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A heat flow for special metrics

Hartmut Weiß^a, Frederik Witt^{b,*}^a *Mathematisches Institut der Universität München, Theresienstraße 39, D-80333 München, Germany*^b *Mathematisches Institut der Universität Münster, Einsteinstraße 62, D-48149 Münster, Germany*

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Abstract

On the space of positive 3-forms on a seven-manifold, we study a natural functional whose critical points induce metrics with holonomy contained in G_2 . We prove short-time existence and uniqueness for its negative gradient flow. Furthermore, we show that the flow exists for all times and converges modulo diffeomorphisms to some critical point for any initial condition sufficiently C^∞ -close to a critical point.

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1. Introduction

A central problem in Riemannian geometry is the construction of metrics with prescribed properties of the Ricci tensor. In this context the metrics we refer to as *special* are particularly interesting. These metrics are induced by a differential form Ω of special algebraic type subject to the non-linear harmonic equation

$$d\Omega = 0, \quad d\star_\Omega \Omega = 0, \quad (1)$$

where \star_Ω is the Hodge operator associated with the induced metric g_Ω . A prototypical example is provided by G_2 -metrics. A theorem of Fernández and Gray [11] asserts (1) to hold if and only

* Corresponding author.

E-mail addresses: weiss@math.lmu.de (H. Weiß), frederik.witt@uni-muenster.de, fwitt_04@uni-muenster.de (F. Witt).

if the holonomy of g_Ω is contained in G_2 . This, in turn, implies g_Ω to be Ricci flat [3]. Other metrics of this type are Spin(7)-metrics (which are also Ricci flat) or quaternionic Kähler metrics (which are Einstein), but more exotic examples such as PSU(3)-metrics [14] (satisfying a less standard condition on the Ricci tensor) also fit into this setting.

In the G_2 -case we consider *positive* 3-forms Ω over an oriented 7-manifold which are special in as far they give rise to a complementary 4-form $\Theta(\Omega)$ so that the volume form $vol_\Omega := \Omega \wedge \Theta(\Omega)/7$ induces the chosen orientation. In fact, once the metric has been constructed from Ω one has $\Theta(\Omega) = \star_\Omega \Omega$. If M is compact, (1) is equivalent to the non-linear Laplace equation $\Delta_\Omega \Omega = 0$, where Δ_Ω is the Hodge Laplacian associated with g_Ω . In this sense, Ω is “self-harmonic”, but we shall stick to the usual G_2 -jargon and refer to positive forms satisfying (1) as *torsion-free*. For closed Ω , Hitchin [14] interpreted the second condition as the Euler–Lagrange equation for the functional on positive 3-forms $\Omega \mapsto \int_M vol_\Omega$ restricted to the cohomology class $[\Omega]$. Existence of critical points, however, is a delicate issue. Since Joyce’s seminal work [16] we know non-trivial compact holonomy G_2 -manifolds to exist, but a Yau–Aubin type theorem which guarantees a priori existence is yet missing.

A natural idea for proving existence of G_2 -metrics is to look for a geometric evolution equation on the space of positive 3-forms, which could evolve forms towards a torsion-free Ω ; cf. [17] for a rather general perspective. The first candidate for a flow equation has been proposed by Bryant in [5], namely the “Laplacian flow”

$$\frac{\partial}{\partial t} \Omega = \Delta_\Omega \Omega.$$

Restricted to closed positive 3-forms, we can think of this flow as the (L^2) -gradient flow of Hitchin’s functional. However, as we are going to show, the resulting flow equation is not even weakly parabolic so that standard parabolic theory does not apply directly (though Bryant and Xu, resp. Xu and Ye have managed to establish short-time existence for the Laplacian flow for *closed* initial conditions, cf. [6,26]). This is reminiscent of the Einstein–Hilbert functional whose negative gradient is difficult to deal with on the same grounds, a fact which subsequently led to the definition of Ricci flow. We therefore consider the negative gradient flow of the “Dirichlet energy” functional

$$\mathcal{D} : \Omega \mapsto \frac{1}{2} \int_M (|d\Omega|_{g_\Omega}^2 + |d\Theta(\Omega)|_{g_\Omega}^2) vol_\Omega$$

whose critical points, as we will show, are precisely given by torsion-free forms. In principle, the definition of \mathcal{D} makes sense for any special metric. The reason to focus on G_2 is twofold. The set of positive 3-forms Ω is an open subset of $\Omega^3(M)$, and G_2 acts transitively on the sphere (as do all reduced holonomy groups of manifolds which are not locally symmetric). Both these features greatly simplify technicalities.

Our first result is short-time existence and uniqueness.

Theorem. *Let M be a closed, oriented 7-manifold. Given a positive 3-form Ω_0 on M , there exist $\epsilon > 0$ and a smooth family of positive 3-forms $\Omega(t)$ for $t \in [0, \epsilon]$ such that*

$$\frac{\partial}{\partial t} \Omega = -\text{grad } \mathcal{D}(\Omega), \quad \Omega(0) = \Omega_0.$$

Furthermore, for any two solutions $\Omega(t)$ and $\Omega'(t)$ we have $\Omega(t) = \Omega'(t)$ whenever defined.

Hence we can speak of *the* Dirichlet energy flow for some initial condition Ω_0 defined on a maximal time-interval $[0, T_{max})$, $0 < T_{max} \leq \infty$. We emphasise that for the Dirichlet energy flow the initial condition does *not* have to be assumed to be closed. The proof is based on DeTurck's trick as introduced in [8], namely to consider a geometric perturbation \tilde{Q} of the negative gradient of \mathcal{D} . Then \tilde{Q} is strongly elliptic and the standard theory of quasilinear parabolic equations applies. Of course, one cannot expect long-time existence and convergence in general as the existence of G_2 -metrics is topologically obstructed. For instance, if M is compact then Ω cannot be exact, hence the third Betti number b_3 must be greater than or equal to 1. However, a meaningful negative gradient flow is certainly expected to exist for all times and to converge near a critical point. Indeed we will prove the subsequent stability result (cf. [Theorem 8.1](#) and [Corollary 8.2](#) for the precise statement).

Theorem. *Let M be a closed, oriented 7-manifold and let $\bar{\Omega}$ be a torsion-free G_2 -form on M . For initial conditions sufficiently C^∞ -close to $\bar{\Omega}$ the Dirichlet energy flow exists for all times and converges modulo diffeomorphisms to a torsion-free G_2 -form.*

This theorem resembles Šešum's corresponding stability result for Ricci flat metrics under Ricci flow [21]. Her result is related to the fact that up to diffeomorphisms Ricci flow may be viewed as a gradient flow of Perelman's \mathcal{F} -functional [20]. Nevertheless, two major differences occur: for stability of the Ricci flow one needs to assume that (a) the Ricci flat metric \bar{g} (which corresponds to the torsion-free $\bar{\Omega}$ in our setting) is a smooth point of the moduli space of Ricci flat metrics (b) the linearisation at \bar{g} of the quasilinear elliptic operator involved (corresponding to our \tilde{Q}) is non-positive. Both conditions are difficult to check in practice unless one makes further assumptions such as special holonomy. In our situation, however, these assumptions are automatically satisfied. In particular, we can show that near $\bar{\Omega}$, the set $\tilde{Q}^{-1}(0)$ provides a slice for the action of diffeomorphisms isotopic to the identity on the space of torsion-free G_2 -metrics. Consequently, the moduli space of torsion-free G_2 -forms is smooth. This has been proven previously by Joyce (see [16]), but our approach is rather different and based on the specific geometry of the zero sets of $\text{grad } \mathcal{D}$ and \tilde{Q} . Second, we note that on a technical level, the linearisation $L_{\bar{\Omega}}$ of \tilde{Q} at $\bar{\Omega}$ does not admit a Weitzenböck formula as does the Lichnerowicz Laplacian which appears in the Ricci flow case. However, we can prove the Gårding type inequality

$$\langle -L_{\bar{\Omega}}\dot{\Omega}, \dot{\Omega} \rangle_{L^2} \geq C \|\dot{\Omega}\|_{W^{1,2}}^2 - \|\dot{\Omega}\|_{L^2}^2,$$

where $\dot{\Omega} \in \Omega^3(M)$ and C is some positive constant. With this coercivity condition we can invoke a result of Lax–Milgram type in order to establish long-time existence following ideas developed in [15]. Finally, convergence follows from a careful analysis of the remainder term $R_{\bar{\Omega}} = \tilde{Q} - L_{\bar{\Omega}}$. The flow is thus, in principle, capable of finding torsion-free G_2 -forms on a 7-manifold if they exist.

2. G_2 -structures

Here and in the sequel, we let M be a connected, closed, oriented manifold of dimension 7. We first recall some basic features of G_2 -geometry to fix notations. Good references are [4] and Chapter 10 in [16].

The group $GL(7)$ acts on $\Lambda^3 \mathbb{R}^{7*}$ and has an open orbit \mathcal{O} diffeomorphic to $GL(7)/G_2$. In fact, \mathcal{O} is an open cone, for $c \cdot \text{Id} \in GL(7)$, $c \neq 0$ acts on any 3-form by multiplication with c^{-3} . Since

G_2 is a subgroup of $SO(7)$, any $\Omega \in \mathcal{O}$ induces an orientation and a Euclidean metric g_Ω on \mathbb{R}^7 . We denote by Λ^3_+ those $\Omega \in \mathcal{O}$ which induce the standard orientation on \mathbb{R}^7 and refer to its elements as *positive forms*. They are acted on transitively by $GL(7)_+$, the orientation-preserving linear isomorphisms of \mathbb{R}^7 .

Let $\Omega^3_+(M)$ denote the open set of sections of Λ^3_+M , the fibre bundle associated with Λ^3_+ . Then a section $\Omega \in \Omega^3_+(M)$ (which exists if and only if the second Stiefel–Whitney class of M vanishes) induces a reduction of the frame bundle to a principal G_2 -bundle. We also refer to the pair (M, Ω) as a G_2 -structure. Such a structure singles out a principal $SO(7)$ -bundle whose associated metric, Hodge star operator and Levi-Civita connection we denote by g_Ω, \star_Ω and ∇^Ω respectively. Locally, there exist so-called G_2 -frames, i.e. local frames (e_1, \dots, e_7) of TM for which Ω has “normal form”. In our convention, we can write Ω with respect to a G_2 -frame as

$$\Omega = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245}$$

with e^{ijk} shorthand for $e^i \wedge e^j \wedge e^k$. Note that a G_2 -frame is orthonormal for the induced G_2 -metric g_Ω .

The holonomy of g_Ω is contained in G_2 (implying that g_Ω is Ricci-flat [3]) if and only if the underlying G_2 -form Ω is parallel, i.e. $\nabla^{g_\Omega} \Omega = 0$. In this case we shall say that the G_2 -structure is *torsion-free* while we call (M, Ω) a *holonomy G_2 -manifold* if the holonomy of g_Ω is actually equal to G_2 . (This convention is by no means universal in the literature.) A torsion-free G_2 -structure has holonomy G_2 if the fundamental group $\pi_1(M)$ is finite (M being compact). By a theorem of Fernández and Gray [11], torsion-freeness is equivalent to $d\Omega = 0$ and $\delta_\Omega \Omega = 0$, where $\delta_\Omega = (-1)^p \star_\Omega d \star_\Omega$ is the induced codifferential on p -forms. We shall therefore refer to any such Ω as a *torsion-free G_2 -form*. The latter equation can be viewed as the Euler–Lagrange equation of a non-linear variational problem set up by Hitchin [14]. Consider the smooth $GL(7)_+$ -equivariant map

$$\phi : \Lambda^3_+ \rightarrow \Lambda^7, \quad \Omega \mapsto vol_\Omega := \star_\Omega 1 = \frac{1}{7} \Omega \wedge \star_\Omega \Omega,$$

whose first derivative at Ω evaluated on $\dot{\Omega} \in \Lambda^3$ is

$$D_\Omega \phi(\dot{\Omega}) = \frac{1}{3} \star_\Omega \Omega \wedge \dot{\Omega}. \tag{2}$$

Integrating ϕ gives the functional

$$\mathcal{H} : \Omega^3_+(M) \rightarrow \mathbb{R}, \quad \Omega \mapsto \int_M \phi(\Omega). \tag{3}$$

In analogy with Hodge theory we can restrict \mathcal{H} to a fixed cohomology class and ask for critical points. From (2) it follows that a closed Ω is a critical point in its cohomology class if and only if $\delta_\Omega \Omega = 0$ [14]. In particular, Ω is torsion-free and thus harmonic with respect to its induced Laplacian $\Delta_\Omega = d\delta_\Omega + \delta_\Omega d$. In passing we note that modulo a constant, $\mathcal{H}(\Omega)$ can be viewed as the norm square of the Euler vector field $\Omega \mapsto \Omega$ on the “prehilbert” manifold $\Omega^3_+(M)$ with induced L^2 -metric

$$\langle \dot{\Omega}_1, \dot{\Omega}_2 \rangle_{L^2_\Omega} := \int_M g_\Omega(\dot{\Omega}_1, \dot{\Omega}_2) vol_\Omega = \int_M \dot{\Omega}_1 \wedge \star_\Omega \dot{\Omega}_2,$$

for elements $\dot{\Omega}_1, \dot{\Omega}_2$ in the tangent space $T_\Omega \Omega_+^3(M) \cong \Omega^3(M)$. We will drop the reference to Ω whenever this can be safely done and simply write $\langle \cdot, \cdot \rangle_{L^2}$ and g . The associated norms are then denoted by $\| \cdot \|$ and $| \cdot |$ respectively.

3. Representation theory

Next we recall some elements of G_2 -representation theory. Most of the material is standard (mainly taken from [5,7] and [18]) or follows from straightforward computations.

The group G_2 acts irreducibly in its vector representation $\Lambda^1 \cong \mathbb{R}^7$ (in the presence of a metric, we tacitly identify vectors with their duals). This action extends to the exterior algebra in the standard fashion, though Λ^p , the G_2 -representation over p -forms, is no longer irreducible for $2 \leq p \leq 5$. More precisely, we have orthogonal decompositions

$$\Lambda^2 = \Lambda_7^2 \oplus \Lambda_{14}^2, \quad \Lambda^3 = \Lambda_1^3 \oplus \Lambda_7^3 \oplus \Lambda_{27}^3,$$

where the subscript indicates the dimension of the module. We denote the corresponding components by $[\alpha^p]_q$. For $p = 2, 3$ they satisfy

$$\begin{aligned} \Lambda_7^2 &= \{ \alpha \in \Lambda^2 \mid \star_\Omega(\alpha \wedge \Omega) = 2\alpha \}, & \Lambda_{14}^2 &= \{ \alpha \in \Lambda^2 \mid \star_\Omega(\alpha \wedge \Omega) = -\alpha \}, \\ \Lambda_1^3 &= \{ \star_\Omega(X \wedge \Omega) \mid X \in \Lambda^1 \}, & \Lambda_{27}^3 &= \{ \alpha \in \Lambda^3 \mid (\star_\Omega \Omega) \wedge \alpha = 0, \Omega \wedge \alpha = 0 \}. \end{aligned} \tag{4}$$

The Lie algebra of \mathfrak{g}_2 sitting inside $\mathfrak{so}(7) \cong \Lambda^2$ corresponds to Λ_{14}^2 , while Λ_1^3 simply consists of multiples of Ω . Note that by equivariance, \star_Ω induces isomorphisms $\Lambda_q^p \cong \Lambda_q^{7-p}$ from which an analogous decomposition of Λ^4 and Λ^5 follows. This and the characterisations (4) are obtained from a routine application of Schur’s lemma. For illustration, we derive for $\eta \in \Lambda^2$ the identity

$$(\eta_\perp \Omega)_\perp \Omega = 3[\eta]_7. \tag{5}$$

Here, \perp denotes the extension of the metric contraction to $\perp: \Lambda^k V^* \otimes \Lambda^l V^* \rightarrow \Lambda^{l-k} V^*$, e.g. $e^{12}_\perp e^{12345} = e^{345}$ etc. Now $\eta \mapsto \eta_\perp \Omega$ is a G_2 -equivariant map taking values in the irreducible module $\Lambda^1 = \Lambda_7^1$ so that by Schur $\Lambda_{14}^2 \subset \ker \perp \Omega$, whence $\eta_\perp \Omega = [\eta]_7 \Omega$. Therefore, the identity (5) needs only to be checked for one nontrivial element in Λ_7^2 (again by Schur). Fixing a G_2 -frame as in the previous section, we find $e_{1\perp} \Omega = e_{27} + e_{35} - e_{46} \in \Lambda_7^2$, hence $(e_{1\perp} \Omega)_\perp \Omega = 3e_1$. In the same vein, we can prove

$$\begin{aligned} (\star_\Omega(\alpha \wedge \Omega)) \wedge \Omega &= -4 \star_\Omega \alpha, \\ (\star_\Omega(\alpha \wedge \star_\Omega \Omega)) \wedge \star_\Omega \Omega &= 3 \star_\Omega \alpha, \\ (\star_\Omega(\alpha \wedge \star_\Omega \Omega)) \wedge \Omega &= 2\alpha \wedge \star_\Omega \Omega. \end{aligned} \tag{6}$$

If the manifold M is endowed with a G_2 -structure Ω , all these decompositions and identities acquire global meaning. In particular we can speak of Ω_q^p -forms, where $\Omega_q^p(M) = C^\infty(\Lambda_q^p T^*M)$ are smooth sections of the bundles with fibre Λ_q^p . As in the case for Kähler manifolds, this decomposition gives rise to G_2 -analogues of the Cauchy–Riemann operator, provided the G_2 -structure is torsion-free. The subsequent formulæ were derived by Bryant and Harvey and can be found in [5]. We briefly describe their results. First, we fix reference modules for the irreducible G_2 -representations occurring in the exterior algebra Λ^* , namely $\Omega_1 = \Omega_1^0(M)$, $\Omega_7 = \Omega_7^1(M)$, $\Omega_{14} = \Omega_{14}^2(M)$ and $\Omega_{27} = \Omega_{27}^3(M)$. Any form $\alpha \in \Omega^p(M)$ can be written in terms of G_2 -equivariant maps applied to elements of these reference modules. For instance

Table 1
Exterior derivative formulæ.

df	$=$	$d_7^1 f$		
$d(f\Omega)$	$=$	$d_7^1 f \wedge \Omega$		
$d(f\star_\Omega \Omega)$	$=$	$d_7^1 f \wedge \star_\Omega \Omega$		
$d\alpha$	$=$	$\frac{1}{3} \star_\Omega (d_7^1 \alpha \wedge \star_\Omega \Omega)$	$+d_{14}^7 \alpha$	
$d\star_\Omega(\alpha \wedge \star_\Omega \Omega)$	$=$	$-\frac{3}{7} d_7^1 \alpha \cdot \Omega$	$-\frac{1}{2} \star_\Omega (d_7^1 \alpha \wedge \Omega)$	$+d_{27}^7 \alpha$
$d\star_\Omega(\alpha \wedge \Omega)$	$=$	$\frac{4}{7} d_7^1 \alpha \cdot \star_\Omega \Omega$	$+\frac{1}{2} d_7^1 \alpha \wedge \Omega$	$+\star_\Omega d_{27}^7 \alpha$
$d(\alpha \wedge \Omega)$	$=$	$\frac{2}{3} d_7^1 \alpha \wedge \star_\Omega \Omega$	$-\star_\Omega d_{14}^7 \alpha$	
$d(\alpha \wedge \star_\Omega \Omega)$	$=$	$\star_\Omega d_7^1 \alpha$		
$d(\star_\Omega \alpha)$	$=$	$-d_1^7 \alpha \cdot vol_\Omega$		
$d\beta$	$=$	$\frac{1}{4} \star_\Omega (d_{14}^1 \beta \wedge \Omega)$		$+d_{27}^{14} \beta$
$d(\star_\Omega \beta)$	$=$	$\star_\Omega d_{14}^1 \beta$		
$d\gamma$	$=$	$\frac{1}{4} d_{27}^7 \gamma \wedge \Omega$		$+\star_\Omega d_{27}^{27} \gamma$
$d(\star_\Omega \gamma)$	$=$	$-\frac{1}{3} d_{27}^7 \gamma \wedge \star_\Omega \Omega$	$-\star_\Omega d_{14}^{27} \gamma$	

$\dot{\Omega} = [\dot{\Omega}]_1 \oplus [\dot{\Omega}]_7 \oplus [\dot{\Omega}]_{27} \in \Omega^3(M)$ can be written as $\dot{\Omega} = f\Omega \oplus \star_\Omega(\alpha \wedge \Omega) \oplus \gamma$ for $f \in \Omega_1$, $\alpha \in \Omega_7$ and $\gamma \in \Omega_{27}$. There exist first order differential operators $d_q^p : \Omega_p \rightarrow \Omega_q$ such that the identities of Table 1 hold.

