Second neighbourhoods of strongly regular graphs

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Abstract


Some antipodal distance-regular graphs of diameter three arise as the graph induced by the vertices at distance two from a given vertex in a strongly regular graph. We show that if every vertex in a strongly regular graph $G$ has this property, then $G$ is the noncollinearity graph of a special type of semipartial geometry. As these semipartial geometries have all been classified, we obtain a list of the antipodal distance-regular graphs of diameter three that can arise in this way.

1. Introduction

Distance-regular graphs may be divided into the two classes of primitive and imprimitive graphs. (Definitions for all terms may be found in the next section.) Primitive distance-regular graphs have been extensively studied [4], but much less is known about the imprimitive case. Smith [13] showed that imprimitive distance-regular graphs of valency at least three must either be antipodal or bipartite (or both). This work arose whilst studying antipodal distance-regular graphs of diameter three, which are necessarily covers of complete graphs. It was motivated by the following two observations. Payne and Thas [12] noted that a generalized quadrangle of order $(q - 1, q + 1)$ with a spread can be constructed

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from a generalized quadrangle of order $q$ with a regular point, and Brouwer [3]
demonstrates that an antipodal distance-regular graph of diameter three can be
constructed from a generalized quadrangle with a spread. Hence given a
generalized quadrangle $Q$ of order $q$ with a regular point $x$, we may construct an
antipodal distance-regular graph of diameter three. Examination of this construc-
tion shows that the graph constructed is precisely the graph induced by the
vertices at distance two from $x$ in the collinearity graph of $Q$ (that is, the second
neighbourhood of the regular point). In addition, Gardiner [8] proved that the
second neighbourhood of any vertex in a Moore graph of diameter two is an
antipodal distance-regular graph of diameter three.

The collinearity graph of a generalized quadrangle is strongly regular, as is a
Moore graph of diameter two, so these examples prompt the question: are there
further examples of strongly regular graphs with antipodal distance-regular covers
as the second neighbourhood of a vertex?

A further simple infinite family of examples is provided by taking the
complement of the line graph of a complete graph. The remainder of this paper is
devoted to providing a reasonably complete solution to this problem. We start by
showing that the second neighbourhood of a vertex in a strongly regular graph $G$
must be connected and have diameter at most three, unless $G$ is a complete
multipartite graph. Then we demonstrate that if the second neighbourhood $G_2(v)$
of any one vertex $v$ is distance-regular of diameter three, then it must be
antipodal. Furthermore, the parameters of $G$ uniquely determine the parameters
of $G_2(v)$, and conversely. Finally, we make the stronger assumption that the
second neighbourhood of every vertex is an antipodal distance-regular graph, and
under that assumption we show that $G$ is the noncollinearity graph of a
semipartial geometry, with restricted parameters. These semipartial geometries
have all been classified by Hall [10], and therefore we can read off from his list all
the antipodal distance-regular covers that arise in this way.

2. Definitions and notation

A connected graph $G$ is distance-regular if there are numbers $p_{ij}^k$ such that for
any two vertices $u$ and $v$ at distance $k$, there are $p_{ij}^k$ vertices at distance $i$ from $u$
and distance $j$ from $v$ (that is, the number does not depend on which particular
pair of vertices at distance $k$ is chosen). The numbers $p_{ij}^k$ are called the
intersection numbers of $G$, and are completely determined by the subset
$b_i = p_{i+1,1}^k$ and $c_i = p_{i-1,1}^k$. We record these numbers in the intersection array
$\{b_0, b_1, \ldots, b_d; c_1, c_2, \ldots, c_d\}$ (where $d$ is the diameter of $G$). Notice that $b_0$
is the valency of $G$ (so $G$ must therefore be regular) and that $c_1 = 1$. A strongly
regular graph is a distance-regular graph of diameter two. In this special case we
follow commonly accepted practice, and we do not use an intersection array to
represent the intersection numbers. Instead, a strongly regular graph $G$ is
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described as having parameters \((n, k, \lambda, \mu)\) where \(n\) is the number of vertices of \(G\), \(k\) is the valency of \(G\), \(\lambda\) is the number of common neighbours of any pair of adjacent vertices, and \(\mu\) is the number of common neighbours of any pair of non-adjacent vertices. Further details regarding strongly regular and distance-regular graphs may be found in \([1, 2, 4, 5]\).

