + 1 \]

and (2) follows.

The proof of the second inequality of (1) is by induction on \(n\). Again (1) holds for \(n \leq 6\) and for \(\gamma = \lceil n/3 \rceil\). In the latter case the extremal graph is a path. So we can assume that \(n \leq 3\gamma - 3\).

Let \(G\) be a tree and \(v \in V(G)\) such that \(\sigma(v, G)\) is maximum among all trees of order \(n\) and domination number at least \(\gamma\).

Let \(w\) be an eccentric vertex of \(v\). Then \(w\) is an end vertex with a unique neighbour \(u\). By the choice of \(w\), each neighbour of \(u\) except one is an end vertex. On the other hand, \(u\) can be adjacent to only one end vertex since otherwise, if \(w_1, w_2 \in N_G(u)\) are end vertices, the graph \(G' = G - uw_2 + w_1w_2\) has domination number \(\gamma(G') \geq \gamma(G)\) and \(v\) has transmission \(\sigma(v, G') > \sigma(v, G)\). Thus, we have

\[ d_G(u) = 2. \]

Hence, \(G - w - u\) is connected and has domination number at least \(\gamma(G) - 1\). By the induction hypotheses and by (2) we have

\[ \sigma(v, G) = \sigma(v, G - w - u) + d_G(v, w) + d_G(v, u) \leq \sigma(v_1, H_{n-2, \gamma-1}) + (2n - 3\gamma + 1) + (2n - 3\gamma) = \sigma(v_1, H_{n, \gamma}), \]

implying (1).

It remains to prove the uniqueness of the extremal graph. If equality holds in (1), then also in (3). By induction we have

\[ G - w - u = H_{n-2, \gamma-1} \quad \text{and} \quad v = v_1. \]

Together with the fact that the vertices \(w\) and \(u\) have distance \(2n - 3\gamma + 1\) and \(2n - 3\gamma\), respectively, from \(v\), this implies \(G = H_{n, \gamma}\) and \(v = v_1\).

For the sake of completeness we prove that the bound in Lemma 3 holds for every connected graph and we characterize the extremal graphs.
Corollary 1. Let $G$ be a connected graph of order $n$ and domination number $\gamma$. Then, for each vertex $v \in V(G)$,

$$\sigma(v, G) \leq \begin{cases} 
(3\gamma - 1)(n - \frac{3\gamma}{2}) & \text{if } \gamma \leq \frac{n}{3}, \\
(2n - 3\gamma + 1) - \frac{1}{2}(3n - 6\gamma + 3)(3n - 6\gamma + 2) & \text{if } \gamma > \frac{n}{3}.
\end{cases} \quad (4)$$

Equality in (4) holds if and only if $G$ is obtained from $H_{n, \gamma}$ by adding edges of the form $w_iw_j$ and $v = v_1$. Equality in (4) holds if and only if $G = H_{n, \gamma}$ and $v = v_1$.

Proof. By the remark after Lemma 1, $G$ contains a spanning tree $T$ with the same domination number $\gamma$. Since $\sigma(v, G) \leq \sigma(v, T)$ for every vertex $v \in V(G)$, the bound in Lemma 3 applies also to $G$.

In the case of equality, we have $T = H_{n, \gamma}$ and $v = v_1$, and $T$ preserves the distances to $v$. But the only graphs $G$ with $H_{n, \gamma}$ as a subgraph preserving the distances to $v_1$ and having the same domination number are $H_{n, \gamma}$ for $\gamma > n/3$ and the graphs $G$ obtained from $H_{n, \gamma}$ for $\gamma \leq n/3$ by adding edges of the form $w_iw_j$ for $\gamma \leq n/3$. \(\square\)

The preceding lemma enables us to prove the following sharp upper bound on the average distance of a graph with given order $n$ and domination number $\gamma$. Again the shape of the extremal graphs differs according as $\gamma \leq n/3$ or $\gamma > n/3$. We will treat the two cases separately.

Definition 2. For positive integers $n, \gamma$ with $\gamma \leq n/3$ let $G_{n, \gamma}$ denote the graph obtained from a path $P_{3\gamma - 2}$ with end vertices $v_1, v_2$, which are identical if $\gamma = 1$, and two independent sets of vertices $W_1, W_2$ of order $[(n - 3\gamma + 2)/2]$ and $[(n - 3\gamma + 2)/2]$, respectively, by joining each vertex of $W_i$ to $v_i$, $i = 1, 2$.