That such a table must exist follows from the torsion-freeness which is equivalent to finding coordinates x_1, \dots, x_7 with $\partial_{x_i} \Omega(x) = 0$ around any $x \in M$. Put differently, the G_2 -structure (M, Ω) locally osculates to first order to the flat structure on \mathbb{R}^7 . Therefore, the exterior derivative of expressions such as $\star_\Omega(\alpha \wedge \Omega)$ only depends on the 1-jet of the reference forms involved, e.g. α . The operators d_q^p are obtained by compounding d with suitable G_2 -equivariant maps. As an example, $d\alpha = \star_\Omega(\dot{\alpha} \wedge \star_\Omega \Omega) \oplus \beta$ for $\dot{\alpha} \in \Omega_7$ and $\beta \in \Omega_{14}$, where $\dot{\alpha} = \star_\Omega(d\alpha \wedge \star_\Omega \Omega)/3$ and $\beta = [d\alpha]_{14}$. Working out the identities of Table 1 is then a matter of computation using the algebraic formulæ (6), (16) and (17). Our precise definition of the operators d_q^p can be found in Appendix A.

Remark. The operators d_q^p and d_p^q are formally adjoint to each other, i.e.

$$\langle d_q^p \sigma_p, \sigma_q \rangle_{L^2_\Omega} = \langle \sigma_p, d_p^q \sigma_q \rangle_{L^2_\Omega}$$

for any $\sigma_p \in \Omega_p$ and $\sigma_q \in \Omega_q$.

Example. With Table 1 at hand we compute the (co-)differential of $\dot{\Omega} = \dot{f}\Omega \oplus \star_\Omega(\dot{\alpha} \wedge \Omega) \oplus \dot{\gamma} \in \Omega^3(M)$ and find

$$\begin{aligned} d\dot{\Omega} &= \frac{4}{7} d_1^7 \dot{\alpha} \star_\Omega \Omega \oplus \left(d_7^1 \dot{f} + \frac{1}{2} d_7^7 \dot{\alpha} + \frac{1}{4} d_{27}^{27} \dot{\gamma} \right) \wedge \Omega \oplus \star_\Omega (d_{27}^7 \dot{\alpha} + d_{27}^{27} \dot{\gamma}) \\ \delta_\Omega \dot{\Omega} &= \star_\Omega \left(\left(-d_7^1 \dot{f} - \frac{2}{3} d_7^7 \dot{\alpha} + \frac{1}{3} d_{27}^{27} \dot{\gamma} \right) \wedge \star_\Omega \Omega \right) \oplus d_{14}^7 \dot{\alpha} + d_{14}^{27} \dot{\gamma}. \end{aligned} \tag{7}$$

Using Table 1, $d^2 = 0$ implies the following second-order identities of Table 2. We will also need the Laplacians $\Delta_\Omega \sigma_p, \sigma_p \in \Omega_p$. These are given in Table 3.

Table 2
Second order identities.

	$d_7^7 d_7^1 = 0$	$d_{14}^7 d_7^1 = 0$	
$d_1^7 d_7^7 = 0$	$d_7^{14} d_{14}^7 = \frac{2}{3} d_7^7 d_7^7$	$d_{14}^7 d_7^7 + 2d_{14}^{27} d_{27}^7 = 0$	$3d_{27}^{14} d_{14}^7 + d_{27}^7 d_7^7 = 0$
	$d_{27}^{27} d_{27}^7 = d_7^7 d_7^7 + \frac{12}{7} d_7^1 d_7^7$		$2d_{27}^{27} d_{27}^7 - d_{27}^7 d_7^7 = 0$
$d_1^7 d_7^{14} = 0$	$d_7^7 d_7^{14} + 2d_{27}^{27} d_{27}^{14} = 0$		$d_{27}^7 d_7^{14} + 4d_{27}^{27} d_{27}^{14} = 0$
	$3d_7^{14} d_{14}^{27} + d_7^7 d_7^{27} = 0$	$d_{14}^7 d_7^{27} + 4d_{14}^{27} d_{27}^{27} = 0$	
	$2d_{27}^{27} d_{27}^7 - d_7^7 d_7^{27} = 0$		

Table 3
Laplacians.

$\Delta_\Omega f = d_1^7 d_7^1 f$
$\Delta_\Omega \alpha = (d_7^7 d_7^7 + d_7^1 d_7^7) \alpha$
$\Delta_\Omega \beta = (\frac{5}{4} d_{14}^7 d_7^{14} + d_{14}^{27} d_{27}^{14}) \beta$
$\Delta_\Omega \gamma = (\frac{7}{12} d_{27}^7 d_7^{27} + d_{27}^{14} d_{14}^{27} + (d_{27}^{27})^2) \gamma$

Example. If $\dot{\Omega} = \dot{f} \Omega \oplus \star_\Omega(\dot{\alpha} \wedge \Omega) \oplus \dot{\gamma}$, then

$$\Delta_\Omega \dot{\Omega} = \Delta_\Omega \dot{f} \cdot \Omega \oplus \star_\Omega(\Delta_\Omega \dot{\alpha} \wedge \Omega) \oplus \Delta_\Omega \dot{\gamma}.$$

We finish this section with some material on SU(3)-representation theory. This will become relevant later, when we compute the symbols of various differential operators on a manifold equipped with some fixed G₂-structure. The finer SU(3)-representation theory will enable us to derive properties of these symbols such as invertibility and definiteness.

We pick a unit vector $\xi \in \Lambda^1$ in the vector representation of G₂. Since the unit sphere S⁶ is diffeomorphic with G₂/SU(3), ξ gives rise to an SU(3)-representation over ξ^\perp , namely the real representation underlying the complex vector representation C³. In particular, ξ^\perp carries a complex structure. In terms of forms, the group SU(3) can be regarded as the stabiliser of a non-degenerate 2-form $\omega \in \Lambda^2 \xi^\perp$ and a complex volume form $\Psi = \psi_+ + i\psi_- \in \Lambda^{3,0} \xi^\perp$. These forms relate to Ω and $\star_\Omega \Omega$ via

$$\begin{aligned} \Omega &= \omega \wedge \xi + \psi_+, \\ \star_\Omega \Omega &= \psi_- \wedge \xi + \frac{1}{2} \omega^2. \end{aligned} \tag{8}$$

In terms of a G₂-frame with $\xi = e_7$ we find $\omega = e^{12} + e^{34} + e^{56}$, $\psi_+ = e^{135} - e^{146} - e^{236} - e^{245}$ and $\psi_- = e^{136} + e^{145} + e^{235} - e^{246}$. They satisfy the algebraic relations

$$\omega \wedge \psi_\pm = 0, \quad \psi_+ \wedge \psi_- = \frac{2}{3} \omega^3. \tag{9}$$

The decomposition of the exterior algebra over ξ^\perp into irreducibles is given by

$$\lambda^1 = \xi^\perp, \quad \lambda^2 = \lambda_1^2 \oplus \lambda_6^2 \oplus \lambda_8^2, \quad \lambda^3 = \lambda_{1+}^3 \oplus \lambda_{1-}^3 \oplus \lambda_6^3 \oplus \lambda_{12}^3, \tag{10}$$

where $\lambda^i := \Lambda^i \xi^\perp$. As above the numerical subscript keeps track of the dimension. We also use these subscripts to denote the corresponding components of a form, e.g. $\gamma \in \lambda^3$ can be decomposed into the direct sum $\gamma = \gamma_{1+} \oplus \gamma_{1-} \oplus \gamma_6 \oplus \gamma_{12}$. The two trivial representations $\lambda_{1\pm}^3$ are spanned by ψ_+ and ψ_- respectively, while λ_8^2 corresponds to the Lie algebra of $\mathfrak{su}(3)$ sitting inside $\mathfrak{so}(6) \cong \lambda^2$. More importantly for our purposes we can consider the decomposition of the exterior algebra over \mathbb{R}^7 into $SU(3)$ -irreducibles. Here, we shall denote by $(\mathbf{n})_q^p$ the n -dimensional irreducible $SU(3)$ -representation inside Λ_q^p . Then

$$\begin{aligned} \Lambda^1 &\cong (\mathbf{1})_7^1 \oplus (\mathbf{6})_7^1, & \Lambda^2 &\cong (\mathbf{1})_7^2 \oplus (\mathbf{6})_7^2 \oplus (\mathbf{6})_{14}^2 \oplus (\mathbf{8})_{14}^2, \\ \Lambda^3 &\cong (\mathbf{1})_1^3 \oplus (\mathbf{1})_7^3 \oplus (\mathbf{6})_7^3 \oplus (\mathbf{1})_{27}^3 \oplus (\mathbf{6})_{27}^3 \oplus (\mathbf{8})_{27}^3 \oplus (\mathbf{12})_{27}^3, \end{aligned}$$

so that no confusion shall occur. The decomposition of Λ^3 is of particular importance for the sequel. The occurring modules can be characterised as follows:

$$\begin{aligned} (\mathbf{1})_1^3 &= \{a(\omega \wedge \xi + \psi_+) \mid a \in \mathbb{R}\}, \\ (\mathbf{1})_7^3 &= \{b\psi_- \mid b \in \mathbb{R}\}, \\ (\mathbf{1})_{27}^3 &= \{c(-4\omega \wedge \xi + 3\psi_+) \mid c \in \mathbb{R}\}, \\ (\mathbf{6})_7^3 &= \{(X \lrcorner \psi_-) \wedge \xi + (X \lrcorner \omega) \wedge \omega \mid X \in \xi^\perp\}, \\ (\mathbf{6})_{27}^3 &= \{(Y \lrcorner \psi_-) \wedge \xi - (Y \lrcorner \omega) \wedge \omega \mid Y \in \xi^\perp\}, \\ (\mathbf{8})_{27}^3 &= \{\beta_8 \wedge \xi \mid \beta_8 \in \lambda_8^2\}. \end{aligned}$$

For instance, $Y \in \xi^\perp \mapsto A(Y) = (Y \lrcorner \psi_-) \wedge \xi - (Y \lrcorner \omega) \wedge \omega \in \Lambda^3$ is a linear isomorphism onto its image. Further, $(Y \lrcorner \psi_-) \wedge \psi_+ = -(Y \lrcorner \psi_+) \wedge \psi_-$ so that the algebraic relations (9) readily imply that $A(Y) \wedge \Omega = 0$, $A(Y) \wedge \star_\Omega \Omega = 0$, i.e. $\text{im } A \subset \Lambda_{27}^3$. Summarising, we can write any $\hat{\Omega} \in \Lambda^3$ as

$$\begin{aligned} \hat{\Omega} &= [\hat{\Omega}]_1 \oplus [\hat{\Omega}]_7 \oplus [\hat{\Omega}]_{27} \\ &= [\hat{a}(\omega \wedge \xi + \psi_+)] \oplus [\hat{b}\psi_- + (\hat{X} \lrcorner \psi_-) \wedge \xi + (\hat{X} \lrcorner \omega) \wedge \omega] \\ &\quad \oplus [\hat{c}(-4\omega \wedge \xi + 3\psi_+) + (\hat{Y} \lrcorner \psi_-) \wedge \xi - (\hat{Y} \lrcorner \omega) \wedge \omega + \hat{\beta}_8 \wedge \xi + \hat{\gamma}_{12}] \end{aligned} \tag{11}$$

for constants $\hat{a}, \hat{b}, \hat{c} \in \mathbb{R}$, vectors $\hat{X}, \hat{Y} \in \xi^\perp$ and forms $\hat{\beta}_8 \in \lambda_8^2$, $\hat{\gamma}_{12} \in \lambda_{12}^3$. In particular, decomposing $\hat{\Omega} = \hat{\beta} \wedge \xi + \hat{\gamma}$, where $\hat{\beta}$ and $\hat{\gamma}$ are the uniquely determined 2- and 3-forms in $\Lambda^* \xi^\perp$ such that $\xi \lrcorner \hat{\beta}, \hat{\gamma} = 0$, we obtain

$$\hat{\beta} = (\hat{a} - 4\hat{c})\omega \oplus (\hat{X} + \hat{Y}) \lrcorner \psi_- \oplus \hat{\beta}_8 \tag{12}$$

$$\hat{\gamma} = (\hat{a} + 3\hat{c})\psi_+ \oplus \hat{b}\psi_- \oplus ((\hat{X} - \hat{Y}) \lrcorner \omega) \wedge \omega \oplus \hat{\gamma}_{12}. \tag{13}$$

Thus $\hat{\beta}_1 = (\hat{a} - 4\hat{c})\omega$ etc. For later applications, we need for $X \in \xi^\perp$ the identities

$$\star_\Omega((X \lrcorner \psi_-) \wedge \Omega) = X \lrcorner \psi_- + 2X \wedge \xi \tag{14}$$

and

$$g_\Omega(X \lrcorner \psi_-, X \lrcorner \psi_-) = 2g_\Omega(X, X). \tag{15}$$

We prove (14) along the lines of (5), while (15) uses the transitive and isometric action of $SU(3)$ on S^5 . Hence, up to a rotation we may assume that $X = |X|e_1$. Similarly, the transitive action of

G_2 on S^6 implies

$$\langle \xi \wedge \Omega, \xi \wedge \Omega \rangle = 4\langle \xi, \xi \rangle, \quad \langle \xi \wedge \star_\Omega \Omega, \xi \wedge \star_\Omega \Omega \rangle = 3\langle \xi, \xi \rangle \tag{16}$$

for all $\xi \in \Lambda^1$. Furthermore, in conjunction with (4) and (6) we note the useful formulæ

$$\begin{aligned} \langle \tau_7^2 \wedge \star_\Omega \Omega, \tau_7^2 \wedge \star_\Omega \Omega \rangle &= 3\langle \tau_7^2, \tau_7^2 \rangle, \\ \langle \tau_7^2 \wedge \Omega, \tau_7^2 \wedge \Omega \rangle &= 4\langle \tau_7^2, \tau_7^2 \rangle, \\ \langle \tau_7^3 \wedge \Omega, \tau_7^3 \wedge \Omega \rangle &= 4\langle \tau_7^3, \tau_7^3 \rangle \end{aligned} \tag{17}$$

for all $\tau_q^p \in \Lambda_q^p$.

4. The Dirichlet energy functional \mathcal{D}

In this section we introduce the Dirichlet energy functional and study some basic properties. In particular, we compute its first variation.

Definition 4.1. The Dirichlet energy functional $\mathcal{D} : \Omega_+^3(M) \rightarrow \mathbb{R}_{\geq 0}$ is defined by

$$\mathcal{D}(\Omega) = \frac{1}{2} \int_M (|d\Omega|_{g_\Omega}^2 + |d\Theta(\Omega)|_{g_\Omega}^2) \text{vol}_\Omega.$$

Remark. Using the L^2 -inner product and integration by parts we may also write

$$\mathcal{D}(\Omega) = \frac{1}{2} (\|d\Omega\|_{L^2_\Omega}^2 + \|\delta_\Omega \Omega\|_{L^2_\Omega}^2) = \frac{1}{2} \langle \Delta_\Omega \Omega, \Omega \rangle_{L^2_\Omega}.$$

Proposition 4.2. (i) *The functional \mathcal{D} is invariant under orientation preserving diffeomorphisms, i.e. $\mathcal{D}(\varphi^* \Omega) = \mathcal{D}(\Omega)$ for all $\varphi \in \text{Diff}(M)_+$, $\Omega \in \Omega_+^3(M)$.*

(ii) *For $\lambda \in \mathbb{R}_{>0}$, $\mathcal{D}(\lambda \Omega) = \lambda^{\frac{5}{3}} \mathcal{D}(\Omega)$, i.e. \mathcal{D} is positively homogeneous.*

Proof. The first assertion follows directly from $\star_{\varphi^* \Omega} = \varphi^* \star_\Omega \varphi^{-1*}$ for $\varphi \in \text{Diff}(M)_+$. Second, we recall that $\tilde{\Omega} = \lambda \Omega \in \Omega_+^3(M)$ if $\lambda > 0$ and $\Omega \in \Omega_+^3(M)$. Now a G_2 -frame $\{e_i\}$ for Ω gives the G_2 -frame $\{f_i = \lambda^{-1/3} e_i\}$ for $\tilde{\Omega}$. Its dual basis is $\{f^i = \lambda^{1/3} e_i\}$. Hence, $\text{vol}_{\lambda \Omega} = f^1 \wedge \dots \wedge f^7 = \lambda^{\frac{7}{3}} \text{vol}_\Omega$, while for the metric $g_{\tilde{\Omega}}$ induced on Λ^p , we find $g_{\lambda \Omega} = f_{i_1 \dots i_p} \otimes f_{i_1 \dots i_p} = \lambda^{-\frac{2p}{3}} g_\Omega$. Hence $|d\tilde{\Omega}|_{g_{\tilde{\Omega}}}^2 = \lambda^{-\frac{2}{3}} |d\Omega|_{g_\Omega}^2$. To compute $|\delta_{\tilde{\Omega}} \tilde{\Omega}|_{g_{\tilde{\Omega}}}^2 = |d \star_{\lambda \Omega} \lambda \Omega|_{g_{\lambda \Omega}}^2$ we observe that considered as an operator $\Omega^p(M) \rightarrow \Omega^{7-p}(M)$,

$$\star_{\lambda \Omega} \alpha^p = \lambda^{\frac{7-2p}{3}} \star_\Omega \alpha^p,$$

whence $|d \star_{\lambda \Omega} \lambda \Omega|_{g_{\lambda \Omega}}^2 = \lambda^{-\frac{2}{3}} |d \star_\Omega \Omega|_{g_\Omega}^2$. \square

Corollary 4.3. *The space \mathcal{X} of critical points of \mathcal{D} is acted on by $\text{Diff}(M)_+$ and is given by*

$$\mathcal{X} = \{\Omega \in \Omega_+^3(M) \mid d\Omega = 0, \delta_\Omega \Omega = 0\},$$

the torsion-free positive 3-forms on M , which are the absolute minima of \mathcal{D} .

Proof. The first claim follows from diffeomorphism invariance of \mathcal{D} , i.e. **Proposition 4.2(i)**. For the second claim we use **Proposition 4.2(ii)** and apply Euler’s formula for homogeneous functions to get

$$D_\Omega \mathcal{D}(\Omega) = \frac{5}{3} \mathcal{D}(\Omega) = \frac{5}{6} \langle \Delta_\Omega \Omega, \Omega \rangle_{L^2_\Omega} \geq 0. \tag{18}$$

Equality holds precisely if $\Delta_\Omega \Omega = 0$, i.e. $d\Omega = 0$ and $\delta_\Omega \Omega = 0$. Hence, if Ω is critical, then in particular $D_\Omega \mathcal{D}(\Omega) = 0$ and therefore Ω is torsion-free. \square

Next we compute the first variation of \mathcal{D} . To that end we introduce the following piece of notation. Let E be some vector bundle and $A : \Omega^3_+(M) \rightarrow C^\infty(E)$ a differential operator. We write \dot{A}_Ω for the linearisation of A at $\Omega \in \Omega^3_+(M)$ evaluated on some 3-form $\dot{\Omega}$ tangent to Ω , i.e.

$$\dot{A}_\Omega := D_\Omega A(\dot{\Omega}).$$

We illustrate this convention by two examples which will be needed later.

Example. (i) Consider the non-linear, homogeneous map

$$\theta : \Omega^3_+(M) \rightarrow \Omega^4(M), \quad \Omega \mapsto \star_\Omega \Omega.$$

Further, for a fixed G_2 -structure $\Omega \in \Omega^3_+(M)$ we define the linear, self-adjoint isomorphism

$$p_\Omega : \Omega^3(M) \rightarrow \Omega^3(M), \quad \dot{\Omega} \mapsto \frac{4}{3} [\dot{\Omega}]_1 + [\dot{\Omega}]_7 - [\dot{\Omega}]_{27}.$$

With the concrete G_2 -structure in mind we shall simply write p . By Prop. 10.3.5 in [16],

$$\dot{\theta}_\Omega = \star_\Omega(p_\Omega(\dot{\Omega})). \tag{19}$$

In particular, $\dot{\Omega} = \Omega$ gives $\dot{\theta}_\Omega = 4\theta(\Omega)/3$.

(ii) In continuation of the first example we consider the map $F : \Omega^3_+(M) \rightarrow \Omega^3(M)$ defined by $F(\Omega) = \Delta_\Omega \Omega$. Then

$$\begin{aligned} \dot{F}_\Omega &= \dot{\star}_\Omega d \star_\Omega d \Omega + \star_\Omega d \dot{\star}_\Omega d \Omega + \star_\Omega d \star_\Omega d \dot{\Omega} - d \dot{\star}_\Omega d \theta(\Omega) - d \star_\Omega d \dot{\theta}(\Omega) \\ &\stackrel{(i)}{=} \star_\Omega d \star_\Omega d \dot{\Omega} - d \star_\Omega d \star_\Omega p_\Omega \dot{\Omega} + \text{terms of lower order in } \dot{\Omega} \\ &= \delta_\Omega d \dot{\Omega} + d \delta_\Omega p_\Omega \dot{\Omega} + \text{terms of lower order in } \dot{\Omega}. \end{aligned}$$

Proposition 4.4. *We have*

$$\dot{D}_\Omega = \int_M \dot{\Omega} \wedge \star_\Omega (\delta_\Omega d \Omega + p_\Omega d \delta_\Omega \Omega + q_\Omega (\nabla^\Omega \Omega))$$

for some quadratic form q_Ω whose coefficients depend smoothly on Ω .

