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Advances in Mathematics 202 (2006) 602–637

ADVANCES IN Mathematics

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Associative conformal algebras with finite faithful representation ☆

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Received 4 May 2004; accepted 4 April 2005

Communicated by P. Etingof

Available online 10 May 2005

Abstract

The classification of irreducible subalgebras of the associative conformal algebra Cend_N is presented in this paper. The structure theory of associative conformal algebras with finite faithful representation is developed.

MSC: 16S32; 16S99; 16W20

Keywords: Conformal algebra; Weyl algebra; Finite representation

1. Introduction

The notion of a conformal algebra appears as an algebraic language describing singular part of operator product expansion (OPE) in conformal field theory [4]. The explicit algebraic exposition of this theory (see, e.g., [8,15,19]) leads to the notion of a vertex (chiral) algebra. Roughly speaking, conformal algebras correspond to vertex algebras by the same way as Lie algebras correspond to their associative enveloping algebras (see [2,21] for a detailed explanation).

In the recent years a great advance in the structure theory of associative and Lie conformal algebras of finite type has been obtained. In [12,13,17,18,22], simple and

☆ Partially supported by RFBR and SSc-2269.2003.

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doi:10.1016/j.aim.2005.04.001

semisimple Lie conformal (super)algebras of finite type were described (as well as associative ones). The main result of [13] was generalized in [1] for finite pseudoalgebras.

Some features of the structure theory of conformal algebras (and their representations) of infinite type were also considered in a series of works (see [9,10,14,22,24,25,29,30]). One of the most urgent problems in this field is to describe the structure of conformal algebras with faithful irreducible representation of finite type (these algebras could be of infinite type themselves). In [9,22], the conjectures on the structure of such algebras (associative and Lie) were stated. The papers [9,14,30] contain confirmations of these conjectures under some additional conditions.

Another problem is to classify simple and semisimple conformal algebras of linear growth (i.e., of Gel'fand–Kirillov dimension one) [24]. In the papers [24,25,29,30], this problem was solved for finitely generated associative conformal algebras which contain a unit [24,25], or at least an idempotent [29,30]. The objects appearing from the consideration of conformal algebras with faithful irreducible representation of finite type are similar to those examples of conformal algebras stated in these papers.

A combinatorial aspect of the theory of conformal algebras requires the notion of a free conformal (and vertex) algebra. Actually, it was stated in [8] what is to be the free vertex algebra. Studying of free conformal algebras was initiated in [26]. The basic notions of the combinatorial theory of associative conformal algebras (Gröbner–Shirshov bases, Composition–Diamond lemma) were considered in [5–7].

The class of conformal algebras coincides with the class of pseudoalgebras [1,3] over the polynomial Hopf algebra $\Bbbk[D]$, where D is just a formal variable, $\Bbbk$ is a field of zero characteristic. In general, a pseudoalgebra over a cocommutative Hopf algebra H is just an algebra in the pseudotensor category associated with H [3]. Therefore, the notion of a pseudoalgebra provides the common point of view to the theory of ordinary algebras ($H = \Bbbk$) and to the theory of conformal algebras ($H = \Bbbk[D]$). In particular, for every finitely generated H-module V one can define a pseudoalgebra structure on the set of all H-linear (with respect to the second tensor factor) maps $V \to H \otimes H \otimes_H V$ [1, Proposition 10.1]. The pseudoalgebra obtained is denoted by $\operatorname{Cend} V$ (conformal endomorphisms). This is a direct analogue of the ordinary algebra $\operatorname{End} U$ of all linear transformations of a vector space U.

From now on, let $\Bbbk$ be an algebraically closed field of zero characteristic, and let $H = \Bbbk[D]$ be the polynomial algebra in one variable D. By ∂_x we denote the operator of formal derivation with respect to a variable x. We will also use the notation $x^{(n)} = x^n/n!$, $n \in \mathbb{Z}_+$, where $\mathbb{Z}_+$ means the set of non-negative integers.

Our purpose is to develop the structure theory of associative conformal algebras with faithful representation of finite type (as a corollary, we obtain the classification of simple and semisimple associative conformal algebras of finite type [22]). The main point of this theory is an analog of the classical Burnside theorem on irreducible subalgebras of the matrix algebra $M_N(\Bbbk)$. The last statement says that, if a subalgebra $S \subseteq M_N(\Bbbk)$ acts irreducibly on $\Bbbk^N$ (i.e., there are no non-trivial submodules), then $S = M_N(\Bbbk)$. A similar question for conformal algebras was raised by Kac [22]. In this context, the "classical" objects $\Bbbk$, $\Bbbk^N$, $M_N(\Bbbk)$ should be replaced with H, V_N, Cend_N, respectively. Here V_N is the free N-generated H-module, $\operatorname{Cend}_N = \operatorname{Cend} V_N$.

The associative conformal algebra Cend_N (and its adjoint Lie conformal algebra gc_N) plays the same role in the theory of conformal algebras as $M_N(\Bbbk)$ (respectively, $\mathrm{gl}_N(\Bbbk)$) does in the theory of ordinary algebras.

Ref. [9] contains a systematical investigation of the algebra Cend_N: the structure of its left/right ideals and their modules of finite type, classification of (anti-) automorphisms and involutions, etc. Since the algebra Cend_N is the main object of our study, we will also state some of these results.

In [22] the following statement was conjectured: *every irreducible conformal subalgebra $C \subseteq \mathrm{Cend}_N$ is either a left ideal of Cend_N or a conjugate of the current subalgebra* (in the last case, C is of finite type). In [9], the second part of this conjecture (i.e., finite type case) was derived from the well-developed representation theory of Lie conformal algebras of finite type [13]. Also, the conjecture was proved in [9] for $N = 1$ as well as for unital conformal subalgebras.

We prove this conjecture in general (Theorem 5.14). Moreover, although it follows from [13,22] that any simple associative conformal algebra of finite type is isomorphic to the current conformal algebra over $M_n(\Bbbk)$, we focus on the independent proof of this fact.

The paper is organized as follows. In Section 2 we state the definitions of conformal algebras and their representations. Also, we write down the general construction of an associative conformal algebra: given an (ordinary) associative conformal algebra A with a locally nilpotent derivation ∂, the free H-module $H \otimes A$ could be endowed with the structure of an associative conformal algebra [1,22,24,25] denoted by $\mathrm{Diff}\, A$.

In Section 3, we introduce the conformal algebra $\mathrm{Cend}_N = \mathrm{Cend}\, V_N$ (N is a positive integer) which is the main object of our study. This conformal algebra is isomorphic to $\mathrm{Diff}\, M_N(\Bbbk[v])$, where v is a formal variable, and $\partial = \partial_v$ is the usual derivation with respect to v. Therefore, Cend_N could be identified with $M_N(\Bbbk[D, v])$.

As a main tool of our investigation we use a correspondence between conformal subalgebras of Cend_N and subalgebras of $M_N(W)$, where $W = \Bbbk\langle p, q \mid qp - pq = 1\rangle$ is the 1st Weyl algebra. This correspondence is provided by a construction called operator algebra, which is very close to the annihilation algebra [1,22]. Namely, for any conformal subalgebra $C \subseteq \mathrm{Cend}_N$ we consider the ordinary algebra $S(C) \subseteq M_N(W)$ which consists of linear operators on V_N corresponding to all elements of C. If C is a left (right) ideal then so is $S(C)$. Moreover, C has a canonical structure of a (left) $S(C)$-module.

In Section 4, we prove some properties of operator algebras. First, we show that for any irreducible conformal subalgebra $C \subseteq \mathrm{Cend}_N$ its "expanded" operator algebra $\Bbbk[p]S(C)$ is a dense subalgebra of $M_N(W)$ with respect to the finite topology [20]. The rest of the section is devoted to the properties of "small" operator algebras (i.e., those of linear growth) corresponding to conformal subalgebras of finite type. Roughly speaking, we show that if $S = S(C) \subset M_N(W)$ is an operator algebra of linear growth corresponding to an irreducible conformal subalgebra $C \subseteq \mathrm{Cend}_N$, then C is a conjugate of the current subalgebra of Cend_N. In this way, we obtain an independent description of irreducible conformal subalgebras of finite type [9, Theorem 5.2].

Section 5 contains the main results of the paper. The basic point is to describe all irreducible conformal subalgebras of Cend_N. First, we show that if $C \subseteq \mathrm{Cend}_N$ is irreducible, then

$$\sum_{n\geqslant 0} v^n C = \mathrm{Cend}_{N,Q}, \tag{1.1}$$

where $\mathrm{Cend}_{N,Q} = M_N(\Bbbk[D,v])Q(-D+v)$, $Q = Q(v)$ is a non-degenerate matrix with polynomial entries.

It is left to consider three cases:

(1) the sum (1.1) is direct;
(2) $C \cap vC \neq 0$;
(3) sum (1.1) is non-direct, but $C \cap vC = 0$.

The first case is considered in Section 5.2. Using the results of Sections 4.2 and 4.3, we show that C is a conjugate of the current subalgebra of Cend_N. In the second case, we use the method proposed in [9, Section 2] for $N = 1$: in Section 5.3 we reduce the computation to the one-dimensional case and obtain $C = \mathrm{Cend}_{N,Q}$. The third case is impossible, as we show in Section 5.4.

In Section 5.5 we complete the classification of irreducible subalgebras (Theorem 5.14) and proceed with the structure theory of associative conformal algebras with faithful representation of finite type: we describe simple and semisimple ones. Moreover, we show that any conformal algebra of this kind has a maximal nilpotent ideal. As a corollary, we obtain the classification theorem for associative conformal algebras of finite type [22].

2. Conformal algebras and their representations

In this section, we state necessary definitions and constructions from the theory of conformal algebras.

2.1. Definition and examples of conformal algebras

Definition 2.1 (*Kac [21]*). A vector space C endowed with a linear map D and with a family of bilinear operations $\circ_n$, $n \in \mathbb{Z}_+$, is said to be a *conformal algebra*, if

(C1) $a \circ_n b = 0$, for n sufficiently large, $a, b \in C$;
(C2) $Da \circ_n b = -na \circ_{n-1} b$;
(C3) $a \circ_n Db = D(a \circ_n b) + na \circ_{n-1} b$.

Axiom (C1) allows to define so-called *locality function* $\mathcal{N} : C \times C \to \mathbb{Z}_+$. Namely,

$$\mathcal{N}(a,b) = \min\{N \in \mathbb{Z}_+ \mid a \circ_n b = 0 \text{ for all } n \geqslant N\}, \quad a, b \in C. \tag{2.1}$$

Every conformal algebra could be considered as a unital left module over $H = \Bbbk[D]$ endowed with a family of sesquilinear (i.e., satisfying (C2), (C3)) products $\circ_n$, $n \in \mathbb{Z}_+$, such that the locality axiom (C1) holds.

For any pair of D-invariant subspaces (H-submodules) $X, Y \subseteq C$ the space $X \circ_\omega Y = \sum_{n\in\mathbb{Z}_+} X \circ_n Y$ is also an H-submodule.

An ideal of a conformal algebra C is an H-submodule $I \subseteq C$ which is closed under all multiplications by C, i.e., $C \circ_\omega I, I \circ_\omega C \subseteq I$. Left and right ideals are defined analogously.

A conformal algebra C is simple, if $C \circ_\omega C \neq 0$ and there are no non-trivial ideals.

For any ideal I of a conformal algebra C one can define descending sequences of ideals $\{I^{(n)}\}_{n=1}^\infty$ and $\{I^n\}_{n=1}^\infty$ by the usual rule: $I^{(1)} = I^1 = I$, $I^{(n)} = I^{(n-1)} \circ_\omega I^{(n-1)}$, $I^n = \sum_{s=1}^{n-1} I^s \circ_\omega I^{n-s}$, $n > 1$. An ideal I is said to be solvable (resp., nilpotent), if $I^{(n)} = 0$ (resp., $I^n = 0$) for sufficiently large n. If a conformal algebra C has no non-zero solvable ideals then C is called semisimple.

If a conformal algebra C is finitely generated as an H-module, then it is said to be *of finite type* (or *finite* conformal algebra).

A homomorphism $\phi : C_1 \to C_2$ of conformal algebras is a D-invariant linear map such that $\phi(a \circ_n b) = \phi(a) \circ_n \phi(b)$, $a, b \in C_1$, $n \in \mathbb{Z}_+$.

For any conformal algebra C in the sense of Definition 2.1 there exists an ordinary algebra A such that C lies in the space of formal distributions $A[[z, z^{-1}]]$, where $D = \partial_z$ and the $\circ_n$-products are given by

$$a(z) \circ_n b(z) = \operatorname*{Res}_{w=0} a(w)b(z)(w-z)^n, \quad n \in \mathbb{Z}_+. \tag{2.2}$$

Such an algebra A is not unique, but there exists a universal one denoted by Coeff C. Namely, for any algebra A such that $A[[z, z^{-1}]]$ contains C as above, there exists a homomorphism Coeff $C \to A$ such that the natural expansion Coeff $C[[z, z^{-1}]] \to A[[z, z^{-1}]]$ acts on C as an identity [22,26]. The algebra Coeff C is called the *coefficient algebra* of C.

There is a correspondence between identities on Coeff C and systems of conformal identities on C. In particular, Coeff C is associative if and only if C satisfies

$$(a \circ_n b) \circ_m c = \sum_{s\geqslant 0} (-1)^s \binom{n}{s} a \circ_{n-s} (b \circ_{m+s} c) \tag{2.3}$$

or

$$a \circ_n (b \circ_m c) = \sum_{s\geqslant 0} \binom{n}{s} (a \circ_{n-s} b) \circ_{m+s} c, \tag{2.4}$$

for all $n, m \in \mathbb{Z}_+$. The systems of relations (2.3) and (2.4) are equivalent. Also, Coeff C is a Lie algebra if and only if C satisfies

$$a \circ_n b = -\sum_{s\geqslant 0} (-1)^{n+s} D^{(s)}(b \circ_{n+s} a), \tag{2.5}$$

$$a \circ_n (b \circ_m c) - b \circ_m (a \circ_n c) = \sum_{s\geqslant 0} \binom{n}{s} (a \circ_{n-s} b) \circ_{m+s} c \tag{2.6}$$

for all $n, m \in \mathbb{Z}_+$.