Theorem 1. Let $G$ be a connected graph of order $n$ and domination number $\gamma \leq n/3$. Then we have

$$\mu(G) \leq \begin{cases} 
\frac{n + 1}{3} - \frac{(n - 3\gamma)(n - 3\gamma + 2)(2n + 3\gamma - 7)}{6n(n - 1)} & \text{if } n - \gamma \text{ is even}, \\
\frac{n + 1}{3} - \frac{(n - 3\gamma)(n - 3\gamma + 2)(2n + 3\gamma - 7) - 9(\gamma - 1)}{6n(n - 1)} & \text{if } n - \gamma \text{ is odd}.
\end{cases}$$

Equality holds if and only if $G = G_{n, \gamma}$ (Fig. 3).

Proof. The proof is by induction on $n$. It is easy to check that the statement of the theorem holds for $n \leq 6$. Assume the theorem holds for all values smaller than $n$. We will prove the statement for $n$ by induction on $\gamma$. Clearly, the theorem holds for $\gamma = 1$, so we can assume that $n \geq 7$ and $\gamma \geq 2$. If $\gamma = \lceil n/3 \rceil$, then the theorem follows immediately from the fact that the graph $G_{n, \lceil n/3 \rceil}$ is isomorphic to the path $P_n$, the unique graph which has maximum transmission among all connected graphs of order $n$, so let $\gamma < n/3$. \(\square\)
Let $G$ be a connected graph of order $n$ and domination number $\gamma$ that has maximum transmission. As stated in the preceding section, $G$ is a tree.

We first show

If $\gamma(G - xy) = \gamma(G)$ then $d_G(x), d_G(y) \leq 2$. \hfill (A)

Let $G_x$ and $G_y$ denote the connected components of $G - xy$ that contain $x$ and $y$, respectively. With $X = V(G_x)$, $Y = V(G_y)$ we have

\[
\sigma(G) = \sum_{a,b \in X} + \sum_{a,b \in Y} + 2 \sum_{a \in X, b \in Y} d_G(a,b)
\]

\[
= \sigma(G_x) + \sigma(G_y) + 2 \sum_{a \in X, b \in Y} (d_G(a,x) + 1 + d_G(b,y))
\]

\[
= \sigma(G_x) + \sigma(G_y) + 2|X||Y| + 2|Y|\sigma(x,G_x) + 2|X|\sigma(y,G_y).
\]

Suppose that $d_G(x) > 2$ and consequently $x$ is not an end vertex of $G_x$. By Lemma 2 there is a vertex $x' \in N(x)$ with $\sigma(x',G_x) > \sigma(x,G_x)$. Consider the graph $G' = G - xy + x'y$. Clearly, we have $\gamma(G') \in \{\gamma, \gamma - 1\}$. Using the same calculations as above we obtain

\[
\sigma(G') > \sigma(G),
\]

which implies together with the maximality of $\sigma(G)$ that

$\gamma(G') = \gamma(G) - 1$.

Since the theorem holds for $n$ and $\gamma - 1$, we have

$\sigma(G) < \sigma(G') \leq \sigma(G_{n,\gamma - 1}) < \sigma(G_{n,\gamma})$,

which contradicts the maximality of $\sigma(G)$. Similarly, we prove that $y$ is an end vertex of $G_y$ and thus (A) is established.
Now, we are able to accomplish the induction step. By Lemma 1 \( G \) contains an edge \( xy \) whose deletion does not change the domination number. It is easy to see that we can choose \( xy \) such that the components \( G_x \) and \( G_y \) of \( G - xy \) both are nontrivial. By (A) \( x \) and \( y \) are end vertices of \( G_x \) and \( G_y \). Let \( x' \) and \( y' \) be the unique neighbours of \( x \) in \( G_x \) and \( y \) in \( G_y \), respectively. Denote the graph obtained from \( G \) by identifying the vertices \( x, x', y, \) and \( y' \) to a new vertex \( z \) and deleting loops by \( G' \). Clearly, \( G' \) has \( n - 3 \) vertices and