Proof. As in the previous example,

$$\begin{aligned} \dot{D}_\Omega &= \frac{1}{2} \int_M d \dot{\Omega} \wedge \star_\Omega d \Omega + d \Omega \wedge (\dot{\star}_\Omega d \Omega + \star_\Omega d \dot{\Omega}) \\ &\quad + \frac{1}{2} \int_M d \dot{\theta}_\Omega \wedge \star_\Omega d \theta(\Omega) + d \theta(\Omega) \wedge (\dot{\star}_\Omega d \theta(\Omega) + \star_\Omega d \dot{\theta}_\Omega) \\ &= \int_M d \dot{\Omega} \wedge \star_\Omega d \Omega + d \dot{\theta}_\Omega \wedge \star_\Omega d \theta(\Omega) \end{aligned}$$

$$+ \frac{1}{2} \int_M d\Omega \wedge \star_\Omega d\Omega + d\Theta(\Omega) \wedge \star_\Omega d\Theta(\Omega). \tag{20}$$

Now $\Gamma_\Omega(d\Omega) : \dot{\Omega} \mapsto \star_\Omega d\Omega$ is a bundle endomorphism of $\Lambda^3 T^*M$ depending on $d\Omega$ in a fibrewise linear fashion, so we can consider its fibrewise adjoint $(\Gamma_\Omega(d\Omega))^*$. Thus

$$\int_M d\Omega \wedge \star_\Omega d\Omega = \langle \Gamma_\Omega(d\Omega)(\dot{\Omega}), \star_\Omega d\Omega \rangle_{L^2_\Omega} = \langle \dot{\Omega}, (\Gamma_\Omega(d\Omega))^*(\star_\Omega d\Omega) \rangle_{L^2_\Omega}.$$

We deal with the second term of (20) in a similar manner. The last line is therefore of the form $\int_M \dot{\Omega} \wedge q_\Omega(\nabla^\Omega \Omega)$ with q_Ω quadratic in the first derivatives of Ω , as asserted. On the other hand, Stokes implies

$$\begin{aligned} \int_M d\dot{\Omega} \wedge \star_\Omega d\Omega + d\dot{\Theta}_\Omega \wedge \star_\Omega d\Theta(\Omega) &= \int_M \dot{\Omega} \wedge d\star_\Omega d\Omega - \dot{\Theta}_\Omega \wedge d\star_\Omega d\Theta(\Omega) \\ &= \langle \dot{\Omega}, \delta_\Omega d\Omega \rangle_{L^2_\Omega} + \langle \star_\Omega \dot{\Theta}_\Omega, d\delta_\Omega \Omega \rangle_{L^2_\Omega} \\ &= \langle \dot{\Omega}, \delta_\Omega d\Omega + p_\Omega(d\delta_\Omega \Omega) \rangle_{L^2_\Omega}, \end{aligned}$$

whence the assertion. \square

5. Short-time existence

Let

$$Q : \Omega^3_+(M) \rightarrow \Omega^3(M), \quad Q(\Omega) = -\text{grad } \mathcal{D}(\Omega)$$

denote the negative L^2 -gradient of \mathcal{D} in the sense of Definition 4.10 [2], i.e. $\langle Q(\Omega), \dot{\Omega} \rangle_{L^2_\Omega} = -\dot{\mathcal{D}}_\Omega$. In view of Proposition 4.4, we find

$$Q(\Omega) = -\delta_\Omega d\Omega - p_\Omega(d\delta_\Omega \Omega) - q_\Omega(\nabla \Omega). \tag{21}$$

The goal of this section is to prove the existence part of the following.

Theorem 5.1. *Given $\Omega_0 \in \Omega^3_+(M)$, there exist $\epsilon > 0$ and a smooth family $\Omega(t) \in \Omega^3_+(M)$ for $t \in [0, \epsilon]$ such that*

$$\frac{\partial}{\partial t} \Omega = Q(\Omega), \quad \Omega(0) = \Omega_0. \tag{22}$$

Further, if $\Omega(t)$ and $\Omega'(t)$ are solutions to (22), then $\Omega(t) = \Omega'(t)$ whenever defined. Hence $\Omega(t)$ is uniquely defined on a maximal time-interval $[0, T)$ for some $0 < T \leq \infty$.

Remark. We emphasise that, in contrast to the corresponding results for the Laplacian flow in [6,26], the initial condition Ω_0 is not assumed to be closed.

Definition 5.2. We call the negative gradient flow of \mathcal{D} defined by (22) the Dirichlet energy flow with initial condition $\Omega_0 \in \Omega^3_+(M)$.

We will prove short-time existence and uniqueness by invoking the standard theory of quasilinear parabolic equations which we briefly recall. Consider a Riemannian vector bundle $(E, (\cdot, \cdot))$. Let $Q_t : C^\infty(E) \rightarrow C^\infty(E)$ be a family of quasilinear, second order differential

operators, that is locally, $Q_t(u)(x) \stackrel{\text{loc}}{=} (a_\beta^{\alpha ij}(t, x, u, \nabla u)\partial_i\partial_j u^\beta + b^\alpha(t, x, u, \nabla u))s_\alpha$ for smooth functions $a_\beta^{\alpha ij}$ and b^α and a local basis $\{s_\alpha\}$ of E . We say that the induced flow equation

$$\frac{\partial}{\partial t}u = Q_t(u), \quad u(0) = u_0 \tag{23}$$

is *strongly parabolic* at u_0 if there exists a constant $\lambda > 0$ such that the linearisation $D_{u_0}Q_0$ of Q_0 at u_0 satisfies

$$- (\sigma(D_{u_0}Q_0)(x, \xi)v, v) \geq \lambda|\xi|^2|v|^2 \tag{24}$$

for all $(x, \xi) \in TM$, $\xi \neq 0$, and $v \in E_x$. Here, the minus sign in (24) stems from our definition of the principal symbol. Namely, for a k -th order linear differential operator Q we define

$$\sigma(Q)(x, \xi)v = \frac{i^k}{k!}Q(f^k u)(x)$$

for an $f \in C^\infty(M)$ with $f(x) = 0$, $d_x f = \xi$ and $u \in C^\infty(E)$ with $u(x) = v$.

Theorem 5.3. *If Eq. (23) is strongly parabolic at u_0 , then there exist $\epsilon > 0$ and a smooth family $u(t) \in C^\infty(E)$, $t \in [0, \epsilon]$ such that*

$$\frac{\partial}{\partial t}u = Q_t(u), \quad u(0) = u_0.$$

Further, if $u(t)$ and $u'(t)$ are solutions to (22), then $u(t) = u'(t)$ whenever defined.

Our main reference for this is Chapter 4.4 in [24]. Further useful references are Chapter 4.4.2 in [1], Chapter 7.8 in [19] and Chapter 7.1 in [22].

Next we investigate the operator Q as given in (21).

Lemma 5.4. *The second order non-linear differential operator Q is quasilinear.*

Proof. For instance, up to composition with the linear map p whose coefficients depend solely on Ω , the second term on the right hand side of (21) can be locally written as

$$-d\delta_\Omega \Omega \stackrel{\text{loc}}{=} \partial_i(\star_{\Omega, pq}^{jstuv}(\Omega)\partial_j(\star_{\Omega, stuv}^{abc}(\Omega)\Omega_{abc}))dx^{ipq}.$$

Here, $\star_{\Omega, stuv}^{abc}(\Omega)$ denote the coefficients of $\star_\Omega : \Omega^3(M) \mapsto \Omega^4(M)$ with respect to local coordinates x_1, \dots, x_7 etc. Hence Q is linear in its highest (i.e. second) order derivatives. \square

Lemma 5.5. *The principal symbol $\sigma(D_\Omega Q)(x, \xi) : \Lambda^3 T_x^*M \rightarrow \Lambda^3 T_x^*M$ of the linearisation of Q at $\Omega \in \Omega_+^3(M)$ is given by*

$$\sigma(D_\Omega Q)(x, \xi)\dot{\Omega} = -\xi \lrcorner (\xi \wedge \dot{\Omega}) - p_\Omega(\xi \wedge (\xi \lrcorner p_\Omega(\dot{\Omega}))).$$

Moreover, the symbol is negative semi-definite.

Proof. As the principal symbol involves highest order terms only, we only need to linearise the expression

$$-\delta_\Omega d\Omega - p_\Omega(d\delta_\Omega \Omega) = Q(\Omega) - \text{terms of lower order in } \Omega.$$

In our convention, $\sigma(d)(x, \xi)\dot{\Omega} = i\xi \wedge \dot{\Omega}$ and $\sigma(\delta_\Omega)(x, \xi)\dot{\Omega} = -i\xi \lrcorner \dot{\Omega}$. Hence, from (19) and the standard symbol calculus we get the asserted symbol. Further,

$$\begin{aligned} -g_\Omega(\sigma(D_\Omega Q)(x, \xi)\dot{\Omega}, \dot{\Omega}) &= g_\Omega(\xi \lrcorner (\xi \wedge \dot{\Omega}) + p_\Omega(\xi \wedge (\xi \lrcorner p_\Omega(\dot{\Omega}))), \dot{\Omega}) \\ &= |\xi \wedge \dot{\Omega}|_\Omega^2 + |\xi \lrcorner p_\Omega(\dot{\Omega})|_\Omega^2 \\ &\geq 0 \end{aligned}$$

so that $\sigma(D_\Omega Q)(x, \xi)$ is negative semi-definite. \square

Remark. For $\varphi \in \text{Diff}(M)_+$ we have

$$\varphi^* Q(\Omega) = Q(\varphi^* \Omega) \tag{25}$$

since $\mathcal{D} \circ \varphi^* = \mathcal{D}$ and $\langle \dot{\Omega}_0, \dot{\Omega}_1 \rangle_{L^2_{\varphi^* \Omega}} = \langle \varphi^{-1*} \dot{\Omega}_0, \varphi^{-1*} \dot{\Omega}_1 \rangle_{L^2_\Omega}$. Because of this diffeomorphism invariance we cannot expect the principal symbol to be negative definite. Indeed, using the decomposition $\dot{\Omega} = \dot{\beta} \wedge \xi + \dot{\gamma}$ as in Section 3, $g_\Omega(\sigma(D_\Omega Q)(x, \xi)\dot{\Omega}, \dot{\Omega}) = 0$ implies $\dot{\gamma} = 0$. From (13) we deduce $\dot{a} = -3\dot{c}$, $\dot{b} = 0$, $\dot{X}_0 = \dot{Y}_0$ and $\dot{\gamma}_{12} = 0$. Consequently, one then gets that $\dot{\beta}_8 = -\xi \lrcorner p(\dot{\Omega}) = 0$, whence

$$\ker \sigma(D_\Omega Q)(x, \xi) = \{(\dot{v}\omega + \dot{V} \lrcorner \psi_-) \wedge \xi \mid \dot{v} \in \mathbb{R}, \dot{V} \in \xi^\perp\}$$

by (11).

In order to apply Theorem 5.3 we use so-called DeTurck’s trick which was originally invented for Ricci flow [8]. Given a family of diffeomorphisms $\partial_t \varphi_t = X_t \circ \varphi_t$ induced by a (time-dependent) vector field X_t on M , differentiating (25) yields the intertwining formula

$$\mathcal{L}_X(Q(\Omega)) = D_\Omega Q(\mathcal{L}_X \Omega). \tag{26}$$

Here \mathcal{L}_X denotes the Lie derivative with respect to X . While the left hand side of (26) is of first order in X , the right hand side is of third order. Passing to symbol level implies

$$\sigma(D_\Omega Q)(x, \xi) \circ \sigma(X \mapsto \mathcal{L}_X \Omega)(x, \xi) = 0. \tag{27}$$

In this way, we can conceive the symbol of the map

$$\Omega \in \Omega^3_+(M) \mapsto X(\Omega) \in C^\infty(TM) \mapsto \Lambda(\Omega) = \mathcal{L}_{X(\Omega)} \Omega \in \Omega^3(M) \tag{28}$$

(where the vector field $X(\Omega)$ depends non-trivially on the 1-jet of Ω) as a kind of projection to the kernel of $\sigma(D_\Omega Q)$. One therefore expects the symbol of the modified operator

$$\tilde{Q}(\Omega) = Q(\Omega) + \Lambda(\Omega) \tag{29}$$

to have trivial kernel for a suitably chosen vector field. For a fixed $\bar{\Omega} \in \Omega^3_+(M)$ we take

$$X_{\bar{\Omega}} : \Omega^3(M) \rightarrow \Omega^1(M), \quad X_{\bar{\Omega}}(\dot{\Omega}) := -(\delta_{\bar{\Omega}} \dot{\Omega}) \lrcorner \bar{\Omega}, \tag{30}$$

where we contract and dualise with respect to the metric $g_{\bar{\Omega}}$. We think of $X_{\bar{\Omega}}$ as a first order, linear differential operator. Subsequently we write $\Lambda_{\bar{\Omega}}$ and $Q_{\bar{\Omega}}$ in (29) to emphasise the choice of $\bar{\Omega}$.

To give some motivation concerning the definition of $X_{\bar{\Omega}}$ we introduce the operator

$$\lambda^*_{\bar{\Omega}} : C^\infty(TM) \rightarrow \Omega^3(M), \quad X \mapsto \mathcal{L}_X \bar{\Omega}. \tag{31}$$

We consider $\lambda_{\bar{\Omega}}^*$ to be the formal L^2 -adjoint with respect to $\langle \cdot, \cdot \rangle_{L^2_{\bar{\Omega}}}$ of $\lambda_{\bar{\Omega}}$, i.e.

$$\begin{aligned} \langle \lambda_{\bar{\Omega}}^*(X), \dot{\Omega} \rangle_{L^2_{\bar{\Omega}}} &= \langle X \lrcorner d\bar{\Omega} + d(X \lrcorner \bar{\Omega}), \dot{\Omega} \rangle_{L^2_{\bar{\Omega}}} \\ &= -\langle d\bar{\Omega}, \dot{\Omega} \wedge X \rangle_{L^2_{\bar{\Omega}}} + \langle \bar{\Omega}, \delta_{\bar{\Omega}} \dot{\Omega} \wedge X \rangle_{L^2_{\bar{\Omega}}} \\ &= \langle -\dot{\Omega} \lrcorner d\bar{\Omega} + \delta_{\bar{\Omega}} \dot{\Omega} \lrcorner \bar{\Omega}, X \rangle_{L^2_{\bar{\Omega}}}, \end{aligned}$$

whence

$$\lambda_{\bar{\Omega}}(\dot{\Omega}) = -X_{\bar{\Omega}}(\dot{\Omega}) - \dot{\Omega} \lrcorner d\bar{\Omega} = -X_{\bar{\Omega}}(\dot{\Omega}) + \text{terms of lower order in } \dot{\Omega}. \tag{32}$$

In analogy with the decomposition of symmetric 2-tensors into a divergence free part and a part tangential to the $Diff(M)_+$ -orbit of some given Riemannian metric we have the following.

Proposition 5.6. *For any $\dot{\Omega} \in \Omega^3(M)$ there exist $X \in C^\infty(TM)$ and $\dot{\Omega}_0 \in \Omega^3(M)$ with $\lambda_{\bar{\Omega}}(\dot{\Omega}_0) = 0$ such that we have an $L^2_{\bar{\Omega}}$ -orthogonal decomposition*

$$\dot{\Omega} = \dot{\Omega}_0 \oplus \mathcal{L}_X \bar{\Omega}. \tag{33}$$

Proof. Put $L = \lambda_{\bar{\Omega}} \lambda_{\bar{\Omega}}^* : C^\infty(TM) \rightarrow C^\infty(TM)$. If we can solve $\lambda_{\bar{\Omega}}(\dot{\Omega}) = L(X)$ for some $X \in C^\infty(TM)$, then taking $\dot{\Omega}_0 = \dot{\Omega} - \mathcal{L}_X \bar{\Omega}$ yields the desired splitting. Since L is symmetric, such an X exists if and only if $\lambda_{\bar{\Omega}}(\dot{\Omega}) \in (\ker L)^\perp$. But $L(Y) = 0$ implies $\lambda_{\bar{\Omega}}^*(Y) = 0$, whence $\langle \lambda_{\bar{\Omega}}(\dot{\Omega}), Y \rangle_{L^2_{\bar{\Omega}}} = 0$ for all $Y \in \ker L$. \square

Remark. (i) The condition $\lambda_{\bar{\Omega}}(\dot{\Omega}) = 0$ should be viewed as a gauge-fixing condition, i.e. a choice of a local slice to the $Diff(M)_+$ -action near $\bar{\Omega}$. If $\bar{\Omega}$ is torsion-free, then $\lambda_{\bar{\Omega}}(\dot{\Omega}) = -X_{\bar{\Omega}}(\dot{\Omega}) = (\delta_{\bar{\Omega}} \dot{\Omega}) \lrcorner \bar{\Omega}$, hence $\lambda_{\bar{\Omega}}(\dot{\Omega}) = 0$ if and only if $[\delta_{\bar{\Omega}} \dot{\Omega}]_7 = 0$. This is precisely the gauge-fixing condition considered by Joyce in [16]; see also the remark following [Corollary 5.9](#).

(ii) The vector field X in the decomposition (33) is unique if there are no non-trivial infinitesimal automorphisms of $\bar{\Omega}$, i.e. vector fields X such that $\mathcal{L}_X \bar{\Omega} = 0$. This holds for holonomy G_2 -manifolds as Ricci-flatness implies any Killing field X to be parallel, so that the holonomy is contained in $SU(3)$ unless $X = 0$. Note further that a generic G_2 -form has no infinitesimal automorphisms as these are automatically Killing for $g_{\bar{\Omega}}$ and a generic metric has no Killing vector fields [9]. An example for a non-generic 3-form is provided by the direct product $M = S^1 \times CY^6$ of S^1 with coordinate vector field $X = \partial_t$ and an almost Calabi–Yau manifold (CY^6, ω, ψ_+) (i.e. ω is a non-degenerate 2-form and ψ_+ is a 3-form of special algebraic type). By (8), $\bar{\Omega} = dt \wedge \omega + \psi_+$ is a G_2 -form and $\mathcal{L}_X \bar{\Omega} = 0$.

Coming back to the mainstream development, we establish strong parabolicity for the flow equation

$$\frac{\partial}{\partial t} \tilde{\Omega} = \tilde{Q}_{\bar{\Omega}}(\tilde{\Omega}), \quad \tilde{\Omega}(0) = \Omega_0. \tag{34}$$

Lemma 5.7. *Eq. (34) is strongly parabolic for Ω_0 sufficiently C^2 -close to $\bar{\Omega}$.*

Proof. Since $X_{\bar{\Omega}}$ is linear in Ω , we find for the linearisation

$$\dot{\lambda}_{\bar{\Omega}} = d(X_{\bar{\Omega}}(\dot{\Omega}) \lrcorner \bar{\Omega}) + \text{lower order terms in } \dot{\Omega}$$

by virtue of Cartan’s formula, whence

$$\begin{aligned} \sigma(D_{\bar{\Omega}}\Lambda)(x, \xi)\dot{\Omega} &= i\xi \wedge (\sigma(X_{\bar{\Omega}})(x, \xi)\dot{\Omega}_\perp\bar{\Omega}) \\ &= -\xi \wedge ((\xi_\perp\dot{\Omega})_\perp\bar{\Omega}_\perp\bar{\Omega}). \end{aligned}$$

Assume without loss of generality that $|\xi|_{\bar{\Omega}} = 1$. Decomposing $\dot{\Omega} = \beta \wedge \xi + \gamma$ as above we deduce from (5)

$$\sigma(D_{\bar{\Omega}}\Lambda)(x, \xi)\dot{\Omega} = -\xi \wedge ((\beta_\perp\bar{\Omega})_\perp\bar{\Omega}) = -3\xi \wedge [\beta]_7.$$

Bearing (4) in mind, the projection of β onto Λ_7^2 is given by

$$\begin{aligned} [\beta]_7 &= \frac{1}{3}(\beta + \star_{\bar{\Omega}}(\beta \wedge \bar{\Omega})) \\ &= \beta_1 \oplus \frac{1}{3}(\beta_6 + \star_{\bar{\Omega}}(\beta_6 \wedge \bar{\Omega})) \\ &\stackrel{(12), (14)}{=} \beta_1 \oplus \frac{2}{3}((X_0 + Y_0)_\perp\psi_- + (X_0 + Y_0) \wedge \xi). \end{aligned}$$

Consequently $\xi \wedge [\beta]_7 = \xi \wedge (\beta_1 + \frac{2}{3}(X_0 + Y_0)_\perp\psi_-)$ so that using

$$|\xi \wedge \gamma|_{\bar{\Omega}}^2 = |\gamma|_{\bar{\Omega}}^2 \quad \text{and} \quad g_{\bar{\Omega}}(\xi \wedge [\beta]_7, \xi \wedge \beta + \gamma) = |[\beta]_7|_{\bar{\Omega}}^2$$

the computation from Lemma 5.5 implies

$$g_{\bar{\Omega}}(\sigma(D_{\bar{\Omega}}\tilde{Q})(x, \xi)\dot{\Omega}, \dot{\Omega}) = -|\gamma|_{\bar{\Omega}}^2 - |\xi_\perp p_{\bar{\Omega}}(\dot{\Omega})|_{\bar{\Omega}}^2 - 3|\beta_1|_{\bar{\Omega}}^2 - 2|(X_0 + Y_0)_\perp\psi_-|_{\bar{\Omega}}^2. \tag{35}$$

Now $\xi_\perp p_{\bar{\Omega}}(\dot{\Omega}) = \sigma \oplus -\beta_8$ with $g_{\bar{\Omega}}(\sigma, \beta_8) = 0$ in view of the decomposition in (11), while by (10), $|\beta|_{\bar{\Omega}}^2 = |\beta_1|_{\bar{\Omega}}^2 + |\beta_6|_{\bar{\Omega}}^2 + |\beta_8|_{\bar{\Omega}}^2$. But (12) gives $|\beta_6|_{\bar{\Omega}}^2 = |(X_0 + Y_0)_\perp\psi_-|_{\bar{\Omega}}^2$, whence

$$-g_{\bar{\Omega}}(\sigma(D_{\bar{\Omega}}\tilde{Q})(x, \xi)\dot{\Omega}, \dot{\Omega}) \geq (|\beta|_{\bar{\Omega}}^2 + |\gamma|_{\bar{\Omega}}^2) = |\dot{\Omega}|_{\bar{\Omega}}^2$$

by (35). \square

Definition 5.8. We call the flow associated with (34) the Dirichlet–DeTurck flow at $\bar{\Omega}$ with initial condition Ω_0 . If the G_2 -form $\bar{\Omega}$ we use to perturb the Dirichlet energy flow is understood we simply speak of the Dirichlet–DeTurck flow with initial condition Ω_0 .