A conformal algebra C is called associative, if $\operatorname{Coeff} C$ is associative, i.e., if C satisfies (2.3) or (2.4). Analogously, C is called Lie conformal algebra, if $\operatorname{Coeff} C$ is a Lie algebra, i.e., if C satisfies (2.5) and (2.6). It is easy to note that the notion of solvability coincides with nilpotency in the case of associative conformal algebras.

Proposition 2.2 (*See, e.g., Kac [22]*). *Let C be an associative conformal algebra. Then the same H-module C endowed with new operations*

$$[a \circ_n b] = a \circ_n b - \sum_{s\geqslant 0} (-1)^{n+s} D^{(s)}(b \circ_{n+s} a), \quad a, b \in C, \ n \in \mathbb{Z}_+ \tag{2.7}$$

is a Lie conformal algebra denoted by $C^{(-)}$.

Example 2.3. Let $\mathcal{A}$ be an (associative or Lie) algebra. Then formal distributions of the form

$$a(z) = \sum_{n\in\mathbb{Z}} a t^n z^{-n-1} \in \mathcal{A}[t, t^{-1}][[z, z^{-1}]], \quad a \in \mathcal{A},$$

together with all their derivatives span a conformal algebra in $\mathcal{A}[t, t^{-1}][[z, z^{-1}]]$ called the *current conformal algebra* $\operatorname{Cur} \mathcal{A}$. Operations (2.2) are given by

$$a(z) \circ_n b(z) = \begin{cases} (ab)(z), & n = 0, \\ 0, & n > 0. \end{cases}$$

The conformal algebra $\operatorname{Cur} \mathcal{A}$ is associative or Lie if and only if so is $\mathcal{A}$.

Example 2.4. Consider $\mathcal{A} = \Bbbk\langle t, t^{-1}, \partial \mid \partial t - t\partial = 1\rangle$. Then formal distributions of the form

$$v_m(z) = \sum_{n\in\mathbb{Z}} t^n \partial^m z^{-n-1} \in \mathcal{A}[[z, z^{-1}]], \quad m \in \mathbb{Z}_+,$$

together with all their derivatives span a conformal algebra in $\mathcal{A}[[z, z^{-1}]]$ called the *Weyl conformal algebra*.

Example 2.5. Consider the element $L = v_1$ from the previous example. It is easy to compute that the one-generated H-submodule $H \otimes \Bbbk L$ of the Weyl conformal algebra is closed under operations (2.7):

$$[L \circ_0 L] = -DL, \quad [L \circ_1 L] = -2L, \quad [L \circ_n L] = 0, \ n \geqslant 2.$$

The Lie conformal algebra obtained is called the *Virasoro conformal algebra*.

2.2. Representations of associative conformal algebras

Remind that $H = \Bbbk[D]$ is a Hopf algebra, so $H \otimes H$ could be considered as the outer product of regular right H-modules (the action is given by $(f \otimes g) \cdot D = fD \otimes g + f \otimes gD$). If $\{h_i \mid i \in I\}$ is a linear basis of H, then $\{h_i \otimes 1 \mid i \in I\}$ is an H-basis of the right H-module $H \otimes H$ (see, e.g., [1, Lemma 2.3]).

Definition 2.6 (*Bakalov et al. [1]*). Let V be a unital left H-module. A linear map

$$a : V \to (H \otimes H) \otimes_H V$$

is said to be a *conformal endomorphism* of V, if $a(fv) = ((1 \otimes f) \otimes_H 1)a(v)$ for any $f \in H$, $v \in V$.

Denote the set of all conformal endomorphisms of V by $\operatorname{Cend} V$. For any $a \in \operatorname{Cend} V$ there exists a uniquely defined sequence $\{a_n\}_{n=0}^{\infty}$ of (ordinary) $\Bbbk$-linear endomorphisms of V such that:

$$a_n(v) = 0 \quad \text{for } n \text{ sufficiently large}, \quad v \in V, \tag{2.8}$$

$$[a_n, D] = na_{n-1}, \quad n \in \mathbb{Z}_+, \tag{2.9}$$

$$a(v) = \sum_{n \geqslant 0} ((-D)^{(n)} \otimes 1) \otimes_H a_n(v), \quad v \in V. \tag{2.10}$$

Conversely, any sequence $\{a_n\}_{n=0}^{\infty} \subset \operatorname{End}_{\Bbbk} V$ satisfying (2.8) and (2.9) defines a conformal endomorphism $a \in \operatorname{Cend} V$ via (2.10).

For any $a, b \in \operatorname{Cend} V$ define the sequences $\{(Da)_n\}_{n=0}^{\infty}$ and $\{(a \circ_m b)_n\}_{n=0}^{\infty}$, $m \in \mathbb{Z}_+$, by the following rules (see, e.g., [13, Section 3]):

$$(Da)_n = [D, a_n] = -na_{n-1}, \tag{2.11}$$

$$(a \circ_m b)_n = \sum_{s \geqslant 0} (-1)^s \binom{m}{s} a_{m-s} b_{n+s}. \tag{2.12}$$

It is clear that these sequences satisfy (2.8) and (2.9), so the conformal endomorphisms Da, $a \circ_m b$ ($m \in \mathbb{Z}_+$) are defined.

For an arbitrary H-module V, the vector space $\operatorname{Cend} V$ endowed with the operations D and $\circ_m$ ($m \in \mathbb{Z}_+$) given by (2.11) and (2.12) satisfies (C2) and (C3). The locality property (C1) does not hold, in general. However, if V is a finitely generated H-module, then (C1) holds, so $\operatorname{Cend} V$ is an associative conformal algebra.

Let C be a conformal algebra, and let V be a left unital H-module. If $\varrho : C \to \operatorname{Cend} V$ is a linear map such that

$$\varrho(Da) = D\varrho(a), \quad \varrho(a \circ_n b) = \varrho(a) \circ_n \varrho(b), \quad a, b \in C, \ n \in \mathbb{Z}_+,$$

then $\overline{C} = \varrho(C) \subseteq \operatorname{Cend} V$ is a conformal algebra. By abuse of terminology, we will say that ϱ is a homomorphism of conformal algebras (although $\operatorname{Cend} V$ is not necessarily a conformal algebra).

Definition 2.7 (*Bakalov et al. [1], Cheng and Kac [11], Kac [21,22]*). A *representation* of an associative conformal algebra C on a left unital H-module V is a homomorphism $\varrho : C \to \operatorname{Cend} V$ of conformal algebras. If C has a representation on V, then V is said to be a module over conformal algebra C (or C-module, if the context is clear).

The last definition is equivalent to the following one [11]: an H-module V is a module over an associative conformal algebra C, if there is a family of $\circ_n$-products

$$\circ_n : C \otimes V \to V \tag{2.13}$$

such that

(M1) $a \circ_n v = 0$, for n sufficiently large, $a \in C$, $v \in V$;
(M2) $Da \circ_n v = -na \circ_{n-1} v$,
(M3) $a \circ_n Dv = D(a \circ_n v) + na \circ_{n-1} v$,
(M4) $a \circ_n (b \circ_m v) = \sum_{s \geqslant 0} \binom{n}{s} (a \circ_{n-s} b) \circ_{m+s} v$.

A representation $\varrho : C \to \operatorname{Cend} V$ is called faithful, if ϱ is injective. If V is a finitely generated H-module, then ϱ is said to be a representation of finite type (or finite representation).

In the case of ordinary algebras, every algebra which has a faithful finite-dimensional representation is finite-dimensional itself. It is not the case for conformal algebras: there exist infinite conformal algebras with faithful finite representation.

2.3. Differential conformal algebras

Let $\mathcal{A}$ be an associative algebra with a locally nilpotent derivation ∂. We consider the free H-module $H \otimes \mathcal{A}$ as an associative conformal algebra. Notice that by (C2), (C3) it is sufficient to define the family of $\circ_n$-products between elements of the form $1 \otimes a$, $a \in \mathcal{A}$. Hereinafter we identify $1 \otimes a$ with $a \in \mathcal{A}$.

Proposition 2.8 (*See, e.g., Bakalov et al. [1], Kolesnikov [23], Retakh [24]*). (i) *The family of $\circ_n$-products*

$$a \circ_n b = a\partial^n(b), \quad a, b \in \mathcal{A}, \tag{2.14}$$

defines an associative conformal algebra structure on $H \otimes \mathcal{A}$. The conformal algebra obtained is denoted by $\operatorname{Diff} \mathcal{A}$.

(ii) *The same is true for another family of operations*:

$$(a \circ_{(n)} b) = \sum_{s \geqslant 0} D^{(s)} \otimes \partial^{n+s}(a)b, \quad a, b \in \mathcal{A}. \tag{2.15}$$

The conformal algebra obtained is denoted by $\operatorname{Diff}^\circ \mathcal{A}$.

Proposition 2.9. $\operatorname{Diff} \mathcal{A} \simeq \operatorname{Diff}^\circ \mathcal{A}$.

Proof. The isomorphism $\varphi : \operatorname{Diff} \mathcal{A} \to \operatorname{Diff}^\circ \mathcal{A}$ is given by

$$h \otimes a \mapsto \sum_{s \geqslant 0} D^{(s)} h \otimes \partial^s(a),$$

the inverse $\varphi^{-1} : \operatorname{Diff}^\circ \mathcal{A} \to \operatorname{Diff} \mathcal{A}$ could be defined via

$$h \otimes a \mapsto \sum_{s \geqslant 0} (-D)^{(s)} h \otimes \partial^s(a).$$

Direct computation shows that $\varphi(a \circ_n b) = \varphi(a) \circ_{(n)} \varphi(b)$. $\square$

In [24,25], an associative conformal algebra obtained by either of constructions (2.14), (2.15) is called *differential*.

Example 2.10. Let $\mathcal{A}$ be an (associative) algebra, and let ∂ be the zero derivation: $\partial(\mathcal{A}) = 0$. Then $\operatorname{Diff} \mathcal{A} = \operatorname{Diff}^\circ \mathcal{A}$ is exactly the current conformal algebra $\operatorname{Cur} \mathcal{A}$ (Example 2.3).

Example 2.11. Consider $\mathcal{A} = \Bbbk[v]$, and let $\partial = \partial_v$ be the usual derivation. Then the conformal algebras $\operatorname{Diff} \mathcal{A}$ and $\operatorname{Diff}^\circ \mathcal{A}$ are isomorphic to the Weyl conformal algebra (Example 2.4).

Example 2.12. Let X be a set of symbols (generators), $v \notin X$, and let $\mathcal{A} = \Bbbk\langle X, v\rangle$ be the free associative algebra generated by $X \cup \{v\}$ endowed with the derivation $\partial = \partial_v$. Consider a conformal subalgebra of $\operatorname{Diff} \mathcal{A}$ generated by the set $\{v^{(N-1)}x \mid x \in X\}$, $N \geqslant 1$. This subalgebra is isomorphic to the *free associative conformal algebra* generated by X with respect to the constant locality function $\mathcal{N}(x, y) \equiv N$, $x, y \in X$ (see [5,26]).

The last example shows that every finitely generated associative conformal algebra is a homomorphic image of a subalgebra of a differential conformal algebra.

It was conjectured in [25] that every annihilation-free (see [24] for rigorous explanation) associative conformal algebra is just a subalgebra of some differential conformal algebra. Example 2.12 confirms this conjecture for free associative conformal algebras.

3. Conformal algebra Cend_N

From now on, we will use the term "conformal algebra" for associative conformal algebra, unless stated otherwise.

3.1. Construction

Let us introduce the main object of our study: the conformal algebra $\mathrm{Cend}_N = \mathrm{Cend}\, V_N$, where V_N is the free N-generated H-module. We will use the following presentation of Cend_N.

Consider the algebra $M_N(\Bbbk[v])$ of all $N \times N$-matrices with polynomial entries endowed with the derivation ∂_v with respect to the variable v. Denote the conformal algebras $\mathrm{Diff}\, M_N(\Bbbk[v])$ and $\mathrm{Diff}^{\circ} M_N(\Bbbk[v])$ by $\mathfrak{A}_N$ and $\mathfrak{A}_N^{\circ}$, respectively. Also, there is the usual multiplication on $H \otimes M_N(\Bbbk[v])$ as on $M_N(\Bbbk[D, v])$, and the following property of $\mathfrak{A}_N$ is clear:

$$v(a \circ_n b) = va \circ_n b, \quad a \circ_n vb = va \circ_n b + na \circ_{n-1} b. \tag{3.1}$$

Let us fix an H-basis $\{e_1, \ldots, e_N\}$ of V_N, and let $U \subset V_N$ be the finite-dimensional subspace spanned by this basis. Since $U \simeq \Bbbk^N$, the free H-module V_N could be identified with $H \otimes \Bbbk^N = H^N$. For any element $a = \sum_{s\geqslant 0} (-D)^{(s)} \otimes A_s(v) \in \mathfrak{A}_N$, $A_s(v) \in M_N(\Bbbk[v])$, consider the following sequence of $\Bbbk$-linear endomorphisms of V_N:

$$a_n : u \mapsto \sum_{s\geqslant 0} \binom{n}{s} A_s(D)\partial_D^{n-s}(u), \quad u \in V_N,\ n \in \mathbb{Z}_+. \tag{3.2}$$

It is easy to see that this sequence satisfies (2.8) and (2.9), so it defines a conformal endomorphism $\tilde{a} \in \mathrm{Cend}_N$.

Proposition 3.1 (*Bakalov et al. [1], D'Andrea and Kac [13], Kac [21], Retakh [24]*). *The map $a \mapsto \tilde{a}$ is an isomorphism of conformal algebras* $\mathfrak{A}_N$ and Cend_N.