\[
\gamma(G') = \gamma(G) - 1
\]

and, consequently, \( \gamma(G') \leq |V(G')|/3 \). Let \( X = V(G_r) - \{x, x'\} \), \( Y = V(G_s) - \{y, y'\} \), \( p = |V(G_r)| \), \( q = \gamma(G_r) \), and \( d'(a, b) = d_G(a, b) \). By induction we have

\[
\sigma(G) = \left( \sum_{a, b \in X} + \sum_{a, b \in Y} + 2 \sum_{a \in X, b \in Y} \right) d_G(a, b) + \sigma(G[\{x, x', y', y\}])
\]

\[
+ 2 \sum_{a \in X \cup Y} (d_G(a, x') + d_G(a, x) + d_G(a, y') + d_G(a, y))
\]

\[
- \left( \sum_{a, b \in X} + \sum_{a, b \in Y} + 2 \sum_{a \in X, b \in Y} \right) d'(a, b) + 6|X||Y| + 20
\]

\[
+ 2 \sum_{a \in X \cup Y} d'(a, z) + 6 \sum_{a \in X} (d_G(a, x) + 1) + 6 \sum_{a \in Y} (d_G(a, y) + 1)
\]

\[
= \sigma(G') + 6|X||Y| + 26 + 6(\sigma(x, G_r) + \sigma(y, G_s) - 2 + |X| + |Y|)
\]

\[
= \sigma(G') + 6n - 16 + 6(|X|(|n - 4 - |X|) + \sigma(x, G_x) + \sigma(y, G_y))
\]

\[
\leq \sigma(G_{n-3, \gamma-1}) + 6n - 16
\]

\[
+ 6[(p - 2)(n - p - 2) + \sigma(v_1, H_{p, q}) + \sigma(v_1, H_{n-p, \gamma-q})].
\]  

(5)

Denote the term in square brackets by \( F(p, q) \). In order to maximize \( F \) we have to distinguish two cases.

Case 1: \( q > p/3 \) or \( \gamma - q > (n - p)/3 \). First, let \( q > p/3 \). It follows that \( \gamma - q < (n - p)/3 \). The derivative of \( F \) with respect to \( p \) equals \( -n - 3\gamma - 3(p - 3q) - \frac{3}{2} \) and is positive since \( \gamma < n/3 \). Hence, we obtain from (5) the contradiction

\[
0 \geq \sigma(G_{n-3, \gamma-1}) + 6n - 16 + 6F(p, q) - \sigma(G)
\]

\[
\leq \sigma(G_{n-3, \gamma-1}) + 6n - 16 + 6F(3q, q) - \sigma(G_{n, \gamma})
\]

\[
= \begin{cases} 
-\frac{3}{2}n^2 - \frac{27}{2}n + 9\gamma n - 12 & \text{if } n - \gamma \text{ is even}, \\
-\frac{5}{2}n^2 - \frac{27}{2}n + 9\gamma n - 3 & \text{if } n - \gamma \text{ is odd}, 
\end{cases}
\]

(*)

\[
\leq \begin{cases} 
-12 & \text{if } n - \gamma \text{ even}, \\
-3 & \text{if } n - \gamma \text{ odd}, 
\end{cases}
\]

\(< 0 \).
where we have to take into account that the expressions (*) are monotonically increasing in $\gamma$ and that $\gamma \leq n/3$. The case $\gamma - q > (n - p)/3$ can be treated analogously.

Case 2: $q \leq p/3$ and $\gamma - q \leq (n - p)/3$. It is easy to verify that $F$ attains its maximum for $p = 3q + \lceil(n - 3\gamma)/2\rceil$. Hence

$$F(p, q) \leq F(3q + \lceil \frac{1}{2}n - \frac{3}{2}\gamma \rceil, q)$$

$$= \begin{cases} 
\frac{1}{4}n^2 - \frac{9}{4}\gamma^2 + \frac{3}{2}\gamma n - 3n + \frac{3}{2}\gamma + 4 & \text{if } n - \gamma \text{ is even,} \\
\frac{1}{4}n^2 - \frac{9}{4}\gamma^2 + \frac{3}{2}\gamma n - 3n + \frac{3}{2}\gamma + \frac{15}{4} & \text{if } n - \gamma \text{ is odd.} 
\end{cases}$$