For $\Omega_0 \in \Omega_+^3(M)$ consider the Dirichlet–DeTurck flow at Ω_0 with initial condition Ω_0 . By Theorem 5.3, the flow $\tilde{\Omega}(t)$ exists on some time interval $[0, \epsilon]$. Let φ_t be the family of diffeomorphisms determined by

$$\partial_t \varphi_t = -X_{\Omega_0}(\tilde{\Omega}(t)) \circ \varphi_t, \quad \varphi_0 = \text{Id}_M. \tag{36}$$

Then $\Omega(t) = \varphi_t^* \tilde{\Omega}(t)$ is a solution to the Dirichlet energy flow (22) with same initial condition Ω_0 for

$$\begin{aligned} \frac{\partial}{\partial t} \Omega &= \varphi_t^* \left(\frac{\partial}{\partial t} \tilde{\Omega} + \mathcal{L}_{-X_{\Omega_0}(\tilde{\Omega})} \tilde{\Omega} \right) \\ &\stackrel{(29)}{=} \varphi_t^* Q(\tilde{\Omega}) \\ &\stackrel{(25)}{=} Q(\Omega). \end{aligned}$$

Moreover, the initial condition is satisfied as $\Omega(0) = \text{Id}_M^* \Omega_0 = \Omega_0$.

Corollary 5.9 (Existence). *For any $\Omega_0 \in \Omega_+^3(M)$ there exists an $\epsilon > 0$ such that the Dirichlet energy flow (22) exists for $t \in [0, \epsilon]$.*

Remark. (i) The idea of DeTurck’s trick is to break the diffeomorphism invariance by modifying the flow along the $Diff(M)_+$ -orbits via the additional term Λ_{Ω_0} . To see this happening in a geometrical way, assume for simplicity that Ω_0 is a torsion-free G_2 -form. By (5), $\dot{\Lambda}_{\Omega_0} = d(X_{\Omega_0}(\dot{\Omega})_{\perp} \Omega_0) = -3d[\delta_{\Omega_0} \dot{\Omega}]_7$. In particular, $\dot{\Lambda}_{\Omega_0} = 0$ if and only if $[\delta_{\Omega_0} \dot{\Omega}]_7 = 0$, for

$$\langle d[\delta_{\Omega_0} \dot{\Omega}]_7, \dot{\Omega} \rangle_{L^2_{\Omega_0}} = \|[\delta_{\Omega_0} \dot{\Omega}]_7\|_{L^2_{\Omega_0}}^2.$$

On the other hand, the tangent space at Ω_0 of the $Diff(M)_+$ -orbit \mathcal{O}_{Ω_0} is given by $\{\mathcal{L}_X \Omega_0 \mid X \in C^\infty(TM)\}$ (cf. also Lemma 7.3). Since

$$\langle \dot{\Omega} - \Omega_0, \mathcal{L}_X \Omega_0 \rangle_{L^2_{\Omega_0}} = \langle [\delta_{\Omega_0} \dot{\Omega}]_7, X_{\perp} \Omega_0 \rangle_{L^2_{\Omega_0}},$$

the form $\dot{\Omega} - \Omega_0$ is perpendicular to $T_{\Omega_0} \mathcal{O}_{\Omega_0}$ if and only if $\dot{\Lambda}_{\Omega_0} = 0$.

(ii) To become strongly parabolic after perturbing with Λ_{Ω_0} is a particular feature of the Dirichlet energy flow. In contrast consider the gradient flow of the Hitchin functional \mathcal{H} restricted to the cohomology class $[\Omega_0]$; cf. Eq. (3). Upon rescaling \mathcal{H} , the resulting flow is

$$\frac{\partial}{\partial t} \alpha = \delta_{\Omega_0 + d\alpha}(\Omega_0 + d\alpha), \quad \alpha(0) = 0$$

for α in a suitably small open neighbourhood of $0 \in \Omega^2(M)$ so that $\Omega_0 + d\alpha \in \Omega_+^3(M)$. The solutions are in 1–1 correspondence with solutions to the Laplacian flow

$$\frac{\partial}{\partial t} \Omega = F(\Omega) = \Delta_{\Omega} \Omega, \quad \Omega(0) = \Omega_0.$$

Now $\sigma(D_{\Omega} F)(x, \xi) \dot{\Omega} = |\xi|_{\Omega}^2 \dot{\Omega} + \xi \wedge (\xi_{\perp}([\dot{\Omega}]_1/3 - 2[\dot{\Omega}]_{27}))$ and one can easily compute that the kernel is also given by

$$K = \ker \sigma(D_{\Omega} F)(x, \xi) = \{(\dot{v}\omega + \dot{V}_{\perp} \psi_{-}) \wedge \xi \mid \dot{v} \in \mathbb{R}, \dot{V} \in \xi^{\perp}\}.$$

Since by (27), the symbol of $X \mapsto \mathcal{L}_X \Omega$ takes values in K , DeTurck’s trick cannot modify the component $\text{pr}_{K^{\perp}} \circ \sigma(D_{\Omega} F)|_{K^{\perp}} : K^{\perp} \rightarrow K^{\perp}$. However, the eigenvectors $\dot{\Omega}_1 = \dot{\beta}_8 \wedge \xi$ and $\dot{\Omega}_2 = \psi_{-}$ in K^{\perp} give $g_{\Omega}(\dot{\Omega}_1, \sigma(D_{\Omega} F)(x, \xi) \dot{\Omega}_1) = -|\xi|_{\Omega}^2 |\dot{\beta}_8|_{\Omega}^2 < 0$ and $g_{\Omega}(\dot{\Omega}_2, \sigma(D_{\Omega} F)(x, \xi) \dot{\Omega}_2) = 4|\xi|_{\Omega}^2 > 0$ respectively. Hence, the linearisation of $\tilde{F} = F + \Lambda$ will be indefinite no matter how the vector field X in (28) is chosen (though the linearisation of \tilde{F} might have trivial kernel). We therefore deal with a heat equation of mixed forwards/backwards type for which short-time existence is in general not expected unless further conditions are imposed. For the Laplacian flow this has recently been achieved in [6] and [26] for closed initial conditions.

6. Uniqueness

We now settle the uniqueness part of Theorem 5.1 along the lines of the uniqueness proof for Ricci flow.

As shown by Corollary 5.9, a solution to the Dirichlet–DeTurck flow $\tilde{\Omega}(t)$ with initial condition Ω_0 yields a solution to the Dirichlet energy flow $\Omega(t) = \varphi_t^* \tilde{\Omega}(t)$ with same initial

condition by integrating the time-dependent vector field in (36). Conversely, substituting $\tilde{\Omega}(t)$ by $\varphi_t^{-1*}\Omega(t)$ turns the ordinary differential equation (36) into the partial differential equation

$$\frac{\partial}{\partial t}\varphi_t = -X_{\Omega_0}(\varphi_t^{-1*}\Omega(t)) \circ \varphi_t, \quad \varphi_0 = \text{Id}_M. \tag{37}$$

A curve $\varphi_t \in \text{Diff}(M)_+$ which solves (37) for a Dirichlet energy flow solution $\Omega(t)$ with initial condition Ω_0 yields the Dirichlet–DeTurck flow solution $\tilde{\Omega}(t) = \varphi_t^{-1*}\Omega(t)$ with same initial condition. Indeed, let Y_t be the time-dependent vector field defined by $Y_t \circ \varphi_t^{-1} = \partial_t \varphi_t^{-1}$. Then differentiating the constant curve $\varphi_t^{-1} \circ \varphi_t(x) = x$ gives

$$Y_t(x) = -d_{\varphi_t(x)}\varphi_t^{-1}(-X_{\Omega_0}(\varphi_t^{-1*}\Omega(t)) \circ \varphi_t(x)) = \varphi_{t*}^{-1}X_{\Omega_0}(\varphi_t^{-1*}\Omega(t))(x), \tag{38}$$

where for $\varphi \in \text{Diff}(M)$ and $X \in C^\infty(TM)$,

$$(\varphi_*X)(x) := d_{\varphi^{-1}(x)}\varphi(X(\varphi^{-1}(x))).$$

As a consequence, we get

$$\begin{aligned} \frac{\partial}{\partial t}\varphi_t^{-1*}\Omega(t) &= \varphi_t^{-1*}\left(\frac{\partial}{\partial t}\Omega(t) + \mathcal{L}_{Y_t}\Omega(t)\right) \\ &= Q(\varphi_t^{-1*}\Omega(t)) + \mathcal{L}_{\varphi_{t*}Y_t}\varphi_t^{-1*}\Omega(t) \\ &= Q(\varphi_t^{-1*}\Omega(t)) + \mathcal{L}_{X_{\Omega_0}(\varphi_t^{-1*}\Omega(t))}\varphi_t^{-1*}\Omega(t) \end{aligned}$$

by (38). We can then deduce uniqueness of the Dirichlet energy flow from uniqueness of the Dirichlet–DeTurck flow; see Corollary 6.3.

Remark. Eq. (37) should be considered as an analogue of the harmonic map heat flow

$$\frac{\partial}{\partial t}\varphi_t = \tau_{g(t),g_0}(\varphi_t)$$

introduced by Eells and Sampson [10], albeit with a time-dependent tension field $\tau_{g(t),g_0}(\varphi_t)$. We can think of $\tau_{g(t),g_0}(\varphi_t)$ as a differential operator defined by Riemannian metrics $g(t)$ and g_0 on M , taking a smooth map $\varphi : (M, g(t)) \rightarrow (M, g_0)$ to a section $\tau_{g(t),g_0}(\varphi) \in C^\infty(\varphi^*TM)$.

We need to prove short-time existence of a solution to (37). By and large, we proceed as in the harmonic map heat flow case; cf. [10]. Let

$$\begin{aligned} P_t &= P_{\Omega(t),\Omega_0} : \varphi \in \text{Diff}(M)_+ \subset C^\infty(M, M) \\ &\mapsto -\varphi_*X_{\varphi_*\Omega_0}(\Omega(t)) \circ \varphi \in C^\infty(\varphi^*TM). \end{aligned} \tag{39}$$

Since $\text{Diff}(M)_+$ is open in $C^\infty(M, M)$, a solution to the flow equation

$$\frac{\partial}{\partial t}\varphi_t = P_t(\varphi_t), \quad \varphi_0 = \text{Id}_M$$

yields the desired solution to (37).

To get formally in a situation to apply Theorem 5.3, we first choose an embedding $f_0 : M \rightarrow \mathbb{R}^n$ and identify M with its image under f_0 . In particular, all tensors on M pushed forward to $f_0(M)$ will be denoted by the same symbol. Let $\mathcal{N} \subset \mathbb{R}^n$ be a tubular neighbourhood of M which we think of as an open neighbourhood inside the normal bundle $\pi : \nu M \rightarrow M$, the normal bundle taken with respect to the Euclidean metric on \mathbb{R}^n . By choosing a fibre metric h

and a compatible connection $\nabla^{\nu M}$ on νM we obtain the induced metric $\pi^*g_{\Omega_0} + h$ on \mathcal{N} which we extend to \mathbb{R}^n using a partition of unity. In particular, this makes f_0 an isometry. Similarly, we extend Ω_0 by $\pi^*\Omega_0$ to \mathcal{N} and subsequently to \mathbb{R}^n . In this way, the restriction $f^*\Omega_0$ for f in a suitably small open neighbourhood $\mathcal{U} \subset C^\infty(M, \mathbb{R}^n)$ of embeddings close to f_0 is still a positive 3-form on M . Consequently, we can extend P_t to an operator

$$P_t : \mathcal{U} \subset C^\infty(M, \mathbb{R}^n) \rightarrow C^\infty(M, \mathbb{R}^n), \quad f \mapsto -df(X_{f^*\Omega_0}(\Omega(t))).$$

Lemma 6.1. *The operator P_t is a quasilinear, second order differential operator.*

Proof. Let e_1, \dots, e_n be the standard basis of \mathbb{R}^n and x^1, \dots, x^7 be local coordinates on $U \subset M$. The components \star_{opqr}^{ijk} of $\star_{f^*\Omega_0} : \Omega^3(U) \mapsto \Omega^4(U)$ depend on the components of $f^*\Omega_0$ given by $\Omega_{0,\alpha\beta\gamma} \partial_{x_i} f^\alpha \partial_{x_m} f^\beta \partial_{x_n} f^\gamma$. Schematically,

$$\star_{f^*\Omega_0} \Omega(t) \stackrel{\text{loc}}{=} \star_{opqr}^{ijk}(x, \partial_{x_i} f^\alpha) \Omega(t)_{ijk} dx^{opqr}$$

so that by the chain rule

$$d \star_{f^*\Omega_0} \Omega(t) \stackrel{\text{loc}}{=} (a_{opqrs,\beta}^{ij} (t, x, \partial_{x_i} f^\alpha) \partial_{x_i} \partial_{x_j} f^\beta + b_{opqrs}(t, x, \partial_{x_i} f^\alpha)) dx^{opqrs}$$

for smooth coefficients a_{\dots} and b_{\dots} . Applying once more $\star_{f^*\Omega_0}$ and contracting the result with $f^*\Omega_0$ lead to

$$(\star_{f^*\Omega_0} d \star_{f^*\Omega_0} \Omega(t)) \lrcorner f^*\Omega_0 \stackrel{\text{loc}}{=} (\tilde{a}_{k,\beta}^{ij} (t, x, \partial_{x_i} f^\alpha) \partial_i \partial_j f^\beta + \tilde{b}_{kl}(t, x, \partial_{x_i} f^\alpha)) dx^k.$$

Finally, dualising and contracting with $df = \partial_{x_i} f^\gamma dx^i \otimes e_\gamma$ show that P_t is a quasilinear, second order differential operator. \square

Lemma 6.2. *There exist $\epsilon > 0$ and a smooth family of embeddings $f(t) \in C^\infty(M, \mathbb{R}^n)$, $t \in [0, \epsilon]$ such that*

$$\frac{\partial}{\partial t} f(t) = P_t(f(t)), \quad f(0) = f_0. \tag{40}$$

Furthermore, $f(t)(M) \subset f_0(M)$ for all t .

Proof. As above let $\mathcal{U} \subset C^\infty(M, \mathbb{R}^n)$ be an open neighbourhood of f_0 . By transversality of $f_0(M)$ to the fibres of the normal bundle we may assume that f is an embedding for $f \in \mathcal{U}$ with $\varphi_f := \pi \circ f \in \text{Diff}(M)_+$, shrinking \mathcal{U} if necessary. We put $\sigma_f(x) := f(x) - \varphi_f(x) \in \nu_{\varphi_f(x)} M$ so that $f(x) = \varphi_f(x) + \sigma_f(x)$. In the following we view σ_f as a section of the pull-back bundle $\varphi_f^* \nu M$. Observe that $\sigma_f = 0$ if and only if $f(M) \subset M$.

The operator P_t considered as acting on \mathbb{R}^n -valued functions on M is not elliptic as the computation of its symbol shows below. In order to complement it to an elliptic operator we proceed as follows: for $\varphi \in \text{Diff}(M)_+$ consider the connection Laplacian $\Delta^{\varphi^* \nu M} = (\nabla^{\varphi^* \nu M})^* \nabla^{\varphi^* \nu M}$ on the pull-back bundle $\varphi^* \nu M$. Here $\nabla^{\varphi^* \nu M}$ denotes the pull-back connection on $\varphi^* \nu M$. Its formal adjoint is taken with respect to the metric g_{Ω_0} on M and the pull-back metric $\varphi^* h$ on $\varphi^* \nu M$. Then, reasoning similarly as above, $f \mapsto \Delta^{\varphi_f^* \nu M} \sigma_f : \mathcal{U} \subset C^\infty(M, \mathbb{R}^n) \rightarrow C^\infty(M, \varphi_f^* \nu M) \subset C^\infty(M, \mathbb{R}^n)$ is a quasilinear, second order differential operator, and so is

$$\tilde{P}_t : \mathcal{U} \subset C^\infty(M, \mathbb{R}^n) \rightarrow C^\infty(M, \mathbb{R}^n), \quad f \mapsto P_t(f) - \Delta^{\varphi_f^* \nu M} \sigma_f.$$

We wish to establish short-time existence and uniqueness of the associated flow equation

$$\frac{\partial}{\partial t} f(t) = \tilde{P}_t(f(t)), \quad f(0) = f_0. \tag{41}$$

To compute the linearisation $D_{f_0} \tilde{P}_0(Y)$ we take a curve $f_s \subset \mathcal{U}$ through f_0 with

$$Y(x) = \frac{d}{ds} f_s(x)|_{s=0} \in T_x \mathbb{R}^n \cong \mathbb{R}^n.$$

We write $Y^\parallel(x)$ and $Y^\perp(x)$ for the projections of $Y(x)$ to $T_x M$ and $\nu_x M$. By design of the extension of Ω_0 to \mathbb{R}^n (cf. our convention above), $\varphi_s := \pi \circ f_s \in \text{Diff}(M)_+$ satisfies $f_s^* \Omega_0 = \varphi_s^* \Omega_0$. Furthermore, one clearly has

$$\frac{d}{ds} \varphi_s(x)|_{s=0} = Y^\parallel(x) \quad \text{and} \quad \frac{d}{ds} \sigma_s(x)|_{s=0} = Y^\perp(x) \tag{42}$$

for $\sigma_s(x) := f_s(x) - \varphi_s(x) \in \nu_{\varphi_s(x)} M$.