Remark 3.2. It follows from Proposition 2.9 that

$$\mathfrak{A}_N \simeq \mathfrak{A}_N^{\circ} \simeq \mathrm{Cend}_N.$$

The isomorphism φ from Proposition 2.9 corresponds to the map $\mathfrak{A}_N \to \mathfrak{A}_N^{\circ}$ given by $D \mapsto D$, $v \mapsto D + v$. In [9,22], the second construction (2.15) was used for Cend_N. We will preferably follow [24] and use the $\circ_n$-products (2.14) for Cend_N.

From now on, identify Cend_N with $\mathfrak{A}_N = H \otimes M_N(\Bbbk[v])$, where $A(v) \circ_n B(v) = A(v)\partial_v^n(B(v))$.

Definition 3.3. A conformal subalgebra $C \subseteq \mathrm{Cend}_N$ is called *irreducible*, if there are no non-trivial C-submodules of V_N.

3.2. Operator algebras

Let us fix an H-linear basis of V_N, and identify V_N with $H \otimes \Bbbk^N$ as above. For every $a \in \mathrm{Cend}_N$ and for every $n \in \mathbb{Z}_+$ consider the $\Bbbk$-linear map

$$a(n) = a_n : V_N \to V_N, \quad a(n) : u \mapsto a \circ_n u, \;\; u \in V_N,$$

given by (3.2). Introduce a new variable p instead of D in this context, and set $q = \partial_p$. Thus, we get $a(n) \in M_N(W)$, where $W = \Bbbk\langle p, q \mid qp - pq = 1\rangle$ is the (1st) Weyl algebra. Axiom (C2) implies $(Da)(n) = -na(n-1) = -\partial_q a(n)$, where ∂_q is the partial derivation with respect to q.

In fact, the set $S(\mathrm{Cend}_N) = \{a(n) \mid a \in \mathrm{Cend}_N,\, n \in \mathbb{Z}_+\}$ is a linear space with multiplication (composition) given by

$$a(n)b(m) = \sum_{s\geqslant 0} \binom{n}{s} (a \circ_{n-s} b)(m+s). \tag{3.3}$$

It is easy to note that $S(\mathrm{Cend}_N) = M_N(W)$.

For every H-submodule X of Cend_N one may consider the subspace

$$S(X) = \{a(n) \mid a \in X,\, n \in \mathbb{Z}_+\} \subseteq S(\mathrm{Cend}_N).$$

If C is a conformal subalgebra of Cend_N, then $S(C)$ is a ∂_q-invariant subalgebra of $M_N(W)$. The same is true for left and right ideals.

Definition 3.4. Let $C \subseteq \mathrm{Cend}_N$ be a conformal subalgebra. The subalgebra $S(C)$ of $M_N(W)$ is called the *operator algebra* of C.

Let us consider the topology (called *q-topology*, for short) on $M_N(W)$ defined by the sequence of the left ideals generated by q^n, $n\geqslant 0$:

$$M_N(W) \supset M_N(W)q \supset \cdots \supset M_N(W)q^n \supset \cdots \supset 0.$$

It is clear that this topology is equivalent to the finite topology [20] defined by the canonical action of $M_N(W)$ on V_N.

Definition 3.5. A sequence $\{a_n\}_{n=0}^{\infty}$, $a_n \in M_N(W)$, is called *differential*, if $\partial_q(a_n) = na_{n-1}$ (we mean $\partial_q(a_0) = 0$) and $\lim\limits_{n\to\infty} a_n = 0$ (in the sense of q-topology).

Lemma 3.6. *Let $\{a_n\}_{n=0}^\infty$ be a differential sequence. Then there exist a finite number of matrices $A_s \in M_N(\Bbbk[p])$, $s = 0, \dots, m$, such that*

$$a_n = \sum_{s=0}^{m} \binom{n}{s} A_s q^{n-s}$$

for all $n \in \mathbb{Z}_+$.

Proof. By the definition of a differential sequence, there exists an integer $m \geqslant 0$ such that $a_n \in M_N(W)q$ for all $n > m$. Then we may represent the element a_m in the following form:

$$a_m = \sum_{s=0}^{m} \binom{m}{s} A_s q^{m-s},$$

where $A_0, \dots, A_m \in M_N(\Bbbk[p])$. For every $n \geqslant 0$ we may write

$$a_n = \sum_{s=0}^{m} \binom{n}{s} A_s q^{n-s} + \sum_{k \geqslant m+1} \binom{n}{k} B_k q^{n-k}.$$

The conditions $a_n \in M_N(W)q$ (for $n > m$) and $\partial_q a_n = n a_{n-1}$ imply $B_k = 0$ for all $k > m$. $\square$

Proposition 3.7. *A subalgebra $S \subseteq M_N(W)$ is an operator algebra of some conformal subalgebra $C \subseteq \mathrm{Cend}_N$ if and only if every element of S lies in a differential sequence of elements from S.*

Proof. If $S = S(C)$ for some $C \subseteq \mathrm{Cend}_N$, then every $a(m) \in S$ ($a \in C$, $m \geqslant 0$) lies in the differential sequence $\{a(n)\}_{n=0}^\infty$.

Let us prove the converse. By Lemma 3.6, every differential sequence $\{a_n\}_{n=0}^\infty$ corresponds to a finite family of matrices $A_s \in M_N(\Bbbk[p])$. Consider the elements

$$a = \sum_{s \geqslant 0} (-D)^{(s)} \otimes A_s(v) \in \mathrm{Cend}_N$$

corresponding to all differential sequences in S. It is clear that these elements form a conformal subalgebra C of Cend_N. Indeed, let $a, b \in \mathrm{Cend}_N$ correspond to $\{a_n\}_{n=0}^\infty$ and $\{b_n\}_{n=0}^\infty$, respectively. Then Da corresponds to $\{-\partial_q(a_n)\}_{n=0}^\infty$, and $a \circ_m b$ corresponds to $\{c_n\}_{n=0}^\infty$, where $c_n = \sum_{s \geqslant 0} (-1)^s \binom{m}{s} a_{m-s} b_{n+s}$ (it is easy to see that these c_n's form a differential sequence). It follows from the construction that $S(C) = S$. $\square$

Definition 3.8. Let $S \subseteq M_N(W)$ be a subalgebra satisfying the conditions of Proposition 3.7. Then the conformal algebra C constructed is called the *reconstruction* of S (c.f. [1, Section 11]). We will denote it by $\mathcal{R}(S)$.

In particular, every operator algebra $S = S(C) \subseteq M_N(W)$ is ∂_q-invariant. Hence, the subspace $\Bbbk[p]S = S + pS + p^2S + \cdots$ is also a subalgebra of $M_N(W)$.

Proposition 3.9. *A conformal subalgebra $C \subseteq \mathrm{Cend}_N$ is irreducible if and only if the subalgebra $S_1 = \Bbbk[p]S(C)$ acts irreducibly on V_N.*

Proof. Let C be an irreducible conformal subalgebra. For every $0 \neq u \in V_N$ the following H-module is also a C-module:

$$V_u = H\{a \circ_n u \mid a \in C,\ n \in \mathbb{Z}_+\}.$$

If $V_u = 0$, then $\{u \in V_N \mid V_u = 0\}$ is a non-trivial C-submodule. Hence, $V_u = V_N$. Then for every $w \in V_N$ there exists a finite family $\{a_i, n_i, f_i\}$, $a_i \in C$, $n_i \in \mathbb{Z}_+$, $f_i \in H$, such that $w = \sum_i f_i(a_i \circ_{n_i} u)$. Then for the S-module V_N we have $w = \sum_i f_i(p)a_i(n_i)u \in S_1u$. The converse statement has a similar proof. $\square$

Proposition 3.10. *The algebra $M_N(W)$ acts on the vector space Cend_N. The action is provided by*

$$a(n) \cdot b = a \circ_n b, \quad a, b \in \mathrm{Cend}_N,\ n \in \mathbb{Z}_+. \tag{3.4}$$

Proof. The explicit expression for the action is

$$A(p)q^n \cdot (D^{(s)} \otimes B(v)) = \sum_{t \geqslant 0} \binom{n}{t} D^{(s-t)} \otimes A(v)\partial_v^{n-t} B(v).$$

It is straightforward to check the relation (3.4) and the associativity. $\square$

Corollary 3.11. *For every conformal subalgebra $C \subseteq \mathrm{Cend}_N$ we have $S(C) \cdot C \subseteq C$.*

Corollary 3.12. *If $C \subseteq \mathrm{Cend}_N$ is a conformal subalgebra, then C is a left ideal of $\mathcal{R}(S(C))$.*

Proof. Let $x \in \mathcal{R}(S(C))$ be an element, corresponding to a differential sequence $\{b_n\}_{n=0}^{\infty}$, $b_n \in S(C)$. Then $x(n) = b_n$ for all $n \geqslant 0$. For every $a \in C$ we have $x \circ_n a = b_n \cdot a \in C$. $\square$

3.3. Automorphisms of Cend_N *and* $M_N(W)$

In this section, we consider a relation between automorphisms of Cend_N and $M_N(W)$. We will also obtain the full description of automorphisms of Cend_N, as in [9].

Proposition 3.13. *Every automorphism* Θ *of the conformal algebra* Cend_N *induces an automorphism* θ *of* $M_N(W)$:

$$\theta : a(n) \mapsto \Theta(a)(n). \tag{3.5}$$

This automorphism is ∂_q*-invariant and continuous in the sense of* q*-topology.*

Proof. First, let us show that the map (3.5) is well-defined. Suppose $a(n) = b(m) \in M_N(W)$. Without loss of generality we may assume $n = m$, so it is sufficient to show that $a(n) = 0$ implies $\Theta(a)(n) = 0$. If $a(n) = 0$, then $a = D^{n+1}x$ for some $x \in \mathrm{Cend}_N$, and $\Theta(a) = D^{n+1}\Theta(x)$. Hence, $\Theta(a)(n) = 0$ and θ is well-defined.

It follows from (3.3) that θ is an automorphism of $M_N(W)$. Moreover, since $\{a(n)\}_{n=0}^{\infty}$ is a differential sequence, we may write $\theta(\partial_q(a(n))) = n\theta(a(n-1)) = n\Theta(a)(n-1) = \partial_q(\theta(a(n)))$. Hence, θ is ∂_q-invariant.

Let us consider the image of $e = 1 \otimes \mathrm{I}_N$ under Θ. If $\Theta(e) = a \in \mathrm{Cend}_N$, then $\theta(q^n) = a(n)$. The sequence $\{a(n)\}_{n=0}^{\infty}$ is differential, and $\theta(M_N(\Bbbk[p])) = M_N(\Bbbk[p])$ since θ is ∂_q-invariant. It is easy to note that, under these conditions, θ has to be continuous in the sense of q-topology. $\square$

Example 3.14. Let us consider the map $\Theta_{\alpha,Q} : \mathrm{Cend}_N \to \mathrm{Cend}_N$ such that

$$\Theta_{\alpha,Q}(a(D,v)) = (1 \otimes Q^{-1}(v))a(D, v+\alpha)Q(-D \otimes 1 + 1 \otimes v), \tag{3.6}$$

$\alpha \in \Bbbk$, $Q, Q^{-1} \in M_N(\Bbbk[v])$. It is straightforward to check that $\Theta_{\alpha,Q}$ is an automorphism of Cend_N (see [9]). The corresponding automorphism $\theta_{\alpha,Q}$ of $M_N(W)$ acts as follows:

$$\theta_{\alpha,Q} : a(p,q) \mapsto Q^{-1}(p)a(p+\alpha, q)Q(p).$$

It is shown in [9] that $\Theta_{\alpha,Q}$ exhaust all automorphisms of Cend_N. Later we will also obtain this result.

Lemma 3.15. *Let* θ *be a* ∂_q*-invariant automorphism of* $M_N(W)$. *Then* $\theta(p) = p + \alpha$, $\theta(q) = q - B(p)$, *where* $\alpha \in \Bbbk$, $B(p) \in M_N(\Bbbk[p])$.

Proof. Since $\theta(M_N(\Bbbk[p])) = M_N(\Bbbk[p])$, we have $\theta(p) \in M_N(\Bbbk[p])$. The relation $[p, M_N(\Bbbk[p])] = 0$ implies $\theta(p)$ to be a scalar matrix: $\theta(p) = f(p)\mathrm{I}_N \equiv f(p)$. Analogously, $\partial_q(\theta(q)) = 1$, so $\theta(q) = q - B(p)$ for some $B(p) \in M_N(\Bbbk[p])$. Then $\theta([q,p]) = [q - B(p), f(p)] = f'(p) = 1$, so, $\deg f = 1$. $\square$

Lemma 3.16. (i) *Let θ be an automorphism of $M_N(\Bbbk[p])$ such that $\theta(p) = p$. Then there exists a non-degenerate matrix $Q \in M_N(\Bbbk[p])$ such that $Q^{-1} \in M_N(\Bbbk[p])$ and $\theta(X) = Q^{-1}XQ$ for all $X \in M_N(\Bbbk[p])$.*

(ii) *Let θ be an automorphism of $M_N(W)$ such that $\theta(p) = p$. Then $\theta(q) = q - B(p)$, where $B = h(p) - Q^{-1}Q'$ for some polynomial $h(p) \in \Bbbk[p]$ and invertible matrix $Q \in M_N(\Bbbk[p])$, $Q' = \partial_p(Q)$.*

Proof. (i) Denote $S_0 = \theta^{-1}(M_N(\Bbbk))$. It is easy to see that $\Bbbk[p]S_0 = M_N(\Bbbk[p])$ and the sum

$$\sum_{n \geqslant 0} p^n S_0 = M_N(\Bbbk[p]) \tag{3.7}$$

is direct.

Let us consider the canonical action of S_0 on V_N. Relation (3.7) implies that $S_0 u \neq 0$ for every $0 \neq u \in V_N$. So there exist non-zero finite-dimensional S_0-submodules of V_N, and we may choose a non-zero S_0-submodule U_0 of minimal dimension over $\Bbbk$. Since S_0 is simple, the S_0-module U_0 is faithful and $\dim_{\Bbbk} U_0 = N$.