Note that the above expression does not depend on $q$. From (5) we obtain by an easy calculation

$$0 \leq \sigma(G_{n-3\gamma-1}) + 6n - 16 + 6F(3q + \lceil \frac{1}{2}n - \frac{3}{2}\gamma \rceil, q) - \sigma(G) \leq \sigma(G_{n-3\gamma-1}) + 6n - 16 + 6F(3q + \lceil \frac{1}{2}n - \frac{3}{2}\gamma \rceil, q) - \sigma(G_{n-3\gamma-1})$$

$$= 0.$$ 

This yields

$$\sigma(G) = \sigma(G_{n, \gamma}).$$

There remains only the uniqueness of the extremal graph $G_{n, \gamma}$ to be shown. Obviously, equality in the above inequality implies equality in (5) and thus $\sigma(G') = \sigma(G_{n-3\gamma-1})$, hence

$$G' = G_{n-3\gamma-1}.$$ 

Furthermore, we have $\sigma(x, G_{x}) = \sigma(v, H_{p,q})$ and $\sigma(y, G_{y}) = \sigma(v, H_{n-p-y-q})$ and thus by Lemma 3,

$$G_{x} = H_{p,q} \quad \text{and} \quad G_{y} = H_{n-p-y-q}.$$ 

It is easy to see that only the graph $G_{n, \gamma}$ has these properties and so the proof of Theorem 1 is complete. $\square$

We now determine the extremal graphs for $\gamma(G) > n/3$.

**Definition 3.** For positive integers $n, \gamma$ with $\gamma > n/3$ let $G_{n, \gamma}$ denote the graph obtained from a path $P = (v_{1}, v_{2}, \ldots, v_{2n-3\gamma})$ and two independent sets of vertices $W_{1} = \{w_{1}, w_{2}, \ldots, w_{\lceil(3\gamma-n)/2\rceil}\}$ and $W_{2} = \{w'_{1}, w'_{2}, \ldots, w'_{\lceil(3\gamma-n)/2\rceil}\}$, respectively, by joining $v_{j}$ to $w_{j}$ and $v_{2n-3\gamma+1-j}$ to $w'_{j}$ for $j = 1, \ldots, \lceil(3\gamma-n)/2\rceil$. 


Theorem 2. Let \( G \) be a graph of order \( n \) with domination number \( \gamma \geq n/3 \). Then the average distance \( \mu(G) \) is at most

\[
\frac{n + 1}{3} - \frac{(3\gamma - n)(3\gamma - n - 2)(5n - 6\gamma - 4)}{3n(n - 1)} \quad \text{if } n - \gamma \text{ is even.}
\]

\[
\frac{n + 1}{3} - \frac{(3\gamma - n - 1)((3\gamma - n - 3)(5n - 6\gamma - 2) + 6(2n - 3\gamma - 1))}{3n(n - 1)} \quad \text{if } n - \gamma \text{ is odd.}
\]

Equality holds if and only if \( G = G_{n, \gamma} \) (Fig. 4).

Proof. The proof is by induction on \( n \). Since the bound is strictly decreasing in \( \gamma \), it suffices to prove it for all graphs with domination number greater than or equal to a given number \( \gamma \). The theorem is immediate for \( \gamma = \lfloor n/3 \rfloor \), since \( G_{n, \gamma} = P_n \) is the unique graph with maximum transmission among the connected graphs of order \( n \). It is easy to check that the theorem holds for \( n < 8 \), so let \( n \geq 9 \). Moreover, we will assume that \( \gamma > \lfloor n/3 \rfloor \).

Let \( G \) be a connected graph of order \( n \) and domination number \( \gamma(G) \geq \gamma \) with maximum transmission. As noted in the first section, \( G \) is a tree.

In order to accomplish an induction step similar to the proof of Theorem 1, we will show that \( G \) contains an induced subgraph isomorphic to \( P_k \circ K_1 \) for some \( k \). Shrinking this graph to \( P_{k-1} \circ K_1 \) will yield a graph of order \( n - 2 \) and domination number \( \gamma(G) - 1 \), to which the induction hypothesis can be applied.