First we compute the linearisation of P_0 . Using the naturality of the vector field X_{Ω_0} , i.e. $\varphi_* X_{\varphi^* \Omega_0}(\Omega(t)) = X_{\Omega_0}(\varphi^{-1*} \Omega(t))$ for $\varphi \in \text{Diff}(M)_+$, we obtain

$$\begin{aligned} \frac{d}{ds} P_0(f_s)|_{s=0} &= -\frac{d}{ds} df_s(X_{\varphi_s^* \Omega_0}(\Omega_0))|_{s=0} \\ &= -\frac{d}{ds} df_s(\varphi_s^{-1*} X_{\Omega_0}(\varphi_s^{-1*} \Omega_0))|_{s=0}. \end{aligned}$$

For Y tangent to M , i.e. $Y(x) = Y^\parallel(x)$ for all $x \in M$, we may in view of Eq. (42) assume that $f_s = \varphi_s \in \text{Diff}(M)_+$ for all s and we get

$$\begin{aligned} \frac{d}{ds} P_0(f_s)|_{s=0} &= -\frac{d}{ds} X_{\Omega_0}(\varphi_s^{-1*} \Omega_0)|_{s=0} \\ &= X_{\Omega_0}(\mathcal{L}_Y \Omega_0) \\ &= -(\delta_{\Omega_0} d(Y \lrcorner \Omega_0)) \lrcorner \Omega_0 + \text{terms of lower order in } Y. \end{aligned}$$

For Y perpendicular to M , i.e. $Y(x) = Y^\perp(x)$ for all $x \in M$, we may assume that $\varphi_s = \text{Id}$ for all s , again using Eq. (42). Hence

$$\frac{d}{ds} P_0(f_s)|_{s=0} = -\frac{d}{ds} df_s X_{\Omega_0}(\Omega_0) = dY(X_{\Omega_0}(\Omega_0)),$$

which is of lower order in Y and hence does not contribute to the symbol. For general $Y = Y^\parallel + Y^\perp$ we therefore find

$$D_{f_0} P_0(Y) = -(\delta_{\Omega_0} d(Y \lrcorner \Omega_0)) \lrcorner \Omega_0 + \text{terms of lower order in } Y.$$

For the linearisation of $f \mapsto \Delta^{\varphi_f^* \nu M} \sigma_f$ we again assume Y to be tangent to M first. For a curve $f_s = \varphi_s \in \text{Diff}(M)_+$ as above we have in particular that $\sigma_s = 0$ for all s . Hence $\frac{d}{ds} \Delta^{\varphi_s^* \nu M} \sigma_s|_{s=0} = 0$. For Y perpendicular to M we have $\sigma_s(x) = f_s(x) - x$ for a curve f_s with $\varphi_s \equiv \text{Id}$ as above. Hence $\frac{d}{ds} \Delta^{\varphi_s^* \nu M} \sigma_s|_{s=0} = \Delta^{\nu M} Y$. For general $Y = Y^\parallel + Y^\perp$ we get

$$D_{f_0}(f \mapsto \Delta^{\varphi_f^* \nu M} \sigma_f)(Y) = \Delta^{\nu M} Y^\perp.$$

It follows that the symbol of the linearised operator $D_{f_0} \tilde{P}_0$ is

$$\sigma(D_{f_0} \tilde{P}_0)(x, \xi) Y = -(\xi \lrcorner (\xi \wedge (Y \lrcorner \Omega_0))) \lrcorner \Omega_0 - |\xi|_{\Omega_0}^2 Y^\perp.$$

To check strong parabolicity we assume $|\xi|_{\Omega_0} = 1$ and write $Y^\parallel = a\xi + Y_0$, $a \in \mathbb{R}$, $Y_0 \in \xi^\perp$ and $\Omega_0 = \omega \wedge \xi + \psi_+$. Then

$$\begin{aligned} & -(\pi^*g_{\Omega_0} + h)(\sigma(D_{f_0}\tilde{P}_0)(x, \xi)Y, Y) = g_{\Omega_0}((\xi_\perp(\xi \wedge (Y^\parallel \lrcorner \Omega_0)))_\perp \Omega_0, Y^\parallel) + |Y^\perp|_h^2 \\ & = g_{\Omega_0}(\Omega_0, (\xi_\perp(\xi \wedge (Y^\parallel \lrcorner \Omega_0))) \wedge Y^\parallel) + |Y^\perp|_h^2 \\ & = g_{\Omega_0}(Y^\parallel \lrcorner \Omega_0, \xi_\perp(\xi \wedge (Y^\parallel \lrcorner \Omega_0))) + |Y^\perp|_h^2 \\ & = g_{\Omega_0}(a\omega + Y_{0\perp}\psi_+ + (Y_{0\perp}\omega) \wedge \xi, a\omega + Y_{0\perp}\psi_+) + |Y^\perp|_h^2 \\ & = 3|a|_{\Omega_0}^2 + |Y_{0\perp}\psi_+|_{\Omega_0}^2 + |Y^\perp|_h^2 \\ & \stackrel{(15)}{=} 3|a|^2 + 2|Y_0|^2 + |Y^\perp|_h^2 \\ & \geq (\pi^*g_{\Omega_0} + h)(Y, Y). \end{aligned}$$

Theorem 5.3 applies once again to yield short-time existence and uniqueness of (41). Last we show that a solution $f(t)$ to (41) satisfies $f(t)(M) \subset M$ for all t . As in this case $\sigma_{f(t)}$ is just the zero section of $\varphi(t)^*\nu M$, we obtain the desired solution to (40). We proceed as in the harmonic map heat flow case, see for instance Part IV in [13]: consider the bundle endomorphism $r : \nu M \rightarrow \nu M$ which is multiplication by -1 in each fibre. We claim that $\tilde{P}_t \circ r = dr \circ \tilde{P}_t$. This is clear for P_t since $(r \circ f)^*\Omega_0 = f^*\Omega_0$. Furthermore, $\varphi_{r \circ f} = \varphi_f$ and $\sigma_{r \circ f} = r \circ \sigma_f$, such that by linearity of the connection Laplacian one has

$$\Delta^{\varphi_{r \circ f}^* \nu M} \sigma_{r \circ f} = r \circ \Delta^{\varphi_f^* \nu M} \sigma_f = dr \circ \Delta^{\varphi_f^* \nu M} \sigma_f,$$

where the last equality follows by viewing a section of νM as a vertical vector field on the total space of that bundle. This proves our claim and implies that for a solution $f(t)$ to (41) the composition $r \circ f(t)$ is again a solution with initial condition f_0 . Now if $f(t)(M)$ were not contained in M for some t , then $r \circ f$ would yield a second, different solution with same initial condition, contradicting uniqueness. \square

A solution $f(t) \in \mathcal{U}$ to (39) yields a solution $\varphi_t = f_0^{-1} \circ f(t) \in Diff(M)_+$ to (37) for a given Dirichlet energy flow solution $\Omega(t)$. From there, uniqueness easily follows:

Corollary 6.3. *Suppose that $\Omega(t)$ and $\Omega'(t)$ are two solutions to (22) for $t \in [0, \epsilon]$, $\epsilon > 0$. If $\Omega(0) = \Omega_0 = \Omega'(0)$, then $\Omega(t) = \Omega'(t)$ for all $t \in [0, \epsilon]$.*

Proof. Solving for (37) with $\Omega(t)$ and $\Omega'(t)$ gives two flows φ_t and φ'_t which without loss of generality we assume to be defined on $[0, \epsilon]$. By design $\tilde{\Omega}(t) = \varphi_t^* \Omega(t)$ and $\tilde{\Omega}'(t) = \varphi'_t{}^* \Omega'(t)$ define a solution to (34) at Ω_0 . Uniqueness of the Dirichlet–DeTurck flow implies $\tilde{\Omega}(t) = \tilde{\Omega}'(t)$. Hence φ_t and φ'_t are solutions of the ordinary differential equation

$$\frac{\partial}{\partial t} \psi_t = -X_{\Omega_0}(\tilde{\Omega}(t)) \circ \psi_t. \tag{43}$$

By uniqueness of the solution to (43), we conclude $\varphi_t = \varphi'_t$, whence $\Omega(t) = \Omega'(t)$. \square

7. The second variation of \mathcal{D}

In this section we compute the second variation of \mathcal{D} at some fixed $\tilde{\Omega} \in \mathcal{X} = \mathcal{Q}^{-1}(0)$ (cf. Corollary 4.3). Further, we show that \mathcal{X} is a Fréchet manifold whose tangent space at $\tilde{\Omega}$ is precisely $\ker D_{\tilde{\Omega}}^2 \mathcal{D}$. All metric quantities, projections on G_2 -invariant modules etc. will be taken with respect to this $\tilde{\Omega}$.

For a given vector bundle $E \rightarrow M$ we denote by $W^{k,2}(E)$ the space of sections whose local components have square integrable derivatives up to order k . The associated Sobolev norm will be written $\|\cdot\|_{W^{k,2}}$. Further, we simply write $L^2(E)$ for $W^{0,2}(E)$. More generally, we consider the Hilbert manifolds $W^{k,2}(\xi)$ for a fibre bundle $\xi \rightarrow M$ in order to deal with non-linear differential operators. The integer k will be chosen appropriately when required, but at any rate big enough so that all sections involved are at least of class C^0 and the corresponding function spaces $W^{k,2}(M, \mathbb{R})$ are Banach algebras under pointwise multiplication. According to the Sobolev embedding theorem we therefore need $k > 11/2$ for Q to extend to a smooth map

$$Q_k : W^{k,2}(\Lambda^3_+ T^*M) \rightarrow W^{k-2,2}(\Lambda^3 T^*M).$$

Lemma 7.1. *The space $\mathcal{X}^k := Q_k^{-1}(0)$ is a smooth Banach manifold whose tangent space at $\bar{\Omega} \in \mathcal{X}$ is given by*

$$T_{\bar{\Omega}}\mathcal{X}^k = \{\dot{\Omega} \in W^{k,2}(\Lambda^3 T^*M) \mid d\dot{\Omega} = 0, d\dot{\Theta}_{\bar{\Omega}} = 0\}.$$

Proof. We have $\mathcal{X}^k = \{\Omega \in W^{k,2}(\Lambda^3_+ T^*M) \mid d\Omega = 0, d\Theta(\Omega) = 0\}$, so it remains to show that the extension N_k of

$$N : \Omega \in \Omega^3_+(M) \mapsto (d\Omega, d\Theta(\Omega)) \in \Omega^4(M) \times \Omega^5(M)$$

to $W^{k,2}(\Lambda^3_+ T^*M)$ has 0 as a regular value. By the first example of Section 4, $(\dot{N}_k)_{\bar{\Omega}} = (d\dot{\Omega}, d\star p(\dot{\Omega}))$. Since the range of $(d_p)_k : W^{k,2}(\Lambda^p T^*M) \rightarrow W^{k-1,2}(\Lambda^{p+1} T^*M)$ is closed, $(B^{p+1})^{k-1} = \text{im}(d_p)_k$ is a Banach space. Hence for a given $(d\sigma, d\tau) \in (B^4)^{k-1} \times (B^5)^{k-1}$, we need $\dot{\Omega} \in \Omega^3(M)$ such that $d\dot{\Omega} = d\sigma$ and $d\star p(\dot{\Omega}) = d\tau$. The Hodge decomposition theorem of $[\dot{\Omega}]_q$ gives $\mathcal{H}(\dot{\Omega}_q) \oplus d\dot{\alpha}_q \oplus \delta\dot{\beta}_q$, where $\dot{\alpha}_q \in W^{k+1,2}(\Lambda^2 T^*M)$, $\dot{\beta}_q \in W^{k+1,2}(\Lambda^4 T^*M)$ for $q = 1, 7, 27$, and \mathcal{H} denotes projection on the space of harmonic forms. Similar decompositions hold for σ and τ . Taking $\dot{\beta}_q = [\beta_\sigma]_q$ yields

$$d\dot{\Omega} = \bigoplus_{q \in \{1, 7, 27\}} d\delta\dot{\beta}_q = d\delta\beta_\sigma = d\sigma.$$

On the other hand,

$$d\star p(\dot{\Omega}) = d\star \left(\frac{4}{3}d\dot{\alpha}_1 + d\dot{\alpha}_7 - d\dot{\alpha}_{27} \right) = d\delta\beta_\tau = d\tau,$$

provided we put $\dot{\alpha}_1 = 3\star[\beta_\tau]_1/4$, $\dot{\alpha}_7 = \star[\beta_\tau]_7$ and $\dot{\alpha}_{27} = \star[\beta_\tau]_{27}$. Consequently, $(\dot{N}_k)_{\bar{\Omega}}$ is surjective, whence the result by the Banach space implicit function theorem. \square

Consider the closed linear subspace $V^k_{\bar{\Omega}} = \{\dot{\Omega} \in W^{k,2}(\Lambda^3 T^*M) \mid [\delta\dot{\Omega}]_7 = 0\} \subset \lambda^{-1}(0)$ (cf. (32)) and let

$$S_{\bar{\Omega}} := V^k_{\bar{\Omega}} \cap \mathcal{X}^k.$$

Note that $S_{\bar{\Omega}} \subset \tilde{Q}^{-1}_{\bar{\Omega}}(0)$, whence $S_{\bar{\Omega}} \subset \Omega^3(M)$ for $\tilde{Q}_{\bar{\Omega}}$ is a quasilinear, elliptic operator by Lemma 5.7.

Proposition 7.2. *Near $\bar{\Omega}$, the space $S_{\bar{\Omega}}$ is a smooth submanifold of \mathcal{X}^k . Its tangent space at $\bar{\Omega}$ is naturally isomorphic with the space of $\bar{\Omega}$ -harmonic 3-forms. In particular, $\dim S_{\bar{\Omega}} = b_3$.*

Remark. In Section 8 we actually show that near $\bar{\Omega}$, $\tilde{\mathcal{Q}}_{\bar{\Omega}}^{-1}(0)$ coincides with $S_{\bar{\Omega}}$ (cf. Corollary 8.11).

Before we can prove Proposition 7.2, we need a technical result first. Consider the Hilbert manifold $Diff(M)_0^{k+1}$ obtained from completion of the identity component of $Diff(M)$ with respect to $\|\cdot\|_{W^{k+1,2}}$. Then $Diff(M)_0^{k+1}$ is a smooth Banach manifold and a topological group which acts continuously on \mathcal{X}^k via pull-back. We denote by $I_{\bar{\Omega}}$ the subgroup of $Diff(M)_0^{k+1}$ fixing $\bar{\Omega}$, i.e. $\varphi^*\bar{\Omega} = \bar{\Omega}$ for all $\varphi \in I_{\bar{\Omega}}$; in particular $g = \varphi^*g$ which by [23] implies $\varphi \in Diff(M)_0$ since any $\varphi \in Diff(M)_0^{k+1}$ is of class C^1 . Hence $I_{\bar{\Omega}}$ is contained in $Diff(M)_0$. Let $\mathcal{O}_{\bar{\Omega}}^k$ denote the $Diff(M)_0^{k+1}$ -orbit through $\bar{\Omega}$ in \mathcal{X}^k and put

$$G_{\bar{\Omega}}^{k+1} := Diff(M)_0^{k+1}/I_{\bar{\Omega}}.$$

As in Section 5 of [9] one can prove that $G_{\bar{\Omega}}^{k+1}$ endowed with the quotient topology is a smooth manifold.

Lemma 7.3. *If $k > 11/2$, then $[\varphi] \in G_{\bar{\Omega}}^{k+1} \mapsto \varphi^*\bar{\Omega} \in \mathcal{X}^k$ is an injective immersion with closed image. In particular, $\mathcal{O}_{\bar{\Omega}}^k$ is a closed, smooth submanifold of \mathcal{X}^k with tangent space*

$$T_{\bar{\Omega}}\mathcal{O}_{\bar{\Omega}}^k = \{\mathcal{L}_X\bar{\Omega} \mid X \in W^{k+1,2}(TM)\}.$$

Proof. We can argue as in Ebin’s proof of the corresponding result for the moduli space of Riemannian metrics; cf. Section 6 in [9]. The only remaining issue to check is the injectivity of the symbol of $\lambda^* : C^\infty(TM) \rightarrow \Omega^3(M)$ defined in (31) (this ensures that the extension to a map $W^{k+1,2}(TM) \rightarrow W^{k,2}(\Lambda^3 T^*M)$ has closed range). Indeed,

$$\sigma(\lambda^*)(x, \xi)(v) = i\xi \wedge (v \lrcorner \bar{\Omega}), \tag{44}$$

and this vanishes if and only if $v \lrcorner \bar{\Omega}$ is of the form $\eta \wedge \xi$ for some $\eta \in \Omega^1(M)$. But $v \lrcorner \bar{\Omega} \in \Lambda_7^2$, so that in this case $0 = [\eta \wedge \xi]_{14} = (2\eta \wedge \xi - \star(\eta \wedge \xi \wedge \Omega))/3$. Since on the right hand side the first term contains ξ while the second does not, this can only hold if $\eta \wedge \xi = 0$, i.e. if $\eta = 0$ or equivalently, $v = 0$. \square

Proof (of Proposition 7.2). Since $\mathcal{O}_{\bar{\Omega}}^k \subset \mathcal{X}^k$, the tangent space $T_{\bar{\Omega}}\mathcal{O}_{\bar{\Omega}}^k$ is contained in $T_{\bar{\Omega}}\mathcal{X}^k$. By extending Proposition 5.6 to Sobolev spaces, we deduce $T_{\bar{\Omega}}\mathcal{X}^k + T_{\bar{\Omega}}V_{\bar{\Omega}}^k = W^{k,2}(\Lambda^3 T^*M)$. Hence, the intersection is transversal near $\bar{\Omega}$ so that $S_{\bar{\Omega}}$ is a smooth submanifold of $W^{k,2}(\Lambda^3 T^*M)$ in a neighbourhood of $\bar{\Omega}$. The tangent space at $\bar{\Omega}$ is

$$T_{\bar{\Omega}}S_{\bar{\Omega}} = T_{\bar{\Omega}}V_{\bar{\Omega}}^k \cap T_{\bar{\Omega}}\mathcal{X}^k = \{\dot{\Omega} \in \Omega^3(M) \mid d\dot{\Omega} = 0, d\dot{\Theta}_{\bar{\Omega}} = 0, [\delta\dot{\Omega}]_7 = 0\}.$$

Next, the map $\dot{\Omega} \in T_{\bar{\Omega}}S_{\bar{\Omega}} \mapsto [\dot{\Omega}] \in H^3(M, \mathbb{R})$ is an isomorphism. For injectivity, assume $\dot{\Omega} = d\eta$ so that $[\delta d\eta]_7 = 0$. By Lemma 10.3.2 in [16], this implies $\delta[d\eta]_1 = 0$ and $\delta[d\eta]_7 = 0$, whence $\delta[d\eta]_{27} = 0$ for $d \star p(d\eta) = 0$. Consequently, $\dot{\Omega} = d\eta$ is Δ -harmonic which is impossible unless $\dot{\Omega} = 0$. For surjectivity, recall that the projections on irreducible G_2 -components $\Lambda_q^p T^*M$ commute with the Hodge Laplacian Δ since $\bar{\Omega}$ is torsion-free (cf. for instance Theorem 3.5.3 in [16]). Hence, a p -form is Δ -harmonic if and only if its irreducible components in $\Omega_q^p(M)$ are Δ -harmonic. So, given a cohomology class $c = [\dot{\Omega}] \in H^3(M, \mathbb{R})$ with unique Δ -harmonic representative $\dot{\Omega}$, we have $d \star p(\dot{\Omega}) = 0$ and thus $\dot{\Omega} \in T_{\bar{\Omega}}S_{\bar{\Omega}}$. \square

In particular, we deduce in conjunction with Proposition 5.6 that

$$T_{\bar{\Omega}}\mathcal{X}^k = T_{\bar{\Omega}}\mathcal{O}_{\bar{\Omega}}^k \oplus T_{\bar{\Omega}}S_{\bar{\Omega}}. \tag{45}$$

Next consider the map

$$\Phi^k : ([\varphi], \Omega) \in G_{\bar{\Omega}}^{k+1} \times S_{\bar{\Omega}} \mapsto [\varphi]^*\Omega \in \mathcal{X}^k$$

on a suitable neighbourhood $S_{\bar{\Omega}}$ of $\bar{\Omega}$ in $S_{\bar{\Omega}}$. Following [16] Φ^k is well-defined: indeed, near $\bar{\Omega}$ we can linearise the action of $I_{\bar{\Omega}}$ on $S_{\bar{\Omega}}$ via the exponential map $\exp_{\bar{\Omega}} : T_{\bar{\Omega}}S_{\bar{\Omega}} \cong H^3(M, \mathbb{R}) \rightarrow S_{\bar{\Omega}}$ induced by $\langle \cdot, \cdot \rangle_{L^2}$. The linearised action must be trivial as any element in $I_{\bar{\Omega}}$ is homotopic to the identity. Hence $I_{\bar{\Omega}}$ acts trivially on $S_{\bar{\Omega}}$ close to $\bar{\Omega}$. The Banach space inverse function theorem and (45) imply that Φ^k is a diffeomorphism onto its image near $([\text{Id}_M], \bar{\Omega})$. Hence, shrinking $S_{\bar{\Omega}}$ possibly further, we have shown that $S_{\bar{\Omega}}$ defines a slice for the $\text{Diff}(M)_0^{k+1}$ -action on \mathcal{X}^k near $\bar{\Omega}$. The same statement holds for the C^∞ -topology instead of the Sobolev topologies, whence the following.

Corollary 7.4. (i) *The space \mathcal{X} is a Fréchet manifold whose tangent space at $\bar{\Omega}$ is given by*

$$T_{\bar{\Omega}}\mathcal{X} = \{\dot{\Omega} \in \Omega^3(M) \mid d\dot{\Omega} = 0, d\dot{\Theta}_{\bar{\Omega}} = 0\}.$$

(ii) (Joyce) *The space of torsion-free G_2 -structures modulo diffeomorphisms isotopic to the identity is a smooth manifold of dimension $b_3 = \dim H^3(M, \mathbb{R})$.*

We can now prove the central result of this section.

