View metadata, citation and similar papers at core.ac.uk

brought to you by CORE

provided by Elsevier - Publisher Connector

Journal of Pure and Applied Algebra 53 (1988) 197-199
North-Holland



# NOETHERIAN RINGS OF DIMENSION ONE ARE POLE ASSIGNABLE

F.G.J. WIID

*National Research Institute for Mathematical Sciences, CSIR, Pretoria 0001, South Africa*

Communicated by E.M. Friedlander
Received 1 July 1986
Revised 13 February 1987

It is shown that Noetherian rings of dimension one are pole-assignable.

In this note we prove that every Noetherian ring $R$ of Krull dimension one has the PA-property. Let us briefly recall the definition of the PA-property. Consider the following data:

(i) $F: P \to P$ and $G: U \to P$ are maps between finitely generated projective $R$-modules such that $G \oplus FG \oplus \cdots \oplus F^{n-1}G : \oplus U \to P$ is surjective, where $n = \operatorname{rank} P$;

(ii) $\lambda_1, \ldots, \lambda_n$ is a sequence in $R$ of length $n = \operatorname{rank} P$.

We say that the *poles* $\lambda_1, \ldots, \lambda_n$ in (ii) can be assigned to the *reachable system* $(F, G)$ in (i) if there exists a map $K: P \to U$ such that $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $F + GK : P \to P$. The ring $R$ has the PA-property if, given arbitrary data $(F, G)$, $(\lambda_1, \ldots, \lambda_n) \in R^n$ satisfying (i) and (ii), the poles $\lambda_1, \ldots, \lambda_n$ can be assigned to $(F, G)$.

We refer the reader to [3] and [4] where these concepts are discussed and related to the theory of systems.

We shall assume that the reader is familiar with the fundamental Eisenbud–Evans paper [1]. In the sequel we shall invoke the following result repeatedly: $R$ has the PA-property (or $R$ is PA for short) if every basic finitely generated $R$-submodule $B$ of a finitely generated projective $R$-module $P$ contains a rank one summand of $P$ (cf. [3, 4]).

**Proposition 1.** *Let $R$ be a reduced Noetherian ring of dimension one and let $B$ be a basic submodule of a projective $R$-module $P$. Then $B$ contains a rank one summand of $P$. Hence $R$ is* PA.

**Proof.** We may assume that $P = R^n$ and that $B$ is generated by $a_1, \ldots, a_m$. If $B$ is 2-fold basic at every minimal prime $\varrho$, then the Eisenbud–Evans theorem applies to

0022-4049/88/$3.50 © 1988, Elsevier Science Publishers B.V. (North-Holland)

ensure the existence of a basic element of $P$ in $B$ – this basic element then generates the required rank one summand of $P$.

Otherwise, let

$$S=\{\varrho\in\operatorname{spec}R \mid \varrho \text{ is minimal in } \operatorname{spec}R \text{ and } B \text{ is at most 1-fold basic at } \varrho\}.$$

It is well known that the local rings of a reduced ring at minimal primes are fields. This means that the statement '$B$ is $t$-fold basic in $P$ at $q$' is equivalent to

$$\mu(R_q, B_q)\geq t$$

at minimal primes in a reduced ring.

It follows that

$$\mu(R_\varrho, B_\varrho)=1 \quad \text{for all } \varrho\in S.$$

Hence $U=\{q \mid \mu(R_q,B_q)=1\}$ is an open neighbourhood of $S$.

Let $Y=\operatorname{spec}R-U$. Then, for $q\in Y$ we know that $B$ is $(\dim(q)+1)$-fold basic in $P$. But then there is some element $\alpha$ of $B$ that is basic in $P$ at every $q\in Y$: to see this, apply a trivially modified version of the proof of the Eisenbud–Evans theorem as given in [1] – in fact, $\alpha$ is basic in $P$ in some open neighbourhood $V$ of $Y$.

For every $R$-module $M$, let $\tilde{M}$ denote the sheaf on $\operatorname{spec}R$ associated with $M$. As $B_q=R_q\alpha$ for every $q$ in $U\cap V$, we are in a position to glue the sheaves $\tilde{B}\mid U$ and $\tilde{R\alpha}\mid V$ to obtain a subsheaf $F$ of $\tilde{B}$ on $\operatorname{spec}R$. The sheaf $F$ is evidently a coherent, invertible sheaf. By [2, 11.5.5], $F=\tilde{M}$ for some submodule of $B$, and $M$ is a rank one projective module. Moreover, $\tilde{R^n/M}=O^n/F$ is a locally free coherent sheaf, so $R^n/M$ is projective, where $O$ is the structure sheaf on $\operatorname{spec}R$. The validity of the final statement of Proposition 1 now follows from the results of [4]. □

**Corollary.** *A Noetherian* 1*-dimensional ring* $R$ *is* PA.

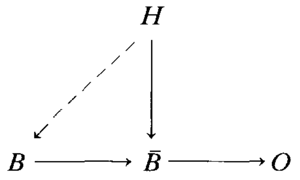
**Proof.** Let $B\leq P$ be $R$-modules with $P$ finitely generated and projective, and let $B$ be basic in $P$. We show that $B$ contains a rank one summand of $P$ – according to the results of [4], we can then conclude that $R$ is PA. The validity of the following statement is well known:

(*) The Picard group of $R/N$ lifts to the Picard group of $R$

($N$ is the nilradical of $R$).

Observe that $\bar{B}=B+NP/NP$ is a basic submodule of $\bar{P}=P/NP$. By Proposition 1, $\bar{B}$ contains a rank one summand $\bar{H}$ of $\bar{P}$.

By (*) there is a rank one projective $R$-module $H$ such that $H/NH\cong\bar{H}$. Then we have the following diagram of $R$-modules:




which can be completed by $(\varphi : H \to B) \in \mathrm{Hom}_R(H, B)$ to become commutative. The map $H \to B \to P$ induces a split injection $H/NH \to P/NP$ so it must also be a split injection. Hence $\mathrm{Im}\,\varphi$ is a rank one summand of $P$.

## References

[1] D. Eisenbud and E.G. Evans Jr., Generating modules efficiently: theorems from algebraic $K$-theory, J. Algebra 27 (1973) 273.

[2] R. Hartshorne, Algebraic Geometry, Graduate Texts in Mathematics (Springer, Berlin, 1977).

[3] M.I.J. Hautus and E.D. Sontag, New results on pole shifting for parametrized families of systems, J. Pure Appl. Algebra 40 (1986) 229–244.

[4] E. Minnaar, C.G. Naudé, G. Naudé and F. Wiid, Pole assignability of rings of low dimension, J. Pure Appl. Algebra 51 (1988) 197–203.