We first show that

No vertex of \( G \) is adjacent to more than one end vertex. \hfill (6)

Suppose a vertex \( v \) is adjacent to two end vertices \( u, w \). Then \( G' = G - uw + uv \) has domination number \( \gamma(G') \geq \gamma(G) \) and \( \sigma(G') > \sigma(G) \), contradicting the choice of \( G \).

Now let \( a, z \) be diametrical vertices. Then \( a \) is an end vertex with a unique neighbour \( b \). Since \( b \) is adjacent to at most one non-end vertex, (6) yields that \( b \) is adjacent to only one other vertex \( c \neq a \).
Case 1: $d_G(c) = 2$. Then $G' = G - \{a, b, c\}$ is connected and has domination number $\gamma(G) - 1$. With $V' = V(G')$ we have by Lemma 3 and the induction hypotheses

$$\sigma(G) = \left( \sum_{x,y \in V(G')} + 2 \sum_{x \in V', y \in V(G) - V'} + \sum_{x,y \in V(G) - V'} \right) d_G(x,y)$$

$$= \sigma(G') + 2 \sum_{y \in V'} (3d_G(a,y) - 3) + 8$$

$$= \sigma(G') + 6\sigma(a,G) - 18 - 6(n-3) + 8$$

$$\leq \sigma(G_{n-3,\gamma-1}) + 6\sigma(v_1,H_{n,\gamma}) - 6n + 8. \tag{7}$$

By the choice of $G$ we have $\sigma(G) \geq \sigma(G_{n,\gamma})$. From that we obtain by some calculations, the details of which we omit,

$$0 \geq \sigma(G_{n,\gamma}) - \sigma(G_{n-3,\gamma-1}) - 6\sigma(v_1,H_{n,\gamma}) + 6n - 8$$

$$= \begin{cases} 
3n^2 + 27\gamma^2 - 18\gamma n + 12n - 36\gamma + 12 & \text{if } n - \gamma \text{ is even}, \\
3n^2 + 27\gamma^2 - 18\gamma n + 12n - 36\gamma + 9 & \text{if } n - \gamma \text{ is odd}. 
\end{cases}$$

Denote the obtained expression by $F(n,\gamma)$. A simple differentiation shows that for constant $n$ and $\gamma \geq (n+2)/3$ the function $F(n,\gamma)$ is strictly increasing in $\gamma$. Hence, by our assumption $\gamma \geq (n+3)/3$, we have

$$0 \geq F(n,\gamma) \geq F(n, (n+3)/3) = \begin{cases} 
3 & \text{if } n - \gamma \text{ is even}, \\
0 & \text{if } n - \gamma \text{ is odd}. 
\end{cases}$$

This is clearly impossible unless $n - 3\gamma - 3$ which implies that $n - \gamma$ is odd and equality holds in (7). In particular, this implies

$$\sigma(a,G) = \sigma(v_1,H_{3\gamma-3,\gamma}),$$

and thus by Lemma 3

$$G = H_{3\gamma-3,\gamma} = G_{3\gamma-3,\gamma}.$$  

Hence the theorem follows.

Case 2: $d_G(c) \geq 3$. We first prove that $c$ is adjacent to an end vertex. Assume that $c$ is not adjacent to an end vertex. By Lemma 2(ii), $G - a - b$ contains a vertex $x$ with $\sigma(x,G - a - b) > \sigma(c,G - a - b)$ that is adjacent to an end vertex. Then the graph $G' = G - bc + bx$ has domination number $\gamma(G') \geq \gamma(G)$ and transmission $\sigma(G') > \sigma(G)$, contradicting the choice of $G$. Hence $c$ is adjacent to an end vertex, denote it by $d$. 


Hence $G$ contains an induced subgraph $H$, namely $H = G[\{a, b, c, d\}]$, with the following properties:

\[
(*) \quad \begin{cases} 
H \text{ is isomorphic to } P_k \circ K_1 \text{ for some } k \geq 2, \\
\text{bound}_G(V(H)) \subseteq \{u\} \text{ for some vertex } u \in V(H) \text{ with } d_H(u) = 2.
\end{cases}
\]

where $\text{bound}_G(M)$ is the set of all vertices of $M, M \subseteq V(G)$, that are adjacent to a vertex not in $M$.

Among all induced subgraphs $H$ of $G$ with properties ($\ast$) and $k \leq (3\gamma - n + 1)/2$ choose one of maximum order.