Proposition 7.5. *Let $\bar{\Omega} \in \mathcal{X}$.*

(i) *We have*

$$D_{\bar{\Omega}}^2\mathcal{D}(\dot{\Omega}, \dot{\Omega}) = \int_M (|d\dot{\Omega}|^2 + |d\dot{\Theta}_{\bar{\Omega}}|^2) \text{vol}_{\bar{\Omega}}.$$

In particular, the second variation of \mathcal{D} at $\bar{\Omega}$ is a positive semi-definite bilinear form with

$$\ker D_{\bar{\Omega}}^2\mathcal{D} = T_{\bar{\Omega}}\mathcal{X}.$$

(ii) *The linearisation $L_{\bar{\Omega}} := D_{\bar{\Omega}}\tilde{Q}_{\bar{\Omega}}$ of $\tilde{Q}_{\bar{\Omega}}$ is a symmetric, non-positive and elliptic operator given by*

$$L_{\bar{\Omega}}\dot{\Omega} = -\delta d\dot{\Omega} - pd\delta p\dot{\Omega} - 3d[\delta\dot{\Omega}]_7.$$

More precisely, writing $\dot{\Omega} = \dot{f}\bar{\Omega} \oplus \star(\dot{\alpha} \wedge \bar{\Omega}) \oplus \dot{\gamma}$, we have

$$\begin{aligned} -L_{\bar{\Omega}}\dot{\Omega} &= \Delta\dot{\Omega} + \frac{34}{21}d_1^7d_1^7\dot{f} \cdot \bar{\Omega} + \star(d_7^7d_7^7\dot{\alpha} \wedge \bar{\Omega}) + d_{27}^7d_{27}^7\dot{\gamma} \\ &\quad - \frac{2}{21}d_1^7d_7^{27}\dot{\gamma} \cdot \bar{\Omega} - \frac{2}{3}d_{27}^7d_7^1\dot{f} \end{aligned} \tag{46}$$

for the G_2 -differential operators d_q^p introduced in Section 3. In particular

$$\langle L_{\bar{\Omega}}\dot{\Omega}, \dot{\Omega} \rangle_{L_{\bar{\Omega}}^2} = -\|d\dot{\Omega}\|_{L_{\bar{\Omega}}^2}^2 - \|\delta p\dot{\Omega}\|_{L_{\bar{\Omega}}^2}^2 - 3\|[\delta\dot{\Omega}]_7\|_{L_{\bar{\Omega}}^2}^2 \quad \text{and} \quad \ker L_{\bar{\Omega}} = T_{\bar{\Omega}}S_{\bar{\Omega}}.$$

Proof. (i) We compute the second variation of \mathcal{D} at a critical point $\bar{\Omega}$ by differentiating Eq. (20) once more. Thus we obtain

$$D_{\bar{\Omega}}^2 \mathcal{D}(\dot{\Omega}, \dot{\Omega}) = \int_M d\dot{\Omega} \wedge \star d\dot{\Omega} + d\dot{\Theta}_{\bar{\Omega}} \wedge \star d\dot{\Theta}_{\bar{\Omega}} = \int_M (|d\dot{\Omega}|^2 + |d\dot{\Theta}_{\bar{\Omega}}|^2) \text{vol} \geq 0,$$

for the remaining terms involve either $d\bar{\Omega}$ or $d\Theta(\bar{\Omega})$. But these terms vanish since $\bar{\Omega}$ is a critical point and hence torsion-free. Furthermore, the kernel $D_{\bar{\Omega}}^2 \mathcal{D}$ is precisely $T_{\bar{\Omega}}\mathcal{X}$ by [Corollary 7.4](#).

(ii) Since $D_{\bar{\Omega}}^2 \mathcal{D}(\dot{\Omega}, \dot{\Omega}) = \langle \text{Hess}_{\bar{\Omega}} \mathcal{D} \dot{\Omega}, \dot{\Omega} \rangle_{L^2}$ and $D_{\bar{\Omega}} \mathcal{Q} = -\text{Hess}_{\bar{\Omega}} \mathcal{D}$, it follows from (i) that $D_{\bar{\Omega}} \mathcal{Q} = -\delta d\dot{\Omega} - pd\delta p\dot{\Omega}$. On the other hand, using again that $\bar{\Omega}$ is torsion-free, we get

$$D_{\bar{\Omega}} \Lambda_{\bar{\Omega}}(\dot{\Omega}) = d(X_{\bar{\Omega}}(\dot{\Omega}) \lrcorner \bar{\Omega}).$$

Bearing (5) in mind, we obtain $\dot{\Lambda}_{\bar{\Omega}} = -3d[\delta\dot{\Omega}]_7$, whence

$$L_{\bar{\Omega}}(\dot{\Omega}) = -\delta d\dot{\Omega} - pd\delta p\dot{\Omega} - 3d[\delta\dot{\Omega}]_7.$$

This operator is clearly symmetric and non-positive. Ellipticity was proven in [Lemma 5.7](#). To compute (46) we start from (7) and use [Tables 1](#) and [2](#). Then

$$\begin{aligned} \delta d\dot{\Omega} &= \frac{1}{7}(4d_1^7 d_7^1 \dot{f} + d_7^1 d_7^{27} \dot{\gamma}) \cdot \bar{\Omega} \oplus \star \left(\left(d_7^1 d_1^7 \dot{\alpha} + \frac{1}{2} d_7^1 d_7^1 \dot{\alpha} + \frac{1}{4} d_7^1 d_7^{27} \dot{\gamma} \right) \wedge \bar{\Omega} \right) \\ &\quad \oplus d_{27}^7 d_7^1 \dot{f} + d_{27}^7 d_7^1 \dot{\alpha} + \frac{1}{4} d_{27}^7 d_7^{27} \dot{\gamma} + d_{27}^{27} d_{27}^{27} \dot{\gamma} \\ d\delta\dot{\Omega} &= \frac{1}{7}(3d_1^7 d_7^1 \dot{f} - d_7^1 d_7^{27} \dot{\gamma}) \cdot \bar{\Omega} \oplus \star \left(\frac{1}{2} \left(d_7^1 d_7^1 \dot{\alpha} - \frac{1}{2} d_7^1 d_7^{27} \dot{\gamma} \right) \wedge \bar{\Omega} \right) \\ &\quad \oplus -d_{27}^7 d_7^1 \dot{f} - d_{27}^7 d_7^1 \dot{\alpha} + \frac{1}{3} d_{27}^7 d_7^{27} \dot{\gamma} + d_{27}^{14} d_{27}^{27} \dot{\gamma} \end{aligned}$$

from which (46) follows by applying [Table 3](#). \square

Let \mathfrak{D} denote the associated Hodge–Dirac operator with respect to the metric induced by $\bar{\Omega} \in \mathcal{X}$, i.e.

$$\mathfrak{D}\dot{\Omega} = d\dot{\Omega} + \delta\dot{\Omega}.$$

In particular, \mathfrak{D} is symmetric and $\mathfrak{D}^2 = \Delta$. In view of long-time existence to be established in the next section we note the following corollary.

Corollary 7.6 (*Gårding Inequality*). *For all $\dot{\Omega} \in \Omega^3(M)$,*

$$\langle -L_{\bar{\Omega}} \dot{\Omega}, \dot{\Omega} \rangle_{L^2} \geq \| \mathfrak{D}\dot{\Omega} \|_{L^2}^2.$$

In particular, we have

$$\langle -L_{\bar{\Omega}} \dot{\Omega}, \dot{\Omega} \rangle_{L^2} \geq C \| \dot{\Omega} \|_{W^{1,2}}^2 - \| \dot{\Omega} \|_{L^2}^2$$

for some constant C independent of $\dot{\Omega}$.

Proof. Writing $\dot{\Omega}$ as in [Proposition 7.5](#), the first inequality follows from

$$\begin{aligned} \langle -L_{\bar{\Omega}} \dot{\Omega}, \dot{\Omega} \rangle_{L^2} &= \| \mathfrak{D}\dot{\Omega} \|_{L^2}^2 + \frac{34}{3} \| d_7^1 \dot{f} \|_{L^2}^2 + 4 \| d_7^1 \dot{\alpha} \|_{L^2}^2 + \| d_7^{27} \dot{\gamma} \|_{L^2}^2 - \frac{4}{3} \langle d_7^{27} \dot{\gamma}, d_7^1 \dot{f} \rangle_{L^2} \\ &\geq \| \mathfrak{D}\dot{\Omega} \|_{L^2}^2 + \frac{34}{3} \| d_7^1 \dot{f} \|_{L^2}^2 + \| d_7^{27} \dot{\gamma} \|_{L^2}^2 - \frac{2}{3} (\| d_7^1 \dot{f} \|_{L^2}^2 + \| d_7^{27} \dot{\gamma} \|_{L^2}^2) \\ &\geq \| \mathfrak{D}\dot{\Omega} \|_{L^2}^2. \end{aligned} \tag{47}$$

The second inequality is just the elliptic estimate $\|\dot{\Omega}\|_{W^{1,2}}^2 \leq C^{-1}(\|\dot{\Omega}\|_{L^2}^2 + \|\mathfrak{D}\dot{\Omega}\|_{L^2}^2)$ for some constant C^{-1} . \square

8. Stability

We continue to fix a torsion-free G_2 -form $\bar{\Omega} \in \mathcal{X}$. From now on, we let $W^{k,2}$ and $W_+^{k,2}$ be shorthand for the Sobolev spaces $W^{k,2}(\Lambda^3 T^*M)$ and $W_+^{k,2}(\Lambda^3 T^*M)$ with respect to the metric $g_{\bar{\Omega}}$. The induced norm will be denoted by $\|\cdot\|_{W^{k,2}}$. Again, k is an integer strictly greater than $11/2$ which for simplicity we assume to be odd (this avoids using fractional Sobolev spaces below). In particular, $W^{k,2}$ embeds continuously into C^2 . The goal of this section is to prove the subsequent stability theorem.

Theorem 8.1 (Stability). *Let $\bar{\Omega} \in \Omega_+^3(M)$ be a torsion-free G_2 -form. For all $\epsilon > 0$ there exists some $\delta = \delta(\epsilon) > 0$ such that for any $\tilde{\Omega}_0$ with $\|\tilde{\Omega}_0 - \bar{\Omega}\|_{W^{k,2}} < \delta$, the Dirichlet–DeTurck flow $\tilde{\Omega}_t$ at $\bar{\Omega}$ with initial condition $\tilde{\Omega}_0$*

1. (**long-time existence**) exists for all $t \in [0, \infty)$,
2. (**a priori estimate**) satisfies the estimate $\|\tilde{\Omega}_t - \bar{\Omega}\|_{W^{k,2}} < \epsilon$ for all $t \in [0, \infty)$, and
3. (**convergence**) converges with respect to the $W^{k,2}$ -norm to a torsion-free G_2 -form $\tilde{\Omega}_\infty$ as $t \rightarrow \infty$.

Since the Dirichlet energy flow exists as long as the Dirichlet–DeTurck flow exists we immediately obtain the following.

Corollary 8.2. *Let $\bar{\Omega} \in \Omega_+^3(M)$ be a torsion-free G_2 -form. For initial conditions sufficiently C^∞ -close to $\bar{\Omega}$ the Dirichlet energy flow Ω_t exists for all times and converges modulo diffeomorphisms to a torsion-free G_2 -form Ω_∞ , i.e. there exists a family of diffeomorphisms $\varphi_t \in \text{Diff}(M)_+$ such that $\varphi_t^* \Omega_t$ converges to Ω_∞ with respect to the C^∞ -topology.*

The proof of [Theorem 8.1](#) will be subdivided into a sequence of intermediate steps.

First, we tackle existence of the Dirichlet–DeTurck flow together with the a priori estimate on arbitrary, but finite time intervals for initial conditions sufficiently close to $\bar{\Omega}$. Here we use the Banach space inverse function theorem, following the approach of Huisken and Polden [15] for geometric evolution equations for hypersurfaces. Let $0 < T < \infty$. If $\pi : M \times [0, T] \rightarrow M$ denotes projection onto the first factor, let $C^\infty(M \times [0, T], \pi^* \Lambda^3)$ denote the space of smooth sections of Λ^3 pulled back to $M \times [0, T]$. For any non-negative integer s we define the Hilbert space $V^s[0, T]$ as the completion of $C^\infty(M \times [0, T], \pi^* \Lambda^3)$ with respect to the inner product given by

$$\langle \dot{\Omega}_1, \dot{\Omega}_2 \rangle_{V^s[0,T]} = \sum_{j \leq s} \int_0^T e^{-2t} \langle \partial_t^j \dot{\Omega}_1, \partial_t^j \dot{\Omega}_2 \rangle_{W^{2(s-j),2}} dt.$$

In particular, $\|\dot{\Omega}\|_{V^0[0,T]}^2 = \int_0^T e^{-2t} \|\dot{\Omega}\|_{L^2}^2 dt$. These spaces are also known as *anisotropic* or *parabolic* Sobolev spaces, where a time derivative has the weight of two space derivatives. The density e^{-2t} is introduced for technical reasons; see below. Similarly, we can consider the Hilbert manifold $V_+^s[0, T]$ consisting of sections of $\pi^* \Lambda_+^3$ of class V^s . Define the map

$$F : V_+^s[0, T] \rightarrow W_+^{2s-1,2} \times V^{s-1}[0, T], \quad F(\tilde{\Omega}) = (\tilde{\Omega}_0, \partial_t \tilde{\Omega}_t - \tilde{Q}_{\bar{\Omega}}(\tilde{\Omega}_t)). \tag{48}$$

As usual, restricting to the boundary is tantamount to invoking a trace theorem, which in this context is stating that the trace map $\dot{\Omega} \mapsto \dot{\Omega}_0$ is continuous from $V^s[0, T]$ to $W^{2s-1,2}$; see [15]. We wish to show that F is a local diffeomorphism near $\dot{\Omega}$. Consider the linearisation at $\dot{\Omega}$ of the map (48), namely

$$D_{\dot{\Omega}}F : V^s[0, T] \rightarrow W^{2s-1,2} \times V^{s-1}[0, T], \quad D_{\dot{\Omega}}F(\dot{\Omega}) = (\dot{\Omega}_0, P\dot{\Omega}), \tag{49}$$

where $P\dot{\Omega} := \partial_t \dot{\Omega}_t - L_{\dot{\Omega}} \dot{\Omega}_t$. Let \mathbf{H} be the completion of $C^\infty(M \times [0, T], \pi^* \Lambda^3)$ with respect to the inner product

$$\langle \dot{\Omega}_1, \dot{\Omega}_2 \rangle_{\mathbf{H}} = \int_0^T e^{-2t} \langle \dot{\Omega}_1, \dot{\Omega}_2 \rangle_{W^{1,2}} dt + \int_0^T e^{-2t} \langle \partial_t \dot{\Omega}_1, \partial_t \dot{\Omega}_2 \rangle_{L^2} dt.$$

Note that the quadratic form $\dot{\Omega} \mapsto \langle -L_{\dot{\Omega}} \dot{\Omega}, \dot{\Omega} \rangle_{L^2}$ is defined on $W^{1,2}$ in view of Eq. (47), so that we can say that $\dot{\Omega} \in \mathbf{H}$ satisfies $P\dot{\Omega} = \dot{\Phi}$ for $\dot{\Phi} \in V^0[0, T]$ weakly, if

$$\langle \partial_t \dot{\Omega}, \dot{\Psi} \rangle_{V^0[0,T]} + \int_0^T e^{-2t} \langle -L_{\dot{\Omega}} \dot{\Omega}, \dot{\Psi} \rangle_{L^2} dt = \langle \dot{\Phi}, \dot{\Psi} \rangle_{V^0[0,T]}$$

holds for all $\dot{\Psi} \in C_0^\infty(M \times (0, T), \pi^* \Lambda^3)$, the space of smooth sections of $\pi^* \Lambda^3$ which vanish near the boundary $M \times \{0, T\}$.

We first show that for $\dot{\Phi} \in V^0[0, T]$ there exists a unique weak solution in \mathbf{H} to the equation

$$D_{\dot{\Omega}}F(\dot{\Omega}) = (0, \dot{\Phi}). \tag{50}$$

Following [15], we use a refined version of the Lax–Milgram lemma; cf. Lemma 7.8 and Theorem 7.9 in [15] or Theorem 16 in Chapter 10 of [12]. As explained in [15], the main point is to check *coercivity* of the bilinear form

$$A(\dot{\Omega}_1, \dot{\Omega}_2) = \int_0^T e^{-2t} \langle \partial_t \dot{\Omega}_1, \partial_t \dot{\Omega}_2 \rangle_{L^2} dt + \int_0^T e^{-2t} \langle -L_{\dot{\Omega}} \dot{\Omega}_1, \partial_t \dot{\Omega}_2 \rangle_{L^2} dt$$

on $C_0^\infty(M \times (0, T), \pi^* \Lambda^3) \subset \mathbf{H}$, i.e. to establish an estimate of the form $A(\dot{\Omega}, \dot{\Omega}) \geq C \|\dot{\Omega}\|_{\mathbf{H}}^2$ for some positive constant C . This will be a consequence of the Gårding inequality for the operator $-L_{\dot{\Omega}}$ of Corollary 7.6. First note that $\dot{\Omega}$ is a solution to $P\dot{\Omega} = \dot{\Phi}$ if and only if $e^{-t} \dot{\Omega}$ is a solution to $(P + 1)e^{-t} \dot{\Omega} = e^{-t} \dot{\Phi}$. Hence by replacing $-L_{\dot{\Omega}}$ by $-L_{\dot{\Omega}} + 1$ we may assume that $-L_{\dot{\Omega}}$ satisfies a strict Gårding inequality of the form $\langle -L_{\dot{\Omega}} \dot{\Omega}, \dot{\Omega} \rangle_{L^2} \geq C \|\dot{\Omega}\|_{W^{1,2}}^2$ with C as in Corollary 7.6. Since $\int_0^T \partial_t (e^{-2t} \langle -L_{\dot{\Omega}} \dot{\Omega}, \dot{\Omega} \rangle_{L^2}) dt = 0$ by assumption, we obtain

$$\int_0^T e^{-2t} \langle -L_{\dot{\Omega}} \dot{\Omega}, \partial_t \dot{\Omega} \rangle_{L^2} dt = \int_0^T e^{-2t} \langle -L_{\dot{\Omega}} \dot{\Omega}, \dot{\Omega} \rangle_{L^2} dt \geq C \int_0^T e^{-2t} \|\dot{\Omega}\|_{W^{1,2}}^2 dt,$$

whence

$$A(\dot{\Omega}, \dot{\Omega}) \geq C \int_0^T e^{-2t} \|\dot{\Omega}\|_{W^{1,2}}^2 dt + \int_0^T e^{-2t} \|\partial_t \dot{\Omega}\|_{L^2}^2 dt \geq C \|\dot{\Omega}\|_{\mathbf{H}}^2,$$

establishing coercivity and the existence of a weak solution.

In order to improve regularity of this weak solution one needs the following estimate.

Lemma 8.3 (Huisken–Polden). *Let $s \geq 0$. If $\dot{\Omega} \in \mathbf{H}$ is a weak solution to the equation $D_{\dot{\Omega}}F(\dot{\Omega}) = (\dot{\Omega}_0, \dot{\Phi})$ with $\dot{\Omega}_0 \in W^{2s+1,2}$ and $\dot{\Phi} \in V^s[0, T]$, then $\dot{\Omega} \in V^{s+1}[0, T]$ and there*

exists a constant $C = C(\bar{\Omega}, s) > 0$ such that

$$\|\dot{\Omega}\|_{V^{s+1}[0,T]}^2 \leq C(\|\dot{\Omega}_0\|_{W^{2s+1,2}}^2 + \|\dot{\Phi}\|_{V^s[0,T]}^2).$$

Proof. See Lemma 7.13 in [15]. \square

For later use we state and prove the ensuing interior estimate.

Corollary 8.4 (Interior Estimate). *Let $s \geq 0$. For all $\delta > 0$ there exists a constant $C = C(\delta, \bar{\Omega}, s) > 0$ such that for $\dot{\Omega} \in V^{s+1}[0, T]$ one has*

$$\|\dot{\Omega}\|_{V^{s+1}[\delta,T]}^2 \leq C(\|\dot{\Omega}\|_{V^0[0,T]}^2 + \|P\dot{\Omega}\|_{V^s[0,T]}^2).$$

Proof. Since $\dot{\Phi} = P\dot{\Omega}$ is of class V^s , Lemma 8.3 gives

$$\|\dot{\Omega}\|_{V^{s+1}[0,T]}^2 \leq C(\bar{\Omega}, s)(\|\dot{\Omega}_0\|_{W^{2s+1,2}}^2 + \|P\dot{\Omega}\|_{V^s[0,T]}^2).$$

Put $\Omega' := \varphi \cdot \dot{\Omega}$, where $\varphi : [0, T] \rightarrow \mathbb{R}$ is a smooth cut-off function satisfying $\varphi(t) = 0$ for t close to 0 and $\varphi(t) = 1$ for $t \in [\epsilon, T]$. Then $P\Omega' = \partial_t \varphi \cdot \dot{\Omega} + \varphi \cdot P\dot{\Omega}$ and $\Omega'_0 = 0$, whence

$$\begin{aligned} \|\dot{\Omega}\|_{V^{s+1}[\epsilon,T]} &\leq \|\Omega'\|_{V^{s+1}[0,T]} \leq C(\bar{\Omega}, s)^{\frac{1}{2}} \|\partial_t \varphi \cdot \dot{\Omega} + \varphi \cdot P\dot{\Omega}\|_{V^s[0,T]} \\ &\leq C(\bar{\Omega}, s)^{\frac{1}{2}} \left(\sup_{t \in [0,T]} |\partial_t \varphi(t)| \cdot \|\dot{\Omega}\|_{V^s[0,T]} + \|P\dot{\Omega}\|_{V^s[0,T]} \right). \end{aligned}$$

Induction on s yields the result. \square

Mutatis mutandis one proves along the lines of Theorem 7.14 in [15].