Consider $u_1, \dots, u_N \in V_N$ such that $U_0 = \Bbbk u_1 \oplus \cdots \oplus \Bbbk u_N$. Then these vectors are linearly independent over the field of rational functions $\Bbbk(p)$. Indeed, let there exist $f_1(p), \dots, f_N(p) \in \Bbbk(p)$ such that

$$f_1(p)u_1 + \cdots + f_N(p)u_N = 0 \tag{3.8}$$

(it is sufficient to consider only $f_i(p) \in \Bbbk[p]$). Then we may choose $a_i \in S_0 = \mathrm{End}_{\Bbbk} U_0$ such that $a_i u_j = \delta_{ij} u_j$. Since $[S_0, \Bbbk[p]] = 0$, relation (3.8) implies $f_j(p)u_j = 0$ for every $j = 1, \dots, N$.

In particular, the matrix $P = (u_1, \dots, u_N)$ has non-zero determinant. In general, P^{-1} does not lie in $M_N(\Bbbk[p])$, but we still have $U_0 = P\Bbbk^N$, $P^{-1}S_0P = M_N(\Bbbk)$. Then (3.7) allows to conclude that $P^{-1}M_N(\Bbbk[p])P = M_N(\Bbbk[p])$. The last relation implies $P = g(p)R$, where $R, R^{-1} \in M_N(\Bbbk[p])$, $g(p) \in \Bbbk[p]$. For the matrix R obtained we also have $R^{-1}S_0R = M_N(\Bbbk)$.

Let τ be the automorphism of $M_N(\Bbbk[p])$ defined as the conjugation by R. Then the composition $\theta_1 = \tau \circ \theta^{-1}$ preserves p and $M_N(\Bbbk)$. Hence, θ_1 acts as the conjugation by a non-degenerate matrix, i.e., $\theta_1 : x \mapsto TxT^{-1}$ for some $T \in M_N(\Bbbk)$. Finally, θ is the conjugation by invertible matrix $Q = RT \in M_N(\Bbbk[p])$.

(ii) Since $\partial_q(a) = [a, p]$ for every $a \in M_N(W)$, we may conclude that θ satisfies the conditions of Lemma 3.15.

It follows from (i) that there exists an invertible matrix $Q \in M_N(\Bbbk[p])$ such that the restriction $\theta|_{M_N(\Bbbk[p])}$ is just the conjugation by Q.

If $S_0 = \theta^{-1}(M_N(\Bbbk))$, then for $q + A = \theta^{-1}(q)$ we have $[q + A, S_0] = 0$. So $[Q^{-1}(q + A)Q, M_N(\Bbbk)] = 0$, and the matrix $Q^{-1}(q + A)Q = q + Q^{-1}AQ + Q^{-1}Q'$ has to be scalar. In particular, $Q^{-1}AQ + Q^{-1}Q' = h(p)$ for some polynomial $h(p)$. Hence, $A = h(p) - Q'Q^{-1}$. But $\theta(A) = Q^{-1}AQ$ by (i), so $\theta(q) = q - \theta(A) = q - (h - Q^{-1}Q')$. $\square$

Theorem 3.17. *Let θ be a ∂_q-invariant automorphism of $M_N(W)$. Then*

$$\theta = \theta_{\alpha,Q,h} : a(p,q) \mapsto Q^{-1}(p)a(p+\alpha, q-h(p))Q(p)$$

for some $\alpha \in \Bbbk$, $h(p) \in \Bbbk[p]$, $Q, Q^{-1} \in M_N(\Bbbk[p])$.

The automorphism $\theta_{\alpha,Q,h}$ is continuous if and only if $h(p) = 0$.

Proof. By Lemma 3.15 there exists $\alpha \in \Bbbk$ such that $\theta(p) = p+\alpha$. Consider $\tau_1 = \theta_{\alpha,\mathrm{I}_N,0}$, then the composition $\theta_1 = \tau_1^{-1} \circ \theta$ preserves p: $\theta_1(p) = p$.

Lemma 3.16 implies that there exist a polynomial $h \in \Bbbk[p]$ and an invertible matrix $Q \in M_N(\Bbbk[p])$ such that $\theta_1(q) = q - h(p) + Q^{-1}Q'$ and $\theta_1|_{M_N(\Bbbk[p])}$ is the conjugation by Q. Consider $\tau_2 = \theta_{0,Q,h}$, then $\tau_2(q) = \theta_1(q)$. Since τ_2 preserves p, we may conclude that the composition $\theta_2 = \tau_2^{-1} \circ \theta_1$ is just the identity map. So, $\theta = \tau_1 \circ \tau_2 = \theta_{\alpha,Q,h}$.

It is clear that $\theta_{\alpha,Q,0}$ is continuous in the sense of q-topology. If $h \neq 0$, then $\theta_{\alpha,Q,h}$ is not continuous since the sequence $\theta_{\alpha,Q,h}(q^n) = Q^{-1}(q-h(p))^n Q$ does not converge to zero. $\square$

Corollary 3.18 (*Boyallian et al. [9, Theorem 4.1]*). *Any automorphism of* Cend_N *is of the form $\Theta_{\alpha,Q}$, as in* (3.6).

Proof. If Θ is an automorphism of Cend_N, then the corresponding automorphism θ is of the form $\theta_{\alpha,Q,0}$ by Proposition 3.13 and Theorem 3.17. By the construction,

$$\Theta(a)(n) = \theta_{\alpha,Q,0}(a(n)) = \Theta_{\alpha,Q}(a)(n),$$

so $\Theta = \Theta_{\alpha,Q}$. $\square$

Proposition 3.19. *Let $C \subseteq$* Cend_N *be a conformal subalgebra, and let $\theta : S(C) \to S$ be an isomorphism from $S(C)$ onto a subalgebra $S \subseteq M_N(W)$. If θ is ∂_q-invariant and continuous, then there exists an injective homomorphism $\Theta : C \to \mathcal{R}(S)$ of conformal subalgebras.*

Proof. Since θ is ∂_q-invariant and continuous, the subalgebra S satisfies the conditions of Proposition 3.7.

For every element $a \in C$ the sequence $\{\theta(a(n))\}_{n=0}^{\infty}$ is differential. By Lemma 3.6 there exists an element $\hat{a} \in \mathrm{Cend}_N$ (moreover, $\hat{a} \in \mathcal{R}(S)$) such that $\hat{a}(n) = \theta(a(n))$ for all $n \geqslant 0$. The map $\Theta : a \mapsto \hat{a}$ is an injective homomorphism of conformal algebras. $\square$

3.4. Left and right ideals of Cend_N

In this section, we state some results of [9, Section 1] on the structure of left and right ideals of Cend_N.

The following description of one-sided ideals was obtained by Bakalov.

Proposition 3.20 (*Boyallian et al. [9]*). *The conformal algebra* Cend_N *is simple. Every right ideal of* Cend_N *has the form* $\mathrm{Cend}_{P,N} = P(v)\mathrm{Cend}_N$, *where* $P \in M_N(\Bbbk[v])$. *Every left ideal has the form* $\mathrm{Cend}_{N,Q} = \mathrm{Cend}_N \varphi^{-1}(Q) = \mathrm{Cend}_N Q(-D\otimes 1 + 1\otimes v)$, *where* $Q \in M_N(\Bbbk[v])$, φ *is the isomorphism from Proposition* 2.9.

Lemma 3.21 (*Boyallian et al. [9]*). *Every* $\mathrm{Cend}_{N,Q}$, $\det Q \neq 0$, *is a conjugate of* $\mathrm{Cend}_{N,D}$, *where* D *is the canonical diagonal form of* Q, *i.e.*, $D = \mathrm{diag}(f_1, \dots, f_N)$, *every* f_i *is monic, and* $f_i \mid f_{i+1}$.

Proposition 3.22 (*Boyallian et al. [9]*). *Conformal subalgebra* $\mathrm{Cend}_{N,Q} \subseteq \mathrm{Cend}_N$ *is irreducible if and only if* $\det Q \neq 0$.

Example 3.23. Consider $C = M_N(\Bbbk[D]) \subset \mathrm{Cend}_N$. This subalgebra is just the current conformal algebra $\mathrm{Cur}_N = \mathrm{Cur}\, M_N(\Bbbk)$. It is clear (see also [9]) that Cur_N is irreducible since $S(\mathrm{Cur}_N) = M_N(\Bbbk[q])$ and $\Bbbk[p]S(\mathrm{Cur}_N) = M_N(W)$.

It was conjectured in [22] that $\Theta_{0,T}(\mathrm{Cur}_N)$, $T, T^{-1} \in M_N(\Bbbk[v])$, and $\mathrm{Cend}_{N,Q}$, $Q \in M_N(\Bbbk[v])$, $\det Q \neq 0$, exhaust all irreducible subalgebras of Cend_N. In Section 5, we will prove the conjecture.

Proposition 3.24 (*Boyallian et al. [9]*). *If* $\det Q \neq 0$, *then*:

(i) *every left ideal of* $\mathrm{Cend}_{N,Q}$ *is of the form* $\mathrm{Cend}_{N,TQ}$, $T \in M_N(\Bbbk[v])$;
(ii) $\mathrm{Cend}_{N,Q}$ *is a simple conformal algebra.*

Remark 3.25. The structure of right ideals of $\mathrm{Cend}_{N,Q}$, $\det Q \neq 0$, was also described in [9]. In fact, every right ideal of $\mathrm{Cend}_{N,Q}$ is of the form $P\mathrm{Cend}_{N,Q}$, $P \in M_N(\Bbbk[v])$.

4. Properties of operator algebras

4.1. Density of irreducible subalgebras

Proposition 3.9 shows the relation between irreducible conformal subalgebras of Cend_N and irreducible subalgebras of $M_N(W)$: if $C \subseteq \mathrm{Cend}_N$ is irreducible, then $S_1 = \Bbbk[p]S(C) \subseteq M_N(W)$ acts on V_N irreducibly. The examples of irreducible conformal subalgebras provided by Proposition 3.22 and Example 3.23 correspond to dense (in the sense of q-topology) subalgebras of $M_N(W)$. In this section, we prove that the density of $\Bbbk[p]S(C)$ is a necessary property of an irreducible conformal subalgebra C.

Let us remind the classical Density Theorem by Jacobson. Consider a primitive algebra S over a field $\Bbbk$ with a faithful irreducible left S-module V. One may identify S with its image in $\mathrm{End}_{\Bbbk} V$. The centralizer $\mathcal{D} = \{\phi \in \mathrm{End}_{\Bbbk} V \mid [\phi, S] = 0\}$ is a skew field over $\Bbbk$. The module V could be considered as a right vector space over $\mathcal{D}^{\mathrm{op}}$, where $\mathcal{D}^{\mathrm{op}}$ is anti-isomorphic to $\mathcal{D}$. There exists the natural embedding of S into $\mathcal{E} = \mathrm{End}_{\mathcal{D}^{\mathrm{op}}} V$.

Theorem 4.1 (*Jacobson [20]*). *The algebra S is a dense subalgebra of $\mathcal{E}$ in the sense of finite topology, i.e., for any $\mathcal{D}^{\mathrm{op}}$-independent family $u_1, \dots, u_n \in V$ and for any $w_1, \dots, w_n \in V$ there exists $a \in S$ such that $au_i = w_i$, $i = 1, \dots, n$.*

Theorem 4.2. *Let $S \subseteq M_N(W)$ be a ∂_q-invariant subalgebra such that $pS \subseteq S$. If S acts on V_N irreducibly, then S is a dense subalgebra of $M_N(W)$ in the sense of q-topology.*

Proof. Let us fix $0 \neq a \in S$. By $\mathcal{D} \subseteq \mathrm{End}_{\Bbbk} V_N$ we denote the centralizer of S. For any $\phi \in \mathcal{D}$ the relations $[\phi, a] = [\phi, pa] = 0$ imply

$$[\phi, p]a = 0. \tag{4.1}$$

But for every $b \in S$ we have

$$[[\phi, p], b] = [[\phi, b], p] + [\phi, [p, b]] = 0,$$

since $[p, b] = -\partial_q b \in S$. Hence, $[\phi, p] \in \mathcal{D}$ is either invertible or zero. The first is not the case because of (4.1), so $[\phi, p] = 0$.

Let us fix an H-basis $\{e_1, \dots, e_N\}$ of V_N. Since $[\phi, p] = 0$, the matrix of ϕ in the $\Bbbk$-basis $\{e_1, \dots, e_N, pe_1, \dots, pe_N, p^2e_1, \dots, p^2e_N, \dots\}$ of V_N has the block-triangular form

$$[\phi] = \begin{pmatrix} A_0 & 0 & 0 & \dots \\ * & A_0 & 0 & \dots \\ * & * & A_0 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix},$$

where A_0 is a matrix from $M_N(\Bbbk)$. Let $\lambda \in \Bbbk$ be an eigenvalue of A_0. Then the transformation $\phi - \lambda \mathrm{I}_N$ lies in $\mathcal{D}$, and it is not invertible. Hence, $\phi = \lambda \mathrm{I}_N$, $\lambda \in \Bbbk$.

Thus, $\mathcal{D} = \Bbbk$. Theorem 4.1 implies S to be a dense subalgebra of $\mathrm{End}_{\Bbbk} V_N$ in the sense of finite topology. It is clear that the finite topology is equivalent to the q-topology on $M_N(W)$, so S is dense in the sense of q-topology. $\square$

4.2. Operator algebras of linear growth

In this section, we prepare some additional facts about operator algebras of Gel'fand–Kirillov dimension one. These subalgebras correspond to conformal subalgebras of finite type. We will essentially use the following result of [28].

Theorem 4.3 (*Small and Warfield Jr. [28]*). *Let S be a finitely generated prime unital algebra such that* $\mathrm{GKdim}\, S = 1$. *Then S is a finitely generated module over its center.*

Proposition 4.4. *Assume that $S \subseteq M_N(W)$ is a ∂_q-invariant subalgebra such that $S_1 = \Bbbk[p]S$ is prime and satisfies ascending chain condition (a.c.c.) for right annihilation ideals. Then S is prime.*

Proof. Suppose that S is not prime, i.e., there exist non-zero ideals $I, J \lhd S$ such that $IJ = 0$.