The vertex $u$ has two neighbours in $H$, one end vertex and one of degree at least two in $H$. Denote these by $U'$ and $u$, respectively, and let $w_1, w_2, \ldots, w_r$ be the remaining neighbours of $u$ in $G$. We define a new graph

\[
G' = G - \{u, u'\} + \{uw_1, uw_2, \ldots, uw_r\},
\]

i.e. we delete the vertices $u$ and $u'$ and join the $u$-neighbours in $V(G) - V(H)$ to $v$.

With $X = V(H) \setminus \{u, u'\}$ and $Y = V(G) - V(H)$ we obtain

\[
\sigma(G) = \left( \sum_{x, y \in X} + \sum_{x, y \in Y} + 2 \sum_{x \in X, y \in Y} \right) d_G(x, y)
\]

\[
+ 2 \sum_{x \in V(G) - \{u, u'\}} (d_G(u, x) + d_G(u', x)) + 2
\]

\[
= \left( \sum_{x, y \in X} + \sum_{x, y \in Y} + 2 \sum_{x \in X, y \in Y} \right) d_{G'}(x, y)
\]

\[
+ 2(2k - 2)(n - 2k) + 2 \sum_{x \in V(G) - \{u, u'\}} (2d_G(u', x) - 1) + 2
\]

\[
= \sigma(G') + 2(2k - 2)(n - 2k) + 4(\sigma(u', H) + \sigma(u', G - X)) - 2n - 2.
\]

It is easy to check that $\gamma(G') = \gamma(G) - 1$ and $\gamma(G') \geq |V(G')|/3$, so we can apply the induction hypotheses to $G'$. Together with $\gamma(H) = k$ and $\gamma(G - X) = \gamma(G) - k + 1 \geq |V(G) - X|/3$ and Lemma 3 we obtain from above

\[
\sigma(G) \leq \sigma(G_{n - 2, \gamma - 1}) + 2(2k - 2)(n - 2k) + 4\sigma(v, H_{2k, k})
\]

\[
+ 4\sigma(v, H_{n - 2k + 2, \gamma - k + 1}) - 2n - 2. \quad (8)
\]

Denote the obtained expression by $F(k)$. A simple differentiation shows that for constant $n, \gamma$ and $0 \leq k \leq \lfloor (3\gamma - n + 1)/2 \rfloor$ the function $F(k)$ attains its maximum at
Thus, the inequality of Theorem 2 is proved.
If equality holds in Theorem 2 then we are either in Case 1 and have $n - 3\gamma - 3$ and $G = G_{n,\gamma}$ or we are in Case 2 and equality holds in (8). Then we have $k = ((3\gamma - n + 1)/2)$, $H = P_k \circ K_1$, and $G - X = H_{n-2k+2,\gamma-k+1}$, implying $G = G_{n,\gamma}$. □

The problem of determining a sharp lower bound on the average distance depending on the order and the domination number is unsolved. The following graphs indicate that this bound is very close to 1. Sanchis [16] proved that these graphs are the unique connected graphs with given order and domination number and maximum size.

For $4 \leq \gamma \leq n/2$ let $G_n^{n,\gamma}$ denote the graph consisting of a $(n - \gamma)$-clique together with an independent set of cardinality $\gamma$ such that each of the vertices in the $(n - \gamma)$-clique is adjacent to exactly one of the vertices in the independent set and such that $G_n^{n,\gamma}$ has no isolated vertices. Then, we have

$$
\mu(G_n^{n,\gamma}) = \frac{n - 3 + 2\gamma}{n - 1}.
$$

We remark that the problem restricted to trees is much simpler. Let $T_n^{n,\gamma}$ be the tree obtained from a star with end vertices $v_1, \ldots, v_{n-\gamma}$ by adding vertices $w_1, \ldots, w_{n-1}$ and joining each $w_i$ with $v_i$. It is easy to prove by induction that for every tree $T$ of order $n$ and domination number $\gamma$

$$
\mu(T) \geq \mu(T_n^{n,\gamma}),
$$

with equality if and only if $T = T_n^{n,\gamma}$.

Acknowledgements

I am grateful to Professor Volkmann for leading my attention to the subject of this paper and for many helpful discussions.
References