Theorem 8.5 (Huisken–Polden). *The map $D_{\bar{\Omega}}F$ in (49) is a Banach space isomorphism.*

As a consequence we obtain existence of the Dirichlet–DeTurck flow together with the a priori estimate on a finite time interval $[0, T]$ for initial conditions (depending a priori on T) sufficiently close to $\bar{\Omega}$.

Corollary 8.6. *For all $\epsilon > 0$ and $0 < T < \infty$, there exists $\delta = \delta(\bar{\Omega}, \epsilon, T) > 0$ such that for $\tilde{\Omega}_0 \in W^{k,2}$ with $\|\tilde{\Omega}_0 - \bar{\Omega}\|_{W^{k,2}} < \delta$, the Dirichlet–DeTurck flow $\tilde{\Omega}_t$ at $\bar{\Omega}$ with initial condition $\tilde{\Omega}_0$ exists and satisfies $\|\tilde{\Omega}_t - \bar{\Omega}\|_{W^{k,2}} < \epsilon$ for all $t \in [0, T]$.*

Proof. Since k is odd, $s := (k + 1)/2$ is an integer. Let $\bar{\Omega}$ also denote the pull-back to $M \times [0, T]$. Then $\bar{\Omega} \in V^s[0, T]$ and $F(\bar{\Omega}) = (\bar{\Omega}, 0)$. By virtue of the previous Theorem 8.5 and the Banach space inverse function theorem, the map F in (48) is a local diffeomorphism near $\bar{\Omega}$. The trace maps $\Omega \in V^s[0, T] \mapsto \Omega_t \in W^{k,2}$ are continuous with a uniform bound on their norms since t varies within a compact interval. Hence there exists $C > 0$ such that $\|\tilde{\Omega}_t - \bar{\Omega}\|_{W^{k,2}} \leq C\|\tilde{\Omega} - \bar{\Omega}\|_{V^s[0,T]}$ holds for all $t \in [0, T]$. For suitably chosen $\delta > 0$, the condition $\|\tilde{\Omega}_0 - \bar{\Omega}\|_{W^{k,2}} < \delta$ implies that $(\tilde{\Omega}_0, 0)$ is close enough to $(\bar{\Omega}, 0)$ to ensure that $\tilde{\Omega} = F^{-1}(\tilde{\Omega}_0, 0)$ satisfies $\|\tilde{\Omega} - \bar{\Omega}\|_{V^s[0,T]} < \epsilon/C$. \square

Remark. Using Theorem 8.5 it is also possible to give an alternative proof of short-time existence for the Dirichlet–DeTurck flow; cf. [15] for details.

As in Section 7 let $\mathcal{S}_{\bar{\Omega}} \subset \tilde{Q}_{\bar{\Omega}}^{-1}(0)$ denote a suitably chosen slice around $\bar{\Omega}$. For a positive 3-form $\bar{\Omega}'$ close to $\bar{\Omega}$ we write $\tilde{\Omega}'_t = \bar{\Omega}' + \omega'_t$. Let now

$$L_{\bar{\Omega}'} := D_{\bar{\Omega}'} \tilde{Q}_{\bar{\Omega}}$$

and

$$R_{\bar{\Omega}'}(\omega'_t) := \tilde{Q}_{\bar{\Omega}}(\tilde{\Omega}'_t) - L_{\bar{\Omega}'}\omega'_t.$$

Then we can recast the flow equation into

$$\frac{\partial}{\partial t} \tilde{\Omega}'_t = L_{\bar{\Omega}'}\omega'_t + R_{\bar{\Omega}'}(\omega'_t). \tag{51}$$

The basic idea is that the behaviour of the Dirichlet–DeTurck flow should be dominated by the linear term $L_{\bar{\Omega}'}$. To obtain precise results, we need to control the remainder term $R_{\bar{\Omega}'}$.

In order to analyse $R_{\bar{\Omega}'}$ we introduce the following notation: let E_1, E_2, E and F be tensor bundles over M equipped with the bundle metrics induced by $\bar{\Omega}$. Denote by $\bar{\nabla}$ the covariant derivative associated with $\bar{\Omega}$. Let Ω be a further positive 3-form of class C^2 and let $\bar{\Omega}' \in \mathcal{S}_{\bar{\Omega}}$. Put $\omega' = \Omega - \bar{\Omega}'$ and assume that $\|\omega'\|_{C^2} < \epsilon_1$ and $\|\bar{\Omega}' - \bar{\Omega}\|_{C^2} < \epsilon_2$. For $s \in \Gamma(M, E)$ we will generically write $\otimes_l s$ for sections of the form $A(s) \in \Gamma(M, F)$, where $A \in \Gamma(M, E^* \otimes F)$ is a section of class C^l possibly depending on ω' and $\bar{\Omega}'$ such that $\|A\|_{C^l} \leq C(\epsilon_1, \epsilon_2)$. Using this convention, we have for instance

$$\otimes_k \otimes_l s = \otimes_k s$$

for $k \leq l$ and, by the product rule for $\bar{\nabla}$,

$$\bar{\nabla}(\otimes_l s) = \otimes_{l-1} s + \otimes_l \bar{\nabla}s.$$

Similarly, for a section

$$B \in \Gamma(M, E_1^* \otimes E_2^* \otimes F)$$

with $\|B\|_{C^l} \leq C(\epsilon_1, \epsilon_2)$ and $s_i \in \Gamma(M, E_i)$ we write generically $s_1 \otimes_l s_2$ for the section $B(s_1, s_2) \in \Gamma(M, F)$. For instance we have

$$(\otimes_{l_0} s_0) \otimes_k (\otimes_{l_1} s_1) = s_1 \otimes_k s_2, \quad \text{and} \quad \otimes_k (s_1 \otimes_l s_2) = s_1 \otimes_k s_2$$

for $k \leq l, l_0, l_1$. In all cases we set $\otimes := \otimes_0$. As we will differentiate at most twice we take l to be smaller than or equal to two even if we have C^l -control for higher l .

Example. Since the exterior differential d is obtained by anti-symmetrising the covariant derivative $\bar{\nabla}$, we have $d\omega' = \otimes_2 \bar{\nabla}\omega'$. Hence $\star_{\bar{\Omega}'} d\omega' = \otimes_2 \bar{\nabla}\omega'$ and we get

$$\begin{aligned} d \star_{\bar{\Omega}'} d\omega' &= \otimes_2 \bar{\nabla}(\otimes_2 \bar{\nabla}\omega') \\ &= \otimes_2(\otimes_1 \bar{\nabla}\omega' + \otimes_2 \bar{\nabla}^2\omega') \\ &= \otimes_1 \bar{\nabla}\omega' + \otimes_2 \bar{\nabla}^2\omega'. \end{aligned}$$

The following result gives a rough description of the structure of the remainder term, which however will turn out to be sufficient for our purposes.

Lemma 8.7. *Let Ω be a positive 3-form of class C^2 and $\bar{\Omega}' \in \mathcal{S}_{\bar{\Omega}}$. Let further $\omega' = \Omega - \bar{\Omega}'$. Assume that $\|\omega'\|_{C^2} < \epsilon_1$ and $\|\bar{\Omega}' - \bar{\Omega}\|_{C^2} < \epsilon_2$. Then*

$$R_{\bar{\Omega}'}(\omega') = \omega' \circledast \omega' + \omega' \circledast \bar{\nabla}\omega' + \omega' \circledast \bar{\nabla}^2\omega' + \bar{\nabla}\omega' \circledast \bar{\nabla}\omega'. \tag{52}$$

In particular, there exists $C = C(\epsilon_1, \epsilon_2) > 0$ such that

$$|R_{\bar{\Omega}'}(\omega')| \leq \epsilon_1 C(|\omega'| + |\bar{\nabla}\omega'|).$$

Remark. In particular we absorbed into \circledast any term in (52) of order strictly higher than two in ω' .

Proof. Recall that $\tilde{Q}_{\bar{\Omega}}(\Omega) = Q(\Omega) + A_{\bar{\Omega}}(\Omega)$ where

$$Q(\Omega) = - \left(\Delta_{\Omega}\Omega + \frac{1}{3}[d\delta_{\Omega}\Omega]_1 - 2[d\delta_{\Omega}\Omega]_{27} + q_{\bar{\Omega}}(\bar{\nabla}\Omega) \right),$$

projections being taken with respect to $\bar{\Omega}$, and

$$A_{\bar{\Omega}}(\Omega) = \mathcal{L}_{X(\Omega)}\Omega$$

with $X(\Omega) = -(\delta_{\bar{\Omega}}\Omega)_{\perp}\bar{\Omega}$. In the following we will calculate the difference between $\tilde{Q}_{\bar{\Omega}}$ and its linearisation at $\bar{\Omega}'$ term by term. First write $\Omega = \bar{\Omega}' + \omega'$ and observe that mapping a positive 3-form Ω to the corresponding Hodge operator \star_{Ω} gives rise to a fibre-preserving smooth map $\star : \Lambda^3_+ T^*M \rightarrow \text{End}(\Lambda^* T^*M)$. Hence there exists a fibrewise linear map $A_{\bar{\Omega}'}(\omega') : \Lambda^3 T^*M \rightarrow \text{End}(\Lambda^* T^*M)$, depending smoothly on $\bar{\Omega}'$ and ω' , such that $\star_{\Omega} = \star_{\bar{\Omega}'} + A_{\bar{\Omega}'}(\omega')\omega'$; in particular $(D_{\bar{\Omega}'}\star)(\omega') = A_{\bar{\Omega}'}(0)\omega'$. Using $d\bar{\Omega}' = 0$, we get

$$\begin{aligned} \delta_{\Omega}d\Omega &= \star_{\Omega} d \star_{\Omega} d\omega' \\ &= \star_{\bar{\Omega}'} d \star_{\bar{\Omega}'} d\omega' + \star_{\bar{\Omega}'} d(A_{\bar{\Omega}'}(\omega')\omega')d\omega' + (A_{\bar{\Omega}'}(\omega')\omega')d \star_{\bar{\Omega}'} d\omega' \\ &\quad + (A_{\bar{\Omega}'}(\omega')\omega')d(A_{\bar{\Omega}'}(\omega')\omega')d\omega'. \end{aligned}$$

Subtracting the linear term in ω' , we get

$$\begin{aligned} \delta_{\Omega}d\Omega - \star_{\bar{\Omega}'} d \star_{\bar{\Omega}'} d\omega' &= \star_{\bar{\Omega}'} d(A_{\bar{\Omega}'}(\omega')\omega')d\omega' + (A_{\bar{\Omega}'}(\omega')\omega')d \star_{\bar{\Omega}'} d\omega' + (A_{\bar{\Omega}'}(\omega')\omega')d(A_{\bar{\Omega}'}(\omega')\omega')d\omega' \\ &= \omega' \circledast \bar{\nabla}\omega' + \omega' \circledast \bar{\nabla}^2\omega' + \bar{\nabla}\omega' \circledast \bar{\nabla}\omega'. \end{aligned}$$

Similarly, using $\delta_{\bar{\Omega}'}\bar{\Omega}' = 0$, we get

$$\begin{aligned} d\delta_{\Omega}\Omega &= -d \star_{\Omega} d \star_{\Omega} \Omega \\ &= -d \star_{\bar{\Omega}'} d \star_{\bar{\Omega}'} \omega' - d \star_{\bar{\Omega}'} d(A_{\bar{\Omega}'}(\omega')\omega')\bar{\Omega}' - d \star_{\bar{\Omega}'} d(A_{\bar{\Omega}'}(\omega')\omega')\omega' \\ &\quad - d(A_{\bar{\Omega}'}(\omega')\omega')d \star_{\bar{\Omega}'} \omega' - d(A_{\bar{\Omega}'}(\omega')\omega')d(A_{\bar{\Omega}'}(\omega')\omega')\bar{\Omega}' \\ &\quad - d(A_{\bar{\Omega}'}(\omega')\omega')d(A_{\bar{\Omega}'}(\omega')\omega')\omega'. \end{aligned}$$

Again, subtracting the linear term in ω' , we get

$$\begin{aligned} d\delta_{\Omega}\Omega + d \star_{\bar{\Omega}'} d \star_{\bar{\Omega}'} \omega' + d \star_{\bar{\Omega}'} d(A_{\bar{\Omega}'}(0)\omega')\bar{\Omega}' &= -d \star_{\bar{\Omega}'} d((A_{\bar{\Omega}'}(\omega') - A_{\bar{\Omega}'}(0))\omega')\bar{\Omega}' - d \star_{\bar{\Omega}'} d(A_{\bar{\Omega}'}(\omega')\omega')\omega' \\ &\quad - d(A_{\bar{\Omega}'}(\omega')\omega')d \star_{\bar{\Omega}'} \omega' - d(A_{\bar{\Omega}'}(\omega')\omega')d(A_{\bar{\Omega}'}(\omega')\omega')\bar{\Omega}' \end{aligned}$$

$$\begin{aligned}
 & -d(A_{\bar{\Omega}'}, (\omega')\omega')d(A_{\bar{\Omega}'}, (\omega')\omega')\omega' \\
 & = \omega' \circledast \omega' + \omega' \circledast \bar{\nabla}\omega' + \omega' \circledast \bar{\nabla}^2\omega' + \bar{\nabla}\omega' \circledast \bar{\nabla}\omega'.
 \end{aligned}$$

This takes care of the first three terms in Q (note that the linear projections onto irreducible components are absorbed into \circledast). Furthermore, the term coming from the quadratic form $q_{\bar{\Omega}}$ contributes a remainder term of type $\bar{\nabla}\omega' \circledast \bar{\nabla}\omega'$. Finally, in order to deal with the term $\Lambda_{\bar{\Omega}}(\Omega)$, we observe that $X(\bar{\Omega}') = 0$ since $\bar{\Omega}' \in \mathcal{S}_{\bar{\Omega}}$ which implies $[\delta_{\bar{\Omega}}\bar{\Omega}']_7 = 0$. Hence $X(\Omega) = X(\omega')$ and

$$\mathcal{L}_{X(\omega')}\omega' = \omega' \circledast \bar{\nabla}\omega' + \omega' \circledast \bar{\nabla}^2\omega' + \bar{\nabla}\omega' \circledast \bar{\nabla}\omega'.$$

This finishes the proof. \square

In the following we need some standard results from perturbation theory of linear operators. This is summarised in the following statement.

Lemma 8.8. *For all $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that if $\|\bar{\Omega}' - \bar{\Omega}\|_{W^{k,2}} < \delta$, then*

$$\langle -L_{\bar{\Omega}'}, \omega, \omega \rangle_{L^2} \geq (1 - \epsilon)\langle -L_{\bar{\Omega}}\omega, \omega \rangle_{L^2} - \epsilon\|\omega\|_{L^2}^2$$

for all $\omega \in W^{2,2}$.

Proof. We apply Theorem 9.1 in [25] to the operators $T := -\epsilon L_{\bar{\Omega}}$ and $V := L_{\bar{\Omega}} - L_{\bar{\Omega}'}$. Without loss of generality we may assume that $L_{\bar{\Omega}'}$ is symmetric; otherwise we replace it by the symmetric operator $\frac{1}{2}(L_{\bar{\Omega}'} + L_{\bar{\Omega}'}^*)$, where $*$ denotes the formal adjoint taken with respect to $\bar{\Omega}$. Then by elliptic regularity for the operator T the estimate

$$\|V\omega\|_{L^2} \leq a\|\omega\|_{L^2} + b\|T\omega\|_{L^2}$$

holds for arbitrarily small $a, b > 0$ if $\bar{\Omega}'$ is sufficiently close to $\bar{\Omega}$ with respect to the $W^{k,2}$ -norm. In particular, V will be T -bounded with T -bound less than 1 (see Chapter 5.1 in [25] for the notion of T -boundedness). Since T is non-negative we obtain with $T + V = -L_{\bar{\Omega}'} + (1 - \epsilon)L_{\bar{\Omega}}$ that

$$-L_{\bar{\Omega}'} + (1 - \epsilon)L_{\bar{\Omega}} \geq -\epsilon$$

or equivalently

$$\langle -L_{\bar{\Omega}'}, \omega, \omega \rangle_{L^2} \geq (1 - \epsilon)\langle -L_{\bar{\Omega}}\omega, \omega \rangle_{L^2} - \epsilon\|\omega\|_{L^2}^2$$

for all $\omega \in W^{2,2}$ and $\bar{\Omega}'$ sufficiently close to $\bar{\Omega}$. \square

The family $(L_{\bar{\Omega}'})_{\bar{\Omega}' \in \mathcal{S}_{\bar{\Omega}}}$ is a smooth family of elliptic operators, and hence gives rise to a smooth family of Fredholm operators $L_{\bar{\Omega}'} : W^{2,2} \rightarrow L^2$. Since $\mathcal{S}_{\bar{\Omega}}$ is a smooth (finite-dimensional) manifold and $\mathcal{S}_{\bar{\Omega}} \subset \tilde{Q}_{\bar{\Omega}}^{-1}(0)$, we have $T_{\bar{\Omega}'}\mathcal{S}_{\bar{\Omega}} \subset \ker L_{\bar{\Omega}'}$. Furthermore $T_{\bar{\Omega}}\mathcal{S}_{\bar{\Omega}} = \ker L_{\bar{\Omega}}$ by Proposition 7.5. Usual Fredholm theory implies that $T_{\bar{\Omega}'}\mathcal{S}_{\bar{\Omega}} = \ker L_{\bar{\Omega}'}$ for $\bar{\Omega}'$ sufficiently close to $\bar{\Omega}$, and hence the kernels and ranges of $L_{\bar{\Omega}'}$ form smooth vector bundles (the latter infinite-dimensional). Interpreting $(\ker L_{\bar{\Omega}'})^\perp$ as the fibre at $\bar{\Omega}'$ of the normal bundle of $\mathcal{S}_{\bar{\Omega}}$ with respect to the L^2 -metric, we obtain the following statement.

Lemma 8.9 (Orthogonal Projection). *There exists $\epsilon > 0$ such that if $\|\Omega - \bar{\Omega}\|_{W^{k,2}} < \epsilon$, then there exists a unique $\bar{\Omega}' \in \mathcal{S}_{\bar{\Omega}}$ such that $\omega' = \Omega - \bar{\Omega}' \in (\ker L_{\bar{\Omega}'})^\perp$, the orthogonal complement being taken inside L^2 .*

Remark. In the situation of Lemma 8.9 we obtain using the continuity of the embedding $W^{k,2} \hookrightarrow C^2$ that there exist constants $\epsilon_1, \epsilon_2 > 0$ such that $\|\Omega - \bar{\Omega}\|_{W^{k,2}} < \epsilon$ implies that $\|\omega'\|_{C^2} < \epsilon_1$ and $\|\bar{\Omega}' - \bar{\Omega}\|_{C^2} < \epsilon_2$ with $\epsilon_1, \epsilon_2 \rightarrow 0$ as $\epsilon \rightarrow 0$.

Proposition 8.10. For all $\kappa > 0$ there exists $\epsilon > 0$ such that if $\|\Omega - \bar{\Omega}\|_{W^{k,2}} < \epsilon$, then

$$\|R_{\bar{\Omega}'}(\omega')\|_{L^2} \leq \kappa \|L_{\bar{\Omega}'}\omega'\|_{L^2}$$

with $\bar{\Omega}' \in \mathcal{S}_{\bar{\Omega}}$ and $\omega' = \Omega - \bar{\Omega}'$ as in Lemma 8.9.

Proof. Elliptic regularity implies that

$$L_{\bar{\Omega}'} : (\ker L_{\bar{\Omega}'})^\perp \cap W^{2,2} \rightarrow \text{im } L_{\bar{\Omega}'} \subset L^2$$

is a Banach space isomorphism, the orthogonal complement being taken inside L^2 . In particular, there exists $C > 0$ such that $\|\omega\|_{W^{2,2}} \leq C \|L_{\bar{\Omega}'}\omega\|_{L^2}$ for all $\omega \in (\ker L_{\bar{\Omega}'})^\perp \cap W^{2,2}$. This constant C can be chosen uniform in $\bar{\Omega}'$, since $L_{\bar{\Omega}'}$ is a smooth family of elliptic operators parametrised by a finite-dimensional, hence locally compact manifold. Using Lemma 8.7 we get

$$\|R_{\bar{\Omega}'}(\omega')\|_{L^2} \leq \epsilon_1 C' \|\omega'\|_{W^{2,2}} \leq \epsilon_1 C C' \|L_{\bar{\Omega}'}\omega'\|_{L^2}$$

with $\epsilon_1, \epsilon_2 > 0$ as in the remark following Lemma 8.9 and some constant $C' = C'(\epsilon_1, \epsilon_2) > 0$. For $\epsilon_1 \ll 1$, this constant C' can in fact be chosen independent of ϵ_1 in the above estimate, which implies the result. \square

As a consequence, we obtain that locally $\tilde{Q}_{\bar{\Omega}}^{-1}(0)$ coincides with $\mathcal{S}_{\bar{\Omega}}$, more precisely we have the following.