If either I or J is ∂_q-invariant, then $(\Bbbk[p]I)(\Bbbk[p]J) = 0$, where $\Bbbk[p]I, \Bbbk[p]J \lhd_l S_1$. Since S_1 is prime, either I or J is zero.

Consider the following ascending sequence of ideals of S: $J_0 = 0$, $J_1 = J$, $J_{n+1} = J_n + \partial_q(J_n)$ for $n \geqslant 1$. It is easy to show by induction on n that $IJ_n \subseteq J_{n-1}$. In particular, $I^n J_n = 0$ for every $n \geqslant 0$.

Since S_1 satisfies a.c.c. for right annihilation ideals, there exists $m \geqslant 1$ such that

$$\mathrm{Ann}_{S_1}(I) \subseteq \mathrm{Ann}_{S_1}(I^2) \subseteq \cdots \subseteq \mathrm{Ann}_{S_1}(I^m) = \mathrm{Ann}_{S_1}(I^{m+1}) = \cdots.$$

We have shown that $J_n \subseteq \mathrm{Ann}_{S_1}(I^n)$, so for every $p \geqslant m$ the ideal J_p lies in $\mathrm{Ann}_{S_1}(I^m)$. In particular, $I^m J_\partial = 0$, where $J_\partial = \bigcup_{n \geqslant 0} J_n$ is a ∂_q-invariant ideal. If $I^m \neq 0$, then we obtain a contradiction as it was shown above.

Therefore, it is sufficient to prove that S has no non-zero nilpotent ideals. By the very same reasons as stated above, S has no non-zero ∂_q-invariant nilpotent ideals.

It is easy to note that, if $I^n = 0$, then $(I + \partial_q(I))^{n^2} = 0$. Hence, for every nilpotent ideal $I \lhd S$ and for every $m \geqslant 0$ the ideal $I + \partial_q(I) + \cdots + \partial_q^m(I)$ is also nilpotent. If $I \neq 0$, then we may choose $0 \neq a \in I$ and consider the ideal I_a generated in S by all derivatives of a: $I_a = (a, \partial_q(a), \ldots, \partial_q^m(a)) \lhd S$. This ideal is ∂_q-invariant and lies in $I + \partial_q(I) + \cdots + \partial_q^m(I)$, so I_a is nilpotent. As it was shown above, $I_a = 0$ in contradiction with $a \neq 0$. □

For a subalgebra $S \subseteq M_N(W)$ denote by $Z(S)$ the center of S. If S_1 is a subalgebra of $M_N(W)$ such that $S_1 \supseteq S$, then by $\mathcal{Z}_{S_1}(S)$ we denote the centralizer of S in S_1. If S and S_1 are ∂_q-invariant, then so are $\mathcal{Z}_{S_1}(S)$ and $Z(S)$.

Lemma 4.5. *Let $S \subseteq M_N(W)$ be a finitely generated prime ∂_q-invariant subalgebra such that* $\mathrm{GKdim}\, S = 1$ *and* $\mathcal{Z}_{M_N(W)}(\Bbbk[p]S) = \Bbbk$. *Then there exists a matrix $A \in M_N(\Bbbk[p])$ such that $q + A \in Z(S)$.*

Proof. Although S is not supposed to be unital, it is still possible to apply Theorem 4.3, since $S + \Bbbk$ also satisfies the conditions of the statement. By Theorem 4.3 S is a finitely generated module over its center $Z(S)$. In particular, $Z(S) \neq 0, \Bbbk$. Since $Z(S)$ is ∂_q-invariant, there exists a non-zero element $y \in Z(S) \cap M_N(\Bbbk[p])$. But $[y, p] = 0$, so $y \in \Bbbk$ and $\Bbbk \subset Z(S)$. Now, let $x \in Z(S) \setminus \Bbbk$. We may assume that $\partial_q(x) \in \Bbbk$, hence, $x = q + A$, where $A \in M_N(\Bbbk[p])$. □

Proposition 4.6. *Let $S \subset M_N(W)$ be a ∂_q-invariant subalgebra such that*

$$\bigoplus_{n\geqslant 0} p^n S = M_N(W). \tag{4.2}$$

Then there exists a ∂_q-invariant automorphism θ of $M_N(W)$ such that $\theta(p) = p$ and $\theta(S) = M_N(\Bbbk[q])$.

Proof. It is clear that S is finitely generated and $\operatorname{GKdim} S = 1$ (see, e.g., [27]). Since $M_N(W)$ is (left and right) Noetherian [16], it satisfies a.c.c. for right annihilation ideals. By Proposition 4.4 the algebra S is prime, and by Lemma 4.5 there exists $x = q + A \in Z(S)$.

Moreover, for every ∂_q-invariant ideal $I \lhd S$ its "envelope" $\Bbbk[p]I$ is an ideal of $M_N(W)$. Thus, either $I = 0$ or $\Bbbk[p]I = M_N(W)$. The last relation implies $I = S$ since sum (4.2) is direct. Therefore, S has no non-trivial ∂_q-invariant ideals.

Denote $S_0 = S \cap M_N(\Bbbk[p]) = \operatorname{Ker} \partial_q|_S$. For every $k \geqslant 1$ let us define $S_k = \{a \in S \mid \partial_q(a) \in S_{k-1}\}$. Since ∂_q is locally nilpotent, we have

$$S = \bigcup_{k\geqslant 0} S_k, \qquad S_k = \operatorname{Ker} \partial_q^{k+1}|_S.$$

It is easy to show by induction on k that

$$S_k = S_0 + S_0 x + \cdots + S_0 x^k. \tag{4.3}$$

Indeed, it is clear for $k = 0$. For every $k \geqslant 1$ and for every $a \in S_k$ we have

$$\partial_q^k \left(a - \frac{1}{k} \partial_q(a) x \right) = 0,$$

so $a - \frac{1}{k}\partial_q(a)x \in S_{k-1}$. Since $\partial_q(a) \in S_{k-1}$, we may use the inductive assumption. Hence, (4.3) is true.

It is also clear that for every $k \geqslant 0$ sum (4.3) is direct. So, $S = \bigoplus_{n\geqslant 0} S_0 x^n$ and $\operatorname{GKdim} S_0 = 0$ [27]. The algebra S_0 has no non-trivial ideals since S is differentially simple. Moreover, for this simple finite-dimensional algebra S_0 we have

$$\Bbbk[p]S_0 = \bigoplus_{n\geqslant 0} p^n S_0 = M_N(\Bbbk[p]).$$

Hence, there exists an isomorphism $\theta : S_0 \to M_N(\Bbbk)$ which could be extended via $\theta(x) = q$, $\theta(p) = p$ to a ∂_q-invariant automorphism of $M_N(W)$ such that $\theta(S) = M_N(\Bbbk[q])$. $\square$

4.3. Operator algebras of current-type subalgebras

The main result of this section could be stated as follows: if there exists a ∂_q-invariant automorphism of $M_N(W)$ such that an operator algebra $S = S(C)$ goes onto $M_N(\Bbbk[q])$, then C is a conjugate of Cur_N by an automorphism of Cend_N (i.e., C is a *current-type* conformal subalgebra of Cend_N). To prove this, we will show that one may choose the isomorphism θ in Proposition 4.6 to be continuous.

Let $h \in \Bbbk[p]$ be a polynomial. Consider two sequences of polynomials in p: $\{h_{(n)}\}_{n\geqslant 0}$ and $\{h_{[n]}\}_{n\geqslant 0}$ such that

$$h_{(0)} = h_{[0]} = 1, \quad h_{(n)} = -hh_{(n-1)} + h'_{(n-1)}, \quad h_{[n]} = hh_{[n-1]} + h'_{[n-1]}. \tag{4.4}$$

It is easy to see that these sequences are constructed by the following reason. For any $n \geqslant 0$ we may write

$$(q-h)^n = a_n(p,q)q + h_{(n)}, \quad (q+h)^n = b_n(p,q)q + h_{[n]} \tag{4.5}$$

in the Weyl algebra W.

Lemma 4.7. *For every $k \geqslant 0$ we have*

$$\sum_{s=\xi}^{k} \binom{k-\xi}{s-\xi} h_{[s-\xi]} h_{(k-s)} = \begin{cases} 0, & \xi < k, \\ 1, & \xi = k. \end{cases} \tag{4.6}$$

Proof. The case $\xi = k$ is obvious, so we will concentrate our attention on the case $\xi < k$. For $k = 1$, $\xi = 0$ relation (4.6) is clear. Suppose that (4.6) holds for some integer k and for all $\xi \leqslant k$. Then for $k+1$ instead of k and for $\xi = k$ the corresponding equality in (4.6) is also true. For $\xi < k$ we have

$$\begin{aligned}
&\sum_{s=\xi}^{k+1} \binom{k-\xi+1}{s-\xi} h_{[s-\xi]} h_{(k+1-s)} \\
&\quad = \sum_{s=\xi}^{k} \binom{k-\xi}{s-\xi} h_{[s-\xi]} \big(-hh_{(k-s)} + h'_{(k-s)} \big) \\
&\qquad + \sum_{s=\xi+1}^{k+1} \binom{k-\xi}{s-\xi-1} \big(hh_{[(s-1)-\xi]} + h'_{[(s-1)-\xi]} \big) h_{(k-(s-1))} \\
&\quad = \sum_{s=\xi}^{k} \binom{k-\xi}{s-\xi} \big(h_{[s-\xi]} h'_{(k-s)} + h'_{[s-\xi]} h_{(k-s)} \big) = 0. \qquad \square
\end{aligned}$$

Corollary 4.8. *For any $\xi < k$ we have*

$$\sum_{s=\xi}^{k-1} \binom{k}{s} \binom{s}{\xi} h_{[s-\xi]} h_{(k-s)} = -\binom{k}{\xi} h_{[k-\xi]}. \tag{4.7}$$

Lemma 4.9. *Let*

$$a(n) = \sum_{k \geqslant 0} \binom{n}{k} A_k(p) q^{n-k}$$

be an element of $M_N(W)$. Replace q with $x - h$, where $x = q + h$, and consider the presentation

$$a(n) = \sum_{s \geqslant 0} \binom{n}{s} B_s(p) x^{n-s}.$$

Then

$$A_k = \sum_{s \geqslant 0} \binom{k}{s} B_s(p) h_{[k-s]}. \tag{4.8}$$

Proof. Since $\partial_x = \partial_q = [\cdot, p]$ on $M_N(W)$, we may assume $A_0 = B_0$, $A_1 = B_0 h + B_1$. For greater k, relation (4.8) could be proved by induction. Indeed,

$$B_k = A_k + \sum_{s=0}^{k-1} \binom{k}{s} A_s h_{(k-s)},$$

so by the assumption

$$A_k = B_k - \sum_{s=0}^{k-1} \sum_{\xi=0}^{s} \binom{k}{s} \binom{s}{\xi} B_\xi h_{[s-\xi]} h_{(k-s)},$$

and it is sufficient to apply (4.7). $\square$

Lemma 4.10. *Let $C \subset \mathrm{Cend}_N$ be a conformal subalgebra such that $S(C) = M_N(\Bbbk[q + h(p)])$. Then* $\deg h = 0$ (*or* $h = 0$).

Proof. Consider an arbitrary element $a \in C$:

$$a = \sum_{k=0}^{n} (-D)^{(k)} \otimes A_k(v) \neq 0.$$

Then

$$a(n+1) = \sum_{k=0}^{n} \binom{n+1}{k} A_k q^{n+1-k} \in S(C) = M_N(\Bbbk)[q + h(p)]. \tag{4.9}$$

We may assume that $A_0 \neq 0$: otherwise, $a = D^{(m-1)}b$, $b \in \mathrm{Cend}_N$, $m \geqslant 2$ and it is sufficient to consider $a(n+m)$ instead of $a(n+1)$ (although the element b may not lie in C, the operator $b(n+1)$ lies in $S(C) = M_N(\Bbbk[q + h(p)])$). Rewriting (4.9) by using $q = x - h$ gives

$$B_{n+1} = \sum_{k=0}^{n} \binom{n+1}{k} A_k h_{(n+1-k)} \in M_N(\Bbbk).$$

But Lemmas 4.9 and 4.7 imply that

$$\begin{aligned} B_{n+1} &= \sum_{k,s=0}^{n} \binom{n+1}{k} \binom{k}{s} B_s h_{[k-s]} h_{(n+1-k)} \\ &= \sum_{s=0}^{n} \binom{n+1}{s} B_s \sum_{k=s}^{n} \binom{n+1-s}{k-s} h_{[k-s]} h_{(n+1-k)} \\ &= -\sum_{s=0}^{n} \binom{n+1}{s} B_s h_{[n+1-s]}. \end{aligned} \tag{4.10}$$

If $\deg h > 0$, then the term of highest degree in (4.10) corresponds to $s = 0$:

$$B_{n+1} = -A_0 h_{[n+1]} + \text{terms of lower degree} \quad \notin M_N(\Bbbk).$$

Hence, either $h = 0$ or $\deg h = 0$. $\square$

Theorem 4.11. *Let $S \subset M_N(W)$ be a subalgebra satisfying the conditions of Propositions* 4.6 *and* 3.7, *i.e.*, $S = S(C)$ *for a conformal subalgebra $C \subset \mathrm{Cend}_N$. Then there exists an automorphism Θ of the conformal algebra* Cend_N *such that* $\Theta(C) = \mathrm{Cur}_N$.

Proof. By Proposition 4.6, there exists a ∂_q-invariant automorphism θ of $M_N(W)$ such that $\theta(S) = M_N(\Bbbk[q])$, $\theta(q + A(p)) = q$. The matrix $A(p)$ is defined up to an additive scalar, i.e., instead of $q + A(p)$ one may consider $q + A(p) + \alpha \in Z(S)$, for any $\alpha \in \Bbbk$. Theorem 3.17 implies that $\theta = \theta_{0,Q,h}$, $A = h(p) - Q'_p Q^{-1}$, and we may assume that $h(0) = 0$.