Corollary 8.11. There exists $\epsilon > 0$ such that if $\|\Omega - \bar{\Omega}\|_{W^{k,2}} < \epsilon$, then $\tilde{Q}_{\bar{\Omega}}(\Omega) = 0$ implies that $\Omega \in \mathcal{S}_{\bar{\Omega}}$.

Proof. Let $\bar{\Omega}' \in \mathcal{S}_{\bar{\Omega}}$ be chosen according to Lemma 8.9, i.e. such that $\omega' = \Omega - \bar{\Omega}'$ satisfies $\omega' \in (\ker L_{\bar{\Omega}'})^\perp$. We write $\tilde{Q}_{\bar{\Omega}}(\Omega) = L_{\bar{\Omega}'}\omega' + R_{\bar{\Omega}'}(\omega')$. Now $\tilde{Q}_{\bar{\Omega}}(\Omega) = 0$ implies that $\|R_{\bar{\Omega}'}(\omega')\|_{L^2} = \|L_{\bar{\Omega}'}\omega'\|_{L^2}$, which in view of Proposition 8.10 (with e.g. $\kappa = \frac{1}{2}$) yields $R_{\bar{\Omega}'}(\omega') = L_{\bar{\Omega}'}\omega' = 0$. Since ω' was chosen orthogonal to the kernel of $L_{\bar{\Omega}'}$, we obtain $\omega' = 0$, hence $\Omega \in \mathcal{S}_{\bar{\Omega}}$. \square

Let $\tilde{\Omega}_t$ be a Dirichlet–DeTurck flow solution on $[0, T]$ and let $\tilde{Q}_t := \tilde{Q}_{\bar{\Omega}}(\tilde{\Omega}_t)$. Then we have $\frac{\partial}{\partial t} \tilde{\Omega}_t = \tilde{Q}_t$ and \tilde{Q}_t satisfies the linearised flow equation

$$\frac{\partial}{\partial t} \tilde{Q}_t = L_{\tilde{\Omega}_t} \tilde{Q}_t \tag{53}$$

with $L_{\tilde{\Omega}_t} = D\tilde{\Omega}_t \tilde{Q}_{\bar{\Omega}}$. This is a linear parabolic equation with time-dependent coefficients. We view the operator $L_{\tilde{\Omega}_t}$ as a (non-symmetric) perturbation of the symmetric operator $L_{\bar{\Omega}}$, to which in particular Lemma 8.8 applies.

Lemma 8.12 (L^2 -almost Orthogonality of \tilde{Q}_t). For all $\kappa > 0$ there exists $\epsilon > 0$ such that if $\|\tilde{\Omega}_t - \bar{\Omega}\|_{W^{k,2}} < \epsilon$ for all $t \in [0, T]$, then

$$|\langle \tilde{Q}_t, \omega \rangle_{L^2}| \leq \kappa \|\tilde{Q}_t\|_{L^2} \|\omega\|_{L^2}$$

for all $\omega \in \ker L_{\bar{\Omega}}$.

Proof. Let $\tilde{\Omega}_t \in \mathcal{S}_{\tilde{\Omega}}$ be chosen according to Lemma 8.9, i.e. such that $\omega'_t = \tilde{\Omega}_t - \bar{\Omega}_t$ satisfies $\omega'_t \in (\ker L_{\tilde{\Omega}_t})^\perp$. We write $\tilde{Q}_t = L_{\tilde{\Omega}_t} \omega'_t + R_{\tilde{\Omega}_t}(\omega'_t)$. First we observe that due to the symmetry of $L_{\tilde{\Omega}}$ its range is L^2 -orthogonal to its kernel. Since the ranges of the operators $L_{\tilde{\Omega}_t}$ form a smooth vector bundle for $\tilde{\Omega}' \in \mathcal{S}_{\tilde{\Omega}}$, this implies that they are L^2 -almost orthogonal to $\ker L_{\tilde{\Omega}}$ in the above sense, in particular

$$|\langle L_{\tilde{\Omega}_t} \omega'_t, \omega \rangle_{L^2}| \leq \kappa \|L_{\tilde{\Omega}_t} \omega'_t\|_{L^2} \|\omega\|_{L^2}$$

for all $\omega \in \ker L_{\tilde{\Omega}}$, if $\epsilon > 0$ is chosen sufficiently small. Proposition 8.10 then shows that

$$|\langle R_{\tilde{\Omega}_t}(\omega'_t), \omega \rangle_{L^2}| \leq \|R_{\tilde{\Omega}_t}(\omega'_t)\|_{L^2} \|\omega\|_{L^2} \leq \kappa \|L_{\tilde{\Omega}_t} \omega'_t\|_{L^2} \|\omega\|_{L^2}$$

and hence

$$\begin{aligned} |\langle \tilde{Q}_t, \omega \rangle_{L^2}| &\leq |\langle L_{\tilde{\Omega}_t} \omega'_t, \omega \rangle_{L^2}| + |\langle R_{\tilde{\Omega}_t}(\omega'_t), \omega \rangle_{L^2}| \\ &\leq 2\kappa \|L_{\tilde{\Omega}_t} \omega'_t\|_{L^2} \|\omega\|_{L^2} \leq 2\kappa(1 - \kappa)^{-1} \|\tilde{Q}_t\|_{L^2} \|\omega\|_{L^2} \end{aligned}$$

for all $\omega \in \ker L_{\tilde{\Omega}}$, again if $\epsilon > 0$ is chosen sufficiently small. This proves the result. \square

Lemma 8.13 (*L^2 -exponential Decay of \tilde{Q}_t*). *There exist $\epsilon > 0$ and $\lambda = \lambda(\tilde{\Omega}) > 0$ such that if $\|\tilde{\Omega}_t - \bar{\Omega}\|_{W^{k,2}} < \epsilon$ for all $t \in [0, T]$, then*

$$\|\tilde{Q}_t\|_{L^2}^2 \leq e^{-\lambda t} \|\tilde{Q}_0\|_{L^2}^2$$

for all $t \in [0, T]$.

Proof. Using Eq. (53) we obtain

$$\frac{d}{dt} \frac{1}{2} \|\tilde{Q}_t\|_{L^2}^2 = \langle L_{\tilde{\Omega}_t} \tilde{Q}_t, \tilde{Q}_t \rangle_{L^2}.$$

Let $\lambda_1 > 0$ be the first (positive) eigenvalue of $-L_{\tilde{\Omega}}$. Then one has $\langle -L_{\tilde{\Omega}} \omega, \omega \rangle_{L^2} \geq \lambda_1 \|\omega\|_{L^2}^2$ for all $\omega \in (\ker L_{\tilde{\Omega}})^\perp$, and hence, using Lemma 8.12, that

$$\langle -L_{\tilde{\Omega}} \tilde{Q}_t, \tilde{Q}_t \rangle_{L^2} \geq \frac{3\lambda_1}{4} \|\tilde{Q}_t\|_{L^2}^2$$

for $\epsilon > 0$ sufficiently small. Using Lemma 8.8 one gets

$$\langle -L_{\tilde{\Omega}_t} \tilde{Q}_t, \tilde{Q}_t \rangle_{L^2} \geq (1 - \epsilon') \frac{3\lambda_1}{4} \|\tilde{Q}_t\|_{L^2}^2 - \epsilon' \|\tilde{Q}_t\|_{L^2}^2 \geq \frac{\lambda_1}{2} \|\tilde{Q}_t\|_{L^2}^2$$

if $\epsilon, \epsilon' > 0$ are chosen sufficiently small. Finally we get

$$\frac{d}{dt} \frac{1}{2} \|\tilde{Q}_t\|_{L^2}^2 \leq -\lambda \|\tilde{Q}_t\|_{L^2}^2$$

with $\lambda := \lambda_1/2$ and $\epsilon > 0$ sufficiently small. Now Gronwall's lemma implies the result. \square

Parabolic regularity theory now yields higher derivative estimates.

Proposition 8.14 (*$W^{k,2}$ -exponential Decay of \tilde{Q}_t*). *There exist $\epsilon > 0$ and $C, \lambda > 0$ such that if $\|\tilde{\Omega}_t - \bar{\Omega}\|_{W^{k,2}} < \epsilon$ for all $t \in [0, T]$, then*

$$\|\tilde{Q}_t\|_{W^{k,2}}^2 \leq C e^{-\lambda t}$$

for all $t \in [0, T]$.

Proof. If $\epsilon > 0$ and $\lambda > 0$ are chosen according to Lemma 8.13, then for $0 \leq t_0 < t_1 \leq T$ we get

$$\int_{t_0}^{t_1} \|\tilde{Q}_\tau\|_{L^2}^2 d\tau \leq \|\tilde{Q}_0\|_{L^2}^2 \int_{t_0}^{t_1} e^{-\lambda\tau} d\tau \leq C_0 e^{-\lambda t_0}$$

for $C_0 := \|\tilde{Q}_0\|_{L^2}^2/\lambda$. It is easy to see that the estimate of Corollary 8.4 remains true for $P_t := \partial_t - L_{\tilde{\Omega}_t}$ (with a constant $C_s = C_s(\epsilon, \delta, \tilde{\Omega}) > 0$) if $\epsilon > 0$ is sufficiently small. In particular we get for $s := (k - 1)/2$ that

$$\|\tilde{Q}\|_{V^{s+1}[t_0+\delta, t_1]}^2 \leq C_s \|\tilde{Q}\|_{V^0[t_0, t_1]}^2 \leq C_s \int_{t_0}^{t_1} \|\tilde{Q}_\tau\|_{L^2}^2 d\tau$$

since $P_t \tilde{Q}_t = 0$. Combining these two estimates and using the trace theorem as in the proof of Corollary 8.6 yield the result. \square

We will now describe the choice of the constants:

1. Choose $\epsilon > 0$ such that
 - (a) $\|\tilde{\Omega}_0 - \bar{\Omega}\|_{W^{k,2}} < \epsilon$ implies that the Dirichlet–DeTurck flow at $\bar{\Omega}$ with initial condition $\tilde{\Omega}_0$ exists on $[0, 1]$. This is possible using Corollary 8.6.
 - (b) $\|\tilde{\Omega}_t - \bar{\Omega}\|_{W^{k,2}} < \epsilon$ for $t \in [0, T]$ yields

$$\|\tilde{Q}_t\|_{W^{k,2}} \leq C e^{-\lambda t}$$
 on $[0, T]$ for any $T < T_{max}$, where T_{max} is the maximal life-time of the flow. This is possible using Proposition 8.14.
2. Choose $T > 0$ such that $\int_T^\infty C e^{-\lambda t} dt < \epsilon/2$ with C and λ as above.
3. Choose $\delta > 0$ such that

$$\|\tilde{\Omega}_0 - \bar{\Omega}\|_{W^{k,2}} < \delta$$

implies that the Dirichlet–DeTurck flow at $\bar{\Omega}$ with initial condition $\tilde{\Omega}_0$ exists on $[0, T]$ with $\|\tilde{\Omega}_t - \bar{\Omega}\|_{W^{k,2}} < \epsilon/2$. This again is possible using Corollary 8.6.

The proof of Theorem 8.1 may then be finished as follows.

Let the initial condition $\tilde{\Omega}_0$ be given satisfying

$$\|\tilde{\Omega}_0 - \bar{\Omega}\|_{W^{k,2}} < \delta.$$

Let $T_{max} > T$ such that $[0, T_{max})$ is the maximal time interval on which the Dirichlet–DeTurck flow with initial condition $\tilde{\Omega}_0$ exists. Suppose now that $T_{max} < \infty$. For $T \leq t < T_{max}$ one has

$$\tilde{\Omega}_t = \tilde{\Omega}_T + \int_T^t \frac{\partial}{\partial \tau} \tilde{\Omega}_\tau d\tau$$

and hence, as $\frac{\partial}{\partial t} \tilde{\Omega}_t = \tilde{Q}_t$,

$$\begin{aligned} \|\tilde{\Omega}_t - \tilde{\Omega}_T\|_{W^{k,2}} &= \left\| \int_T^t \tilde{Q}_\tau d\tau \right\|_{W^{k,2}} \\ &\leq \int_T^t \|\tilde{Q}_\tau\|_{W^{k,2}} d\tau \\ &\leq \int_T^\infty C e^{-\lambda\tau} d\tau < \epsilon/2. \end{aligned}$$

Using this for $t_0 = \max\{T, T_{max} - 1/2\}$ and bearing the assumption $\|\tilde{\Omega}_T - \bar{\Omega}\|_{W^{k,2}} < \epsilon/2$ in mind, we obtain

$$\|\tilde{\Omega}_{t_0} - \bar{\Omega}\|_{W^{k,2}} \leq \|\tilde{\Omega}_{t_0} - \tilde{\Omega}_T\|_{W^{k,2}} + \|\tilde{\Omega}_T - \bar{\Omega}\|_{W^{k,2}} < \epsilon.$$

Therefore the Dirichlet–DeTurck flow can be continued at least up to time $t_0 + 1 \geq T_{max} + 1/2$, a contradiction. Hence $T_{max} = \infty$ and we have established long-time existence together with the estimate $\|\tilde{\Omega}_t - \bar{\Omega}\|_{W^{k,2}} < \epsilon$ for all $t \in [0, \infty)$.

Finally we set

$$\tilde{\Omega}_\infty = \tilde{\Omega}_0 + \int_0^\infty \frac{\partial}{\partial t} \tilde{\Omega}_t dt.$$

Using Proposition 8.14 we see that this integral converges, i.e. that $\tilde{\Omega}_t$ converges to $\tilde{\Omega}_\infty$ with respect to the $W^{k,2}$ -norm.

Furthermore, since $\tilde{Q}_{\tilde{\Omega}} : W^{k,2} \rightarrow W^{k-2,2}$ is continuous (even differentiable), we obtain

$$0 = \lim_{t \rightarrow \infty} \tilde{Q}_t = \tilde{Q}_{\tilde{\Omega}}(\tilde{\Omega}_\infty),$$

in other words, $\tilde{\Omega}_\infty \in \tilde{Q}_{\tilde{\Omega}}^{-1}(0)$. Since locally $\tilde{Q}_{\tilde{\Omega}}^{-1}(0) = \mathcal{S}_{\tilde{\Omega}}$ according to Corollary 8.11, we obtain $\tilde{\Omega}_\infty \in \mathcal{S}_{\tilde{\Omega}}$. In particular, $\tilde{\Omega}_\infty$ is a torsion-free G_2 -form. This finishes the proof of Theorem 8.1.

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Appendix. G_2 -differential operators

The G_2 -differential operators d_q^p we use in the text are given as follows:

$$\begin{aligned}
 d_q^1 : \Omega_1 &\rightarrow \Omega_q & \begin{cases} d_1^1 f \equiv 0 \\ d_{14}^1 f \equiv 0 \\ d_7^1 f = df \\ d_{27}^1 f \equiv 0 \end{cases} \\
 d_q^7 : \Omega_1 &\rightarrow \Omega_q & \begin{cases} d_1^7 \alpha = \delta_\Omega \alpha = \frac{1}{4} \star_\Omega d(\star_\Omega(\alpha \wedge \Omega) \wedge \Omega) \\ d_7^7 \alpha = \star_\Omega(d\alpha \wedge \star_\Omega \Omega) = -\frac{1}{2} \star_\Omega(\delta_\Omega(\alpha \wedge \Omega) \wedge \Omega) \\ d_{14}^7 \alpha = [d\alpha]_{14} \\ d_{27}^7 \alpha = [d \star_\Omega(\alpha \wedge \star_\Omega \Omega)]_{27} = [\delta_\Omega(\alpha \wedge \Omega)]_{27} \end{cases} \\
 d_q^{14} : \Omega_1 &\rightarrow \Omega_q & \begin{cases} d_1^{14} \beta \equiv 0 \\ d_{14}^{14} \beta \equiv 0 \\ d_7^{14} \beta = -\star_\Omega([d\beta]_7 \wedge \Omega) \\ d_{27}^{14} \beta = [d\beta]_{27} \end{cases}
 \end{aligned}$$

$$d_q^{27} : \Omega_1 \rightarrow \Omega_q \quad \begin{cases} d_1^{27} \gamma \equiv 0 \\ d_{14}^{27} \gamma = [\delta_{\Omega} \gamma]_{14} \\ d_7^{27} \gamma = \star_{\Omega}(\delta_{\Omega} \gamma \wedge \star_{\Omega} \Omega) \\ d_{27}^{27} \gamma = \star_{\Omega}[d\gamma]_{27} \end{cases}$$

References

- [1] T. Aubin, *Some Nonlinear Problems in Riemannian Geometry*, Springer, Berlin, 1997.
- [2] A. Besse, *Einstein Manifolds*, Springer, Berlin, 1987.
- [3] E. Bonan, Sur les variétés Riemanniennes à groupe d'holonomie G_2 ou $\text{Spin}(7)$, *C. R. Acad. Sci. Paris* 262 (1966) 127–129.
- [4] R. Bryant, Metrics with exceptional holonomy, *Ann. of Math.* 126 (1987) 525–576.
- [5] R. Bryant, Some remarks on G_2 -structures, in: *Gökova Geometry/Topology Conference (GGT)*, pp. 75–109, Gökova, 2006.
- [6] R. Bryant, F. Xu, Laplacian flow for closed G_2 -structures: Short time behavior, Preprint 2011, [arXiv:1101.2004](https://arxiv.org/abs/1101.2004).
- [7] S. Chiossi, S. Salamon, The Intrinsic Torsion of $SU(3)$ and G_2 Structures, World science publishing, 2002, pp. 115–133.
- [8] D. DeTurck, Deforming metrics in the direction of their Ricci tensors, *J. Differential Geom.* 28 (1983) 157–162.
- [9] D. Ebin, The moduli space of Riemannian metrics, in: *Global Analysis*, in: *Proceedings of Symposia in Pure Mathematics*, vol. 15, AMS, 1968, pp. 11–40.
- [10] J. Eells, J. Sampson, Harmonic maps of Riemannian manifolds, *Amer. J. Math.* 86 (1) (1964) 109–160.
- [11] M. Fernández, A. Gray, Riemannian manifolds with structure group G_2 , *Ann. Mat. Pura Appl.* 132 (1982) 19–45.
- [12] A. Friedman, *Partial Differential Equations of Parabolic Type*, Prentice Hall, Englewood Cliffs, 1964.
- [13] R. Hamilton, *Harmonic Maps of Manifolds with Boundary*, in: *Lecture Notes in Mathematics*, vol. 471, Berlin, 1975.
- [14] N. Hitchin, Stable forms and special metrics, *Contemp. Math.* 288 (2001) 70–89.
- [15] G. Huisken, A. Polden, Geometric evolution equations for hypersurfaces, in: S. Hildebrandt, M. Struwe (Eds.), *in: Calculus of Variations and Geometric Evolution Problems LNM*, vol. 1713, Springer, Berlin, 1999, pp. 45–84.
- [16] D. Joyce, *Compact Manifolds with Special Holonomy*, OUP, Oxford, 2000.
- [17] S. Karigiannis, Flows of G_2 -structures, *Q. J. Math.* 60 (4) (2009) 487–522.
- [18] S. Karigiannis, Some notes on G_2 and $\text{Spin}(7)$ geometry, in: *Recent Advances in Geometric Analysis*, in: *Adv. Lect. Math. (ALM)*, vol. 11, Int. Press, Somerville, MA, 2010, pp. 129–146.
- [19] O. Ladyzhenskaya, V. Solonnikov, N. Ural'tseva, *Linear and Quasilinear Parabolic Equations*, Nauka, Moscow, 1967.
- [20] G. Perelman, The entropy formula for the Ricci flow and its geometric applications, Preprint 2002, [arXiv:math/0211159v1](https://arxiv.org/abs/math/0211159v1).
- [21] N. Šešum, Linear and dynamical stability of Ricci-flat metrics, *Duke Math. J.* 133 (1) (2006).
- [22] M. Taylor, *Pseudodifferential Operators and Nonlinear PDE*, Birkhäuser, Basel, 1991.
- [23] M. Taylor, Existence and regularity of isometries, *Trans. Amer. Math. Soc.* 358 (6) (2006) 2415–2423.
- [24] P. Topping, Lectures on the Ricci flow, in: *LMS Lecture Note Series*, vol. 325, Cambridge University Press, Cambridge, 2006.
- [25] J. Weidmann, *Linear Operators in Hilbert Spaces*, Springer, Berlin, 1980.
- [26] F. Xu, R. Ye, Existence, convergence and limit map of the Laplacian flow, Preprint 2009, [arXiv:0912.0074](https://arxiv.org/abs/0912.0074).