Consider the automorphism $\Theta_{0,Q}$ of the conformal algebra Cend_N, and denote $\Theta_{0,Q}(C) = C^Q$. Then the subalgebra

$$S^Q = S(C^Q) = Q^{-1} S(C) Q \subset M_N(W)$$

is also isomorphic to $M_N(\Bbbk[q])$, and the isomorphism is given by $\theta_{0,\mathrm{I}_N,h}$, where $h(0) = 0$.

By Lemma 4.10 $h = 0$, so $\theta = \theta_{0,Q,0}$ is continuous by Theorem 3.17. Proposition 3.19 and Corollary 3.12 imply $C^Q \lhd_l \mathcal{R}(M_N(\Bbbk[q])) = \mathrm{Cur}_N$. It is well-known that every left ideal of Cur_N is of the form $H \otimes I_0$, where $I_0 \lhd_l M_N(\Bbbk)$. If $C^Q = H \otimes I_0$, then $S^Q = M_N(\Bbbk[q]) = I_0 \otimes \Bbbk[q]$; hence, $I_0 = M_N(\Bbbk)$. Thus, we obtain $C^Q = \mathrm{Cur}_N$. $\square$

5. Irreducible conformal subalgebras

5.1. Preliminary notes

Theorem 5.1. *Let $C \subseteq \mathrm{Cend}_N$ be an irreducible conformal subalgebra. Then*

$$\Bbbk[v]C = C + vC + v^2C + \cdots = \mathrm{Cend}_{N,Q}, \tag{5.1}$$

where $Q = Q(v) \in M_N(\Bbbk[v])$ is a non-degenerate matrix.

Proof. By Proposition 3.9, the subalgebra $S_1 = \Bbbk[p]S(C)$ acts irreducibly on V_N. So Theorem 4.2 implies S_1 to be a dense subalgebra of $M_N(W)$ in the sense of q-topology. In particular, for every $A(p)q^n \in M_N(\Bbbk[p])q^n$, and for every integer $M \geqslant n+1$ there exists $a_{A,n,M} \in S_1$ such that $a_{A,n,M} - A(p)q^n \in M_N(W)q^M$, i.e.,

$$a_{A,n,M} = A(p)q^n + x(p,q)q^M$$

for some $x(p,q) \in M_N(W)$.

The H-submodule $C_1 = \Bbbk[v]C \subseteq \mathrm{Cend}_N$ is also a conformal subalgebra of Cend_N (see (3.1)), and $S(C_1) = S_1$. So Corollary 3.11 implies $S_1 \cdot C_1 \subseteq C_1$.

Let us fix an element $b \in C_1$:

$$b = \sum_{s=0}^{m} D^{(s)} \otimes B_s.$$

For any $n \geqslant 0$, choose an integer $M > n + m + \max_{s=0,\dots,m} \deg_v(B_s)$. Then by Proposition 3.10

$$a_{A,n,M} \cdot b = A(v) \circ_n b \in C_1$$

for every $A(p)q^n \in M_N(W)$. Hence, C_1 is a left ideal of Cend_N, and by Proposition 3.20 it is of the form $\mathrm{Cend}_{N,Q}$ for some $Q \in M_N(\Bbbk[v])$. Proposition 3.22 implies $\det Q \neq 0$. $\square$

Lemma 5.2. (i) *If I is a left (right) ideal of C, then $\Bbbk[v]I$ is a left (right) ideal of* $\Bbbk[v]C = \mathrm{Cend}_{N,Q}$.

(ii) *If $0 \neq I \lhd C$, then $\Bbbk[v]I = \mathrm{Cend}_{N,Q}$.*

Proof. Statement (i) easily follows from (3.1).

To prove (ii), it is sufficient to note that $\Bbbk[v]I \neq 0$ is an ideal of $\mathrm{Cend}_{N,Q}$ for every non-zero ideal I of C. By Proposition 3.24(ii), the conformal algebra $\mathrm{Cend}_{N,Q}$ is simple, so $\Bbbk[v]I = \mathrm{Cend}_{N,Q}$. $\square$

Proposition 5.3. *If there exists an element $a \in C$ such that $a \neq 0$ and $v^k a \in C$ for all $k \geqslant 0$, then $C = \mathrm{Cend}_{N,Q}$.*

Proof. It is clear from (3.1) that $I = \{a \in C \mid v^k a \in C,\ k \geqslant 0\}$ is an ideal of C. In particular, $\Bbbk[v]I = I$. If $I \neq 0$, then by Lemma 5.2(ii) we have $C \supseteq I = \Bbbk[v]I = \mathrm{Cend}_{N,Q}$. $\square$

For every irreducible subalgebra $C \subseteq \mathrm{Cend}_N$ there exist three options as follows:

Case 1: Sum (5.1) is direct, i.e.,

$$\mathrm{Cend}_{N,Q} = \bigoplus_{n \geqslant 0} v^n C. \tag{5.2}$$

Case 2: $C \cap vC \neq 0$.

Case 3: Sum (5.1) is non-direct, but $C \cap vC = 0$.

We will show that the first case corresponds to the current-type conformal subalgebras, the second one gives $C = \mathrm{Cend}_{N,Q}$, and the third one is impossible.

Without loss of generality, we may assume Q to be in the canonical diagonal form: $Q = \mathrm{diag}(f_1, \ldots, f_N)$, where f_i are monic polynomials and $f_i \mid f_{i+1}$. Indeed, for an arbitrary $Q \in M_N(\Bbbk[v])$, $\det Q \neq 0$, there exist $U, T \in M_N(\Bbbk[v])$, $\det U, \det T \in \Bbbk \setminus \{0\}$, such that $TQU = D$, where D has the canonical diagonal form. Then for $C^U = \Theta_{0,U}(C)$ we have $\Bbbk[v]C^U = \mathrm{Cend}_{N,D}$, and all the conditions described by Cases 1–3 hold.

5.2. Finite-type case

Now, let us consider Case 1. Throughout this section C is a conformal subalgebra of Cend_N satisfying (5.2) for a fixed matrix $Q = \mathrm{diag}(f_1, \ldots, f_N)$, $0 \neq f_i \in \Bbbk[v]$, $i = 1, \ldots, N$.

Lemma 5.4. (i) *If $h(D)a \in \mathrm{Cend}_{N,Q}$ for some $0 \neq h \in H$, then $a \in \mathrm{Cend}_{N,Q}$.*

(ii) *For $S = S(C)$ we have*

$$M_N(W)Q = \bigoplus_{n \geqslant 0} p^n S. \tag{5.3}$$

Proof. (i) Consider $U = \mathrm{Cend}_N/\mathrm{Cend}_{N,Q}$ as a module over the conformal algebra Cend_N. If $h(D)a \in \mathrm{Cend}_{N,Q}$, then $\bar{a} = a + \mathrm{Cend}_{N,Q}$ lies in the torsion of U. In particular, $\mathrm{Cend}_N \circ_\omega a \in \mathrm{Cend}_{N,Q}$. But there exists $e = 1 \otimes \mathrm{I}_N \in \mathrm{Cend}_N$ such that $e \circ_0 a = a \in \mathrm{Cend}_{N,Q}$.

(ii) Suppose that sum (5.3) is non-direct. Since $(Da)(n) = -na(n-1)$, we may consider

$$0 = a_0(n) + \cdots + p^m a_m(n), \quad a_i \in C, \ n \geqslant 0.$$

Then for $a = a_0 + va_1 + \cdots + v^m a_m \in \mathrm{Cend}_{N,Q}$ we obtain $a(0) = \cdots = a(n) = 0$, i.e., $a = D^{n+1}b$. Statement (i) implies $b \in \mathrm{Cend}_{N,Q}$, so $b = b_0 + vb_1 + \cdots + v^k b_k$, $b_i \in C$. Since (5.1) is direct, we have $k = m$ and $D^{n+1}b_i = a_i$, so $a_i(n) = 0$ for all $i = 0, \ldots, m$. $\square$

Therefore, the operator algebra $S = S(C)$ of the initial conformal subalgebra C satisfies (5.3). Consider the presentations of all elements

$$q^m e_{ij} Q = \sum_{s \geqslant 0} p^s a_{ijm,s},$$
$$i, j = 1, \ldots, N, \ m = 0, \ldots, \max_{j=1,\ldots,N} \deg f_j + 1, \ a_{ijm,s} \in S,$$

where e_{ij} are the matrix units (so $e_{ij}Q = e_{ij} f_j$). The finite set of all $a_{ijm,s}$ together with all their derivatives $\partial_q^n a_{ijm,s}$ generates a ∂_q-invariant subalgebra S_0 of S. Denote $S_{01} = \Bbbk[p]S_0$, then S_{01} is a ∂_q-invariant subalgebra of $S_1 = M_N(W)Q$.

Lemma 5.5. (i) $\mathrm{GKdim}\, S_{01} = 2$;

(ii) $\mathcal{Z}_{M_N(W)}(S_{01}) = \Bbbk$;

(iii) S_{01} *has no non-zero nilpotent ideals.*

Proof. (i) Since $\mathrm{GKdim}\, M_N(W) = 2$, we have $\mathrm{GKdim}\, S_{01} \leqslant 2$. Let us consider the subalgebra S_{00} generated by $f_1 e_{11}$, $qf_1 e_{11}$ in S_{01}. If $\deg f_1 > 0$, then $\mathrm{GKdim}\, S_{00} = 2$, so $\mathrm{GKdim}\, S_{01} = 2$. If $\deg f_1 = 0$, then we obtain that $We_{11} \subseteq S_{01}$, hence, $\mathrm{GKdim}\, S_{01} = 2$ as well.

(ii) Let $a \in \mathcal{Z} = \mathcal{Z}_{M_N(W)}(S_{01})$. Then, in particular, $[e_{ij} f_j, a] = 0$, so a is a diagonal matrix. Moreover, $[pe_{ij} f_j, a] = 0 = [p, a]e_{ij} f_j$ for all $i, j = 1, \ldots, N$. Hence, $[p, a] = 0$. By the very same reasons $[a, q] = 0$. Since all f_j are unital, we conclude that $a = \alpha \mathrm{I}_N$, $\alpha \in \Bbbk$.

(iii) If $0 \neq I \lhd S_{01}$, then for any $a \in I$ and for all $i, j, k, l = 1, \ldots, N$ the element $f_j e_{ij} a f_k e_{lk}$ lies in I. In particular, I contains an element ge_{11} for some $0 \neq g \in W$. Then $I^n \ni g^n e_{11} \neq 0$ for all $n \geqslant 1$. $\square$

Lemma 5.6. (i) $S_{01}\Bbbk[q] = M_N(W)$;

(ii) *if* $I \lhd_r S_{01}$, *then* $I\Bbbk[q] \lhd_r M_N(W)$.

Proof. (i) If $Q = \mathrm{I}_N$ (i.e., $\deg f_1 = \cdots = \deg f_N = 0$), then the statement is clear. If $\deg f_j > 0$ for some $j \in \{1, \ldots, N\}$, then consider

$$S_{01} \ni q^k f_j e_{ij} = f_j q^k e_{ij} + k\partial_p(f_j) q^{k-1} e_{ij} + \cdots + \partial_p^k(f_j) e_{ij}, \quad k = 1, \ldots, \deg f_j.$$

It is clear that $\partial_p^k(f_j)e_{ij} \in S_{01}\Bbbk[q]$ for all $k = 1, \ldots, \deg f_j$, so, $e_{ij} \in S_{01}\Bbbk[q]$.

(ii) It follows from (i). $\square$

Proposition 5.7. *The algebra S_{01} is prime and satisfies a.c.c. for right annihilation ideals.*

Proof. First, let us show that S_{01} satisfies a.c.c. for right ideals of the form $\mathrm{Ann}_{S_{01}}(X) = \{a \in S_{01} \mid Xa = 0\}$, $X \subseteq S_{01}$.

Consider an ascending chain of right annihilation ideals of S_{01}:

$$I_1 \subseteq I_2 \subseteq \cdots, \quad I_k = \mathrm{Ann}_{S_{01}}(X_k), \quad X_k \subseteq S_{01}.$$

By Lemma 5.6(ii), $I_1\Bbbk[q] \subseteq I_2\Bbbk[q] \subseteq \cdots$ is an ascending chain of right ideals of $M_N(W)$. Since $M_N(W)$ is a Noetherian algebra [16], we have $I_n\Bbbk[q] = I_{n+1}\Bbbk[q]$ for a sufficiently large number n. In particular, $I_{n+1} \subseteq I_n\Bbbk[q]$. Thus, $X_n I_{n+1} \subseteq X_n I_n \Bbbk[q] = 0$, so $I_{n+1} = I_n$.

Now, suppose that there exist two ideals $I, J \lhd S_{01}$ such that $IJ = 0$. If $I \neq 0$, then by Lemma 5.5(iii) the ideal I is not nilpotent. By the very same reason as in the proof of Proposition 4.4, we may assume that J is ∂_q-invariant.

If J is a non-zero ∂_q-invariant ideal of S_{01}, then by Lemma 5.6(ii) $J_1 = J\Bbbk[q] \lhd_r M_N(W)$ is also ∂_q-invariant. Moreover, $S_{01}J_1 \subseteq J_1$. Then for every pair (i, j), $i, j = 1, \ldots, N$, there exist $k, l \in \{1, \ldots, N\}$ such that $e_{ik} f_k J_1 e_{lj} f_j \neq 0$. Thus, we have $J_{ij} = J_1 \cap e_{ij}W \neq 0$ for all $i, j = 1, \ldots, N$. It is clear that $A_{ij} = \{a \in W \mid ae_{ij} \in J_{ij}\} \lhd_r W$, and $\partial_q(A_{ij}) \subseteq A_{ij}$. Then $A_{ij} = g_{ij}(p)W$ for some polynomial $0 \neq g_{ij} \in \Bbbk[p]$. In particular, $P = \sum_{i=1}^{N} g_{ii}e_{ii} \in J_1$ is a diagonal matrix such that $\det P \neq 0$.

The relation $IJ = 0$ implies $IJ_1 = (IJ)\Bbbk[q] = 0$, so $IP = 0$. Then $I = 0$ since $\det P \neq 0$. $\square$

Corollary 5.8. $Q = \mathrm{I}_N$, $\mathrm{Cend}_{N,Q} = \mathrm{Cend}_N$,

Proof. By the construction, $S_{01} = \bigoplus_{n \geqslant 0} p^n S_0$, so $\mathrm{GKdim}\, S_0 = 1$. Lemma 4.5 implies that there exists a matrix $A \in M_N(\Bbbk[p])$ such that $q + A \in Z(S_0)$. Then $\partial_p(x) = [x, A]$ for every $x \in S_0$.

Let us consider

$$S_{01} \ni Q = a_0 + pa_1 + \cdots + p^n a_n, \quad a_i \in S_0.$$

Then

$$\partial_p(Q) = \partial_p(a_0) + p\partial_p(a_1) + \cdots + p^n\partial_p(a_n) + a_1 + \cdots + np^{n-1}a_n.$$

On the other hand,

$$[Q, A(p)] = \partial_p(a_0) + p\partial_p(a_1) + \cdots + p^n\partial_p(a_n).$$

So, $\partial_p(Q) - [Q, A(p)] \in S_{01} \subseteq M_N(W)Q$. It is easy to note that it is not possible, if $\deg f_j \neq 0$ for at least one $j \in \{1, \ldots, N\}$. $\square$

Theorem 5.9 (*Boyallian et al. [9, Theorem 4.1]*). *There exists an automorphism* $\Theta = \Theta_{0,P}$ *of* Cend_N *such that* $\Theta(C) = \mathrm{Cur}_N$.

Proof. By Lemma 5.4(ii) and Corollary 5.8, the operator algebra $S = S(C)$ satisfies the conditions of Theorem 4.11. The last statement implies that C is a conjugate of Cur_N. $\square$

5.3. The case $C \cap vC \neq 0$

Let C be an irreducible conformal subalgebra of Cend_N such that $C \cap vC \neq 0$.

Remind the definition of the map φ from Proposition 2.9. For any element $a = f(D) \otimes A(v) \in H \otimes M_N(\Bbbk[v])$ we put

$$\varphi(a) = (f(D) \otimes 1)A(D \otimes 1 + 1 \otimes v),$$
$$\varphi^{-1}(a) = (f(D) \otimes 1)A(-D \otimes 1 + 1 \otimes v).$$

First, let us consider the main technical features of [9] concerning the description of irreducible subalgebras for $N = 1$.

Lemma 5.10 (*Cf. Boyallian et al. [9, Section 2]*). *The following relations hold in* Cend_N:

$$(1 \otimes A) \circ_n \varphi^{-1}(B) = \begin{cases} (1 \otimes A)\varphi^{-1}(B), & n = 0, \\ 0, & n > 0, \end{cases} \tag{5.4}$$

$$(1 \otimes A)\varphi^{-1}(B) \circ_n (1 \otimes A_1) = 1 \otimes A\partial_v^n(BA_1), \tag{5.5}$$

$$(1 \otimes A)\varphi^{-1}(B) \circ_n (1 \otimes A_1)\varphi^{-1}(B_1) = (1 \otimes A\partial_v^n(BA_1))\varphi^{-1}(B_1), \tag{5.6}$$

where $A, B, A_1, B_1 \in M_N(\Bbbk[v])$.

Proof. Note that

$$\varphi^{-1}(B) = \sum_{s\geqslant 0} (-D)^{(s)} \otimes \partial_v^s(B),$$

so (5.4) follows from axiom (C3).

Relation (5.5) follows from (C2) and the Leibnitz formula. In order to obtain (5.6) one may use (5.4), (5.5) and the associativity relation (2.4). □

Theorem 5.11 (*Boyallian et al. [9]*). *Let* $C \subseteq \mathrm{Cend}_1$ *be an irreducible conformal subalgebra, and let* $C \cap v\, \mathrm{Cend}_1 \neq 0$. *Then* $C = \mathrm{Cend}_{1,Q}$ *for some* $0 \neq Q \in \Bbbk[v]$.

Proof. By Theorem 5.1, $\Bbbk[v]C = \mathrm{Cend}_{1,Q}$. Let $0 \neq a \in C$ be an arbitrary element. Let us present it as $a = \sum\limits_{s=0}^{m} D^s \otimes a_s$. Consider $\varphi(a)$, where φ is the isomorphism from Proposition 2.9. Present $b = \varphi(a) = \sum\limits_{t=0}^{n} D^t \otimes b_t$. Here $a_m, b_n \in \Bbbk[v] \setminus \{0\}$. Then

$$a = \varphi^{-1}(b) = \sum_{t=0}^{n} (D^t \otimes 1)\varphi^{-1}(b_t).$$

It follows from (5.4), (C2), (C3) that $\mathcal{N}(a, a) = n + m + 1$ and

$$\begin{aligned} a \circ_{n+m} a &= a \circ_{n+m} \varphi^{-1}(b) \\ &= \left(\sum_{s=0}^{m} D^s \otimes a_s\right) \circ_{n+m} \left(\sum_{t=0}^{n} (D^t \otimes 1)\varphi^{-1}(b_t)\right) \\ &= (-1)^m (m+n)!(1 \otimes a_m)\varphi^{-1}(b_n). \end{aligned}$$

So, $(1 \otimes a_m)\varphi^{-1}(b_n) \in C$. Since $C \cap v\mathrm{Cend}_1 \neq 0$, we may assume $\deg_v a_m > 0$.

Therefore, C contains an element of the form $(1 \otimes g)\varphi^{-1}(f)$, $f, g \in \Bbbk[v] \setminus \{0\}$, $\deg_v g > 0$. Consider two elements of this form in C (they might be equal): $a = (1 \otimes g)\varphi^{-1}(f)$, $b = (1 \otimes g)\varphi^{-1}(h)$. It follows from (5.6) that $\mathcal{N}(a, b) = n + 1$, where $n = \deg_v f + \deg_v g$, and

$$\begin{aligned} &a \circ_n b = \gamma(1 \otimes g)\varphi^{-1}(h), \quad \gamma \in \Bbbk \setminus \{0\}, \\ &a \circ_{n-1} b = \gamma'(1 \otimes (v + \alpha)g)\varphi^{-1}(h), \quad \gamma' \in \Bbbk \setminus \{0\},\ \alpha \in \Bbbk. \end{aligned}$$

So we may conclude that $(1 \otimes vg)\varphi^{-1}(h) \in C$ as well as $(1 \otimes v^k g)\varphi^{-1}(h) \in C$ for all $k \geqslant 0$. Hence, the conditions of Proposition 5.3 hold and $C = \mathrm{Cend}_{1,Q}$. □

We will use this technique in the general case.

Theorem 5.12. *Let $C \subseteq \mathrm{Cend}_N$ be an irreducible conformal subalgebra, and let $C \cap v\ C \neq 0$. Then $C = \mathrm{Cend}_{N,Q}$ for some $Q \in M_N(\Bbbk[v])$, $\det Q \neq 0$.*

Proof. By Theorem 5.1, $\Bbbk[v]C = \mathrm{Cend}_{N,Q}$, $\det Q \neq 0$. Let $I_1 = \{a \in C \mid va \in C\} \neq 0$. It is clear that I_1 is a non-zero ideal of C. Moreover, $I_2 = \{a \in I_1 \mid va \in I_1\}$ is also non-zero: for example, $I_1 \circ_n I_1 \subseteq I_2$ (it follows from Lemma 5.2(ii) that $I_1 \circ_\omega I_1 \neq 0$). One may define $I_k = \{a \in I_{k-1} \mid va \in I_{k-1}\}$ for every $k > 1$. By the same way, $I_m \circ_n I_k \subseteq I_{k+m}$, so all I_k are non-zero ideals of C.

By Theorem 5.1, there exist an integer $m \geqslant 0$ and some elements $a_i \in C$, $i = 0, \dots, m$, such that the element $E_{N,Q} = (1 \otimes e_{NN})\varphi^{-1}(Q) \in \mathrm{Cend}_{N,Q}$ is presented as

$$E_{N,Q} = a_0 + va_1 + \dots + v^m a_m,$$

where e_{NN} is the matrix unit.

Consider an element $0 \neq a \in I_{2m+1}$ such that $e_{NN} a e_{NN} = f(D,v) e_{NN}$, where $0 \neq f(D,v) \in \mathrm{Cend}_1$. (Note that $e_{NN} I_{2m+1} e_{NN} = 0$ is impossible by Lemma 5.2(ii).) It follows from (3.1) that $E_{N,Q} \circ_\omega I_{2m} \circ_\omega E_{N,Q} \subseteq C$. For example,

$$E_{N,Q} \circ_0 va \circ_n E_{N,Q} = \big(v f_N(v) f(D,v) \circ_n \varphi^{-1}(f_N(v))\big)(1 \otimes e_{NN})$$

lies in C for any $n \in \mathbb{Z}_+$. Hence, the conformal algebra C contains matrices of the form

$$0 \neq x = vg(D,v)(1 \otimes e_{NN}),$$

where $g(D,v) \in \mathrm{Cend}_1$. By the very same reasons as in the proof of Theorem 5.11 we conclude that there exists an element $y \in C$ such that $y \neq 0$ and $v^k y \in C$ for all $k \geqslant 0$. Hence, by Proposition 5.3 we have $C = \mathrm{Cend}_{N,Q}$. $\square$

5.4. The case $C \cap vC = 0$

Let $C \cap vC = 0$ but sum (5.1) is still non-direct. Then there exists a minimal $n \geqslant 1$ such that the sum $C + vC + \dots + v^n C$ is direct. Let us use the following notation: for a subspace $X \subseteq \mathrm{Cend}_N$ we denote $X + vX + \dots + v^k X$ by $\big(v^{\leqslant k}\big)X$.

The set $J_1 = \{a \in C \mid v^{n+1} a \in \big(v^{\leqslant n}\big)C\}$ is a non-zero ideal of C. If $J_1 = C$, then $\Bbbk[v]C = C \oplus vC \oplus \dots \oplus v^n C = \mathrm{Cend}_{N,Q}$.

If $J_1 \neq C$, then we may construct $J_{k+1} = \{a \in J_k \mid v^{n+1} a \in \big(v^{\leqslant n}\big)J_k\}$ for $k \geqslant 1$. All these J_k are ideals of C, and it is clear that

$$J_l \circ_\omega J_m \subseteq J_{l+m}, \quad \big(v^{\leqslant n+k}\big)J_k \subseteq \big(v^{\leqslant n+k-j}\big)J_{k-j}, \quad j = 1, \dots, k. \tag{5.7}$$

If $J_k \neq 0$ for some $k \geqslant 1$, then by Lemma 5.2(ii) we have $\Bbbk[v]J_k = \mathrm{Cend}_{N,Q}$. If $J_l \circ_\omega J_m = 0$, then $\Bbbk[v]J_l \circ_\omega J_m = \mathrm{Cend}_{N,Q} \circ_\omega J_m = 0$ and $J_m = 0$. Since $J_1 \neq 0$, relation (5.7) implies $J_k \neq 0$ for all $k \geqslant 1$. It follows from Lemma 5.2(ii) that $\Bbbk[v]J_k =$

$\mathrm{Cend}_{N,Q}$ for all $k \geqslant 1$. In particular, $\Bbbk[v]J_{n+1} = \mathrm{Cend}_{N,Q}$. But (5.7) and (3.1) imply that $C_0 = \left(v^{\leqslant n}\right)J_{n+1}$ is a conformal subalgebra of $\mathrm{Cend}_{N,Q}$. If $n \geqslant 1$, then C_0 satisfies the conditions of Theorem 5.12, so $C_0 = \mathrm{Cend}_{N,Q}$.

In any case, we obtain that there exists a conformal subalgebra $C \subseteq \mathrm{Cend}_{N,Q}$ such that $\mathrm{Cend}_{N,Q} = C \oplus vC \oplus \cdots \oplus v^n C$, $n \geqslant 1$.

Theorem 5.13. *Let* $\mathrm{Cend}_{N,Q} = C \oplus vC \oplus \cdots \oplus v^n C$. *Then* $n = 0$.

Proof. Suppose that $n \geqslant 1$. For every $a \in C$ we may consider a unique presentation

$$v^{(n+1)}a = a_0 + va_1 + \cdots + v^{(n)}a_n, \quad a_i \in C.$$

Define the map $\chi : C \to C$ by the rule $\chi(a) = a_n$. This is an injective H-linear map satisfying the conditions

$$\chi(a \circ_m x) = \chi(a) \circ_m x, \quad \chi(x \circ_m a) = x \circ_m \chi(a) - mx \circ_{m-1} a \tag{5.8}$$

for $a, x \in C$, $m \geqslant 0$. Let us consider $\psi : C \to \mathrm{Cend}_{N,Q}$ defined as follows:

$$\psi(a) = \chi(a) - va, \quad a \in C.$$

This is an injective H-linear map, and $\psi(C) \cap C = 0$. The map ψ satisfies the following conditions:

$$\psi(a \circ_m x) = \psi(a) \circ_m x, \quad \psi(x \circ_m a) = x \circ_m \psi(a), \quad a, x \in C, \quad m \geqslant 0. \tag{5.9}$$

Since $\mathrm{Cend}_{N,Q} = C \oplus vC \oplus \cdots \oplus v^n C$, we may extend ψ to the map $\bar\psi : \mathrm{Cend}_{N,Q} \to \mathrm{Cend}_{N,Q}$ by the rule $\bar\psi(v^k a) = v^k \psi(a)$, $k = 0, \ldots, n$. It is easy to check that this $\bar\psi$ satisfies (5.9) for every $a \in C$, $x \in \mathrm{Cend}_{N,Q}$:

$$\bar\psi(a \circ_m x) = \psi(a) \circ_m x = a \circ_m \bar\psi(x),$$
$$\bar\psi(x \circ_m a) = \bar\psi(x) \circ_m a = x \circ_m \psi(a).$$

In particular, let us put $x = \varphi^{-1}(Q)$, $\bar\psi(x) = \sum\limits_{t \geqslant 0} (-D)^{(t)} \otimes X_t$. Then for an arbitrary element $a \in C$ we have

$$\bar\psi(x \circ_0 a) = x \circ_0 \psi(a) = \bar\psi(x) \circ_0 a.$$

Hence, $Q\psi(a) = X_0 a$, and we may conclude that $\psi(a) = Q^{-1} X_0 a$. Since $\bar\psi$ is defined by $\bar\psi(v^k a) = v^k \psi(a)$, $k = 0, \ldots, n$, $a \in C$, we may assume that $\bar\psi(x) = Q^{-1} X_0 x$

for every $x \in \mathrm{Cend}_{N,Q}$. In particular, it follows that $\bar{\psi}(vx) = v\bar{\psi}(x)$ for every $x \in \mathrm{Cend}_{N,Q}$. It is easy to conclude that

$$\bar{\psi}(x \circ_m y) = \bar{\psi}(x) \circ_m y = x \circ_m \bar{\psi}(y), \quad x, y \in \mathrm{Cend}_{N,Q},\ m \geqslant 0. \tag{5.10}$$

Also, $\bar{\psi}$ is an injective H-linear map.

The only possibility for (5.10) is $\bar{\psi} = \alpha\,\mathrm{id}$, $\alpha \in \Bbbk$. Indeed, let us apply (5.10) for $y = \varphi^{-1}(Q)$ and $x = e_{ij}y$. Denote $B = Q^{-1}X_0 \in M_N(\Bbbk(v))$, and consider

$$\bar{\psi}(x) \circ_0 y = (1 \otimes Be_{ij}Q)\varphi^{-1}(Q) = x \circ_0 \bar{\psi}(y) = (1 \otimes e_{ij}QB)\varphi^{-1}(Q).$$

In particular, we may conclude that

$$Be_{ij}Q = e_{ij}QB, \tag{5.11}$$

so B is a scalar matrix.

Now, consider

$$\begin{aligned} \bar{\psi}(x) \circ_1 y &= (1 \otimes Be_{ij}\partial_v(Q))\varphi^{-1}(Q) = x \circ_1 \bar{\psi}(y) \\ &= (1 \otimes e_{ij}\partial_v(Q)B)\varphi^{-1}(Q) + (1 \otimes e_{ij}Q\partial_v(B))\varphi^{-1}(Q). \end{aligned} \tag{5.12}$$

It follows from (5.11) and (5.12) that $Be_{ij}\partial_v(Q) = e_{ij}\partial_v(QB)$ for all $i, j = 1, \dots, N$. Since B is a scalar matrix in $M_N(\Bbbk(v))$, we obtain $B\partial_v(Q) = \partial_v(BQ)$, so $B = \alpha I_N$, $\alpha \in \Bbbk$. Therefore, $\psi(C) \subseteq C$, and $C \cap vC \neq 0$ in contradiction with $n \geqslant 1$. □

5.5. Associative conformal algebras with finite faithful representation

Let us compile the arguments stated above in order to prove the following:

Theorem 5.14. *Let $C \subseteq \mathrm{Cend}_N$ be an irreducible conformal subalgebra. Then either C is a conjugate of Cur_N or $C = \mathrm{Cend}_{N,Q}$ for some matrix $Q \in M_N(\Bbbk[v])$, $\det Q \neq 0$.*

Proof. It follows from Theorem 5.1 that $\Bbbk[v]C = \sum_{n \geqslant 0} v^n C = \mathrm{Cend}_{N,Q}$ for a suitable matrix Q. If the sum is direct, then by Theorem 5.9 C is a conjugate of Cur_N. If the sum is not direct, then Theorem 5.13 implies $C \cap vC \neq 0$, so $C = \mathrm{Cend}_{N,Q}$ by Theorem 5.12. □

Lemma 5.15. *Let $C \neq 0$ be an associative conformal algebra with a faithful representation of finite type. If $\{a \in C \mid C \circ_\omega a = 0\} = \{a \in C \mid a \circ_\omega C = 0\} = 0$, then C could be embedded into Cend_N for some N in such a way that every proper C-submodule of V_N is not faithful.*

Proof. It is clear that $C \subseteq \mathrm{Cend}_m$ for some $m \geqslant 1$ (see, e.g., [22]). Consider any descending chain of faithful C-submodules of V_m:

$$V_m \supseteq U_1 \supseteq U_2 \supseteq \cdots. \tag{5.13}$$

It is easy to note (see, e.g., [1, Lemma 2.1]) that there exists $n\geqslant 1$ such that for every $k\geqslant 0$ we have $U_{n+k}\supseteq f_kU_n$ for some $f_k\in H$.

In particular, the H-module U_n/U_{n+k} coincides with its torsion, so $C\circ_s U_n\subseteq U_{n+k}$. Since the C-module $C\circ_\omega U_n\subseteq V_m$ is faithful, it is a lower bound of the initial chain (5.13) in the set of faithful C-submodules of V_m. Hence, there exists a minimal faithful C-submodule U of V_m. Since U is finitely generated over H and torsion-free, U is isomorphic to V_N for some $N\geqslant 1$. $\square$

Theorem 5.16. *Let C be a conformal algebra with a faithful representation of finite type. Then*

(i) *there exists a maximal nilpotent ideal $\mathfrak{N}$ of C;*
(ii) *if C is simple, then C is isomorphic to either* Cur_N *or* $\mathrm{Cend}_{N,Q}$, $\det Q\neq 0$;
(iii) *if C is semisimple, then $C=\bigoplus_{s=1}^{n} I_s$, where I_s, $s=1,\dots,n$, are simple ideals of C, described in* (ii).

Proof. (i) Let V be a faithful C-module of finite type. First, consider the ascending chain of submodules $U_k=\{u\in V\mid C^k\circ_\omega u=0\}$, $k\geqslant 1$. Since V is a Noetherian H-module, the H-module V/U_n is torsion-free for sufficiently large $n\geqslant 1$. In particular, V/U_n is isomorphic as an H-module to V_N for a suitable N. The ideal $I=\{a\in C\mid C^n\circ_\omega a=0\}\lhd C$ is nilpotent (or even zero), and C/I acts faithfully on V/U_n.

Therefore, it is sufficient to show that for every ascending chain of nilpotent ideals

$$I_1\subseteq I_2\subseteq\cdots,\quad I_j\lhd C\subseteq \mathrm{Cend}_N,\ j\geqslant 1, \tag{5.14}$$

their upper bound $\bigcup_{j\geqslant 0} I_j$ is also nilpotent.

Let us consider a chain (5.14) of nilpotent ideals. Denote by n_j $(j\geqslant 1)$ the nilpotency index of I_j. Assume that the union of this chain is not nilpotent, so it could be conjectured that $n_j<n_{j+1}$ for all $j\geqslant 1$.

Since the H-module V_N is Noetherian, any ascending chain of submodules becomes stable. In particular,

$$I_1\circ_\omega V_N\subseteq I_2\circ_\omega V_N\subseteq\cdots=W_1=I_j\circ_\omega V_N,\quad j\geqslant l_1,$$
$$I_1^2\circ_\omega V_N\subseteq I_2^2\circ_\omega V_N\subseteq\cdots=W_2=I_j^2\circ_\omega V_N,\quad j\geqslant l_2,$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$I_1^k\circ_\omega V_N\subseteq I_2^k\circ_\omega V_N\subseteq\cdots=W_k=I_j^k\circ_\omega V_N,\quad j\geqslant l_k,$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

It is clear that $W_1\supseteq W_2\supseteq\cdots$, so there exists an integer $k\geqslant 1$ such that W_k/W_{k+j} coincides with its torsion for every $j\geqslant 0$ (see, e.g., [1, Lemma 2.1]). Hence,

$$C\circ_\omega W_k\subseteq W_m\quad\text{for all } m>k. \tag{5.15}$$

Let us fix some $m > k + 1$. Since the sequence $\{n_j\}$ is assumed to be increasing, there exists a number $p \geqslant \max\{l_k, l_m\}$ such that $n_p > m$. Then $W_k = I_p^k \circ_\omega V_N$, $W_m = I_p^m \circ_\omega V_N$. Relation (5.15) implies

$$I_p^{n_p - m} \circ_\omega C \circ_\omega I_p^k \circ_\omega V_N \subseteq I_p^{n_p} \circ_\omega V_N = 0,$$

so

$$0 = I_p^{n_p - m} \circ_\omega C \circ_\omega I_p^k \supseteq I_p^{n_p - m + k + 1}.$$

Therefore, we obtain the contradiction.

(ii) It follows from Lemma 5.15 that C has an irreducible faithful representation of finite type. Theorem 5.14 implies that either $C \simeq \mathrm{Cur}_N$ or $C \simeq \mathrm{Cend}_{N,Q}$, $\det Q \neq 0$.

(iii) By Lemma 5.15, we may assume that $C \subseteq \mathrm{Cend}_N$ and every proper C-submodule of V_N is not faithful.

Let $U \subset V_N$ be a maximal C-submodule. Then $I = \mathrm{Ann}_C(U) = \{a \in C \mid a \circ_\omega U = 0\} \neq 0$ is an ideal of C.

Denote by $J = \mathrm{Ann}_C(I) = \{a \in C \mid I \circ_\omega a = 0\}$ the annihilation ideal of I in C. Since C is semisimple, $I \cap J = 0$. It is clear that $J \circ_\omega V_N \subseteq \mathrm{Ann}_{V_N}(I) = U$ since U is a maximal C-submodule.

Now, note that V_N/U is an irreducible C-module, V_N/U is a faithful I-module, V_N/U is a faithful irreducible C/J-module. Since I could be considered as an ideal of C/J, we have $C = I \oplus J$, and $I \simeq C/J$ has a faithful irreducible representation of finite type.

By (i), $I \simeq \mathrm{Cur}_N$ or $I \simeq \mathrm{Cend}_{N,Q}$, $\det Q \neq 0$. It is clear that J is also semisimple, and J has a faithful representation of finite type.

Therefore, every semisimple conformal subalgebra C of Cend_m could be presented as $C = I \oplus J$, where I is simple and J is semisimple, $I = \mathrm{Ann}_C(J)$. Let us proceed with the decomposition and write $C = I_1 \oplus \cdots \oplus I_n \oplus J_n$, where all I_j, $j = 1, \ldots, n$, are simple and J_n is semisimple, $I_1 \oplus \cdots \oplus I_n = \mathrm{Ann}_C(J_n)$. Since Cend_m is a (left and right) Noetherian conformal algebra, C satisfies a.c.c. for annihilation ideals. Hence, there exists a sufficiently large integer $n \geqslant 1$ such that $J_n = 0$. Then $C = I_1 \oplus \cdots \oplus I_n$. $\square$

Corollary 5.17 (*Cf. Boyallian et al. [9], Kac [22]*). *Let C be an associative conformal algebra of finite type. If C is simple, then $C \simeq \mathrm{Cur}_N$. If C is semisimple, then C is a direct sum of simple ideals.*

6. Open problems

Let us complete the paper with listing some open problems closely related with the considered one.

(I) Describe all irreducible subalgebras of $M_N(W)$ with respect to the canonical action on $(\Bbbk[p])^N$.

Theorems 4.2 and 5.1 describe the class of irreducible subalgebras of $M_N(W)$ related with conformal subalgebras of Cend_N. It is unclear, what would be the answer in general.

(II) Classify irreducible infinite type Lie conformal subalgebras of the Lie conformal algebra $\mathrm{gc}_N = \mathrm{Cend}_N^{(-)}$.

Irreducible Lie conformal subalgebras of finite type were described in [13] (see also [22]). In [9], it is conjectured that the following Lie conformal subalgebras of gc_N exhaust all (infinite type) irreducible Lie conformal subalgebras of gc_N up to the conjugation by an automorphism of Cend_N ($\det Q \neq 0$ everywhere):

$$\mathrm{gc}_{N,Q} = \mathrm{Cend}_{N,Q}^{(-)};$$

$$\mathrm{oc}_{N,Q} = \{a\varphi^{-1}(Q) \mid a \in \mathrm{Cend}_N,\ \sigma(a) = -a\}, \quad Q^t(-v) = Q(v);$$

$$\mathrm{spc}_{N,Q} = \{a\varphi^{-1}(Q) \mid a \in \mathrm{Cend}_N,\ \sigma(a) = a\}, \quad Q^t(-v) = -Q(v).$$

Here σ is the anti-involution of Cend_N defined by the rule [9]

$$\sigma(h \otimes A(v)) = \sum_{s \geqslant 0} (-D)^{(s)} h \otimes \partial_v^s A^t(v), \quad h \in H,\ A(v) \in M_N(\Bbbk[v]).$$

It is proved in [14] that every irreducible Lie conformal subalgebra of infinite type which is an sl_2-module (with respect to the action of a Virasoro-like element of gc_N) is of the type $\mathrm{on}_{N,Q}$ or $\mathrm{spc}_{N,Q}$. The result of [30] shows that the conjecture is true for simple irreducible Lie subalgebras that contain $\mathrm{Cur}(\mathrm{sl}_2)$.

(III) Describe finitely generated simple and semisimple conformal algebras of Gel'fand–Kirillov dimension one.

This problem was partially solved in [24,25,30]. The conjecture stated in the last paper says that $\mathrm{Cend}_{N,Q}$, $\det Q \neq 0$, exhaust all (finitely generated) simple associative conformal algebras of Gel'fand–Kirillov dimension one. We believe it is true, although the similar statement does not hold for Lie conformal algebras: $\mathrm{gc}_{N,Q}$, $\mathrm{oc}_{N,Q}$, $\mathrm{spc}_{N,Q}$, together with $\mathrm{Cur}\,\mathfrak{g}$ (where $\mathfrak{g}$ is a simple finitely generated Lie algebra of Gel'fand–Kirillov dimension one) is not a complete list of simple Lie conformal algebras of Gel'fand–Kirillov dimension one.

Acknowledgements

I thank Leonid Bokut and Seok-Jin Kang for their interest in this work and helpful discussions. Also, I am grateful to the referee for many valuable remarks which led to improvement of the presentation.

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