A graph \(G\) of diameter \(d\) is antipodal if whenever any two vertices \(u, v\) are both at distance \(d\) from a third vertex \(w\), then they are also at distance \(d\) from each other. In an antipodal graph the relation 'is at distance \(d\) from' is an equivalence relation, and hence the vertices may be partitioned into equivalence classes called fibres. If \(G\) is also distance regular and has diameter at least three, then it can be shown that between any two fibres there are either no edges or there is a matching. We may then define a graph \(H\) whose vertices are the fibres of \(G\) and where two vertices are adjacent if there is a matching between the corresponding fibres of \(G\). We say that \(G\) covers \(H\) or that \(G\) is an \(r\)-fold cover of \(H\), where \(r\) is the common size of the fibres. If \(d = 3\) then \(G\) necessarily covers a complete graph. An antipodal distance-regular graph of diameter three has intersection array of the form \(\{k, mc_2, 1; 1, c_2, k\}\). Such a graph has valency \(k\) and fibres of size \(m + 1\) and therefore it is an \((m + 1)\)-fold cover of \(K_{k+1}\). The following lemma \([9]\) gives a useful way for checking whether a graph is an antipodal distance-regular graph of diameter three. (The proof is a straightforward exercise.)

Lemma 2.1. A graph \(G\) is an antipodal distance-regular graph of diameter three if and only if the following conditions hold:

1. the vertex set can be partitioned into \(k + 1 > 2\) independent sets of size \(m + 1 > 1\), called fibres;
2. there is a matching between any two fibres;
3. there is an integer \(c_2 > 0\) such that two non-adjacent vertices in different fibres have \(c_2\) common neighbours.

Given a graph \(G\) of diameter \(d\), we may define the 'distance graphs' \(G_1, G_2, \ldots, G_d\), where the vertices of \(G_i\) are the vertices of \(G\) and two vertices are adjacent in \(G_i\) if and only if they are at distance \(i\) in \(G\). A distance-regular graph is called primitive if all the distance graphs are connected, otherwise imprimitive. If \(G\) is bipartite then \(G_2\) is disconnected, and if \(G\) is antipodal then \(G_d\) is disconnected. Smith \([13]\) demonstrated that these are the only ways in which a distance-regular graph may be imprimitive (and they may occur simultaneously).

Given a graph \(G\) and a vertex \(v\), we denote the set of neighbours of \(v\) by \(G_i(v)\) and in general the set of vertices at distance \(i\) from \(v\) by \(G_i(v)\). (Note: It is important not to confuse \(G_i(v)\), which is a subset of the vertices of \(G\), with \(G_i\), which is a graph defined on the same vertex set as \(G\).) We shall refer to both \(G_2(v)\) and the graph induced by \(G_2(v)\) as the second neighbourhood of \(v\). The distance between two vertices \(u\) and \(v\) is given by \(d_G(u, v)\).
There are two types of partial linear space that play an important role in this paper. A *partial linear space* is a point/line incidence structure with the property that any two points lie on at most one line (if every two points lie on exactly one line then it is a linear space). A *generalized quadrangle* of order \((s, t)\) is a partial linear space satisfying

1. Each line contains \(1 + s\) points, \(s \geq 1\);
2. Each point is on \(1 + t\) lines, \(t \geq 1\);
3. If a point \(p\) is not incident with a line \(L\) then there is precisely one point on \(L\) collinear with \(p\).

A *semipartial geometry* \(\text{SPG}(s, t, \alpha, \mu)\) is a partial linear space satisfying

1. Each line contains \(1 + s\) points, \(s \geq 1\);
2. Each point is on \(1 + t\) lines, \(t \geq 1\);
3. If a point \(p\) is not incident with a line \(L\) then there are 0 or \(\alpha > 0\) points on \(L\) collinear with \(p\);
4. For any two noncollinear points there are \(\mu > 0\) points collinear with both.

A simple relevant example of a semipartial geometry is to take the points to be all the unordered pairs from a set \(X\), and the lines to be the sets of three unordered pairs contained in any three element subset of \(X\). Then it is clear that each line contains three points, each point lies on \(n - 2\) lines, and that a point not on a line is collinear with either two or no points on a line (depending on whether it has a common point with the three element subset defining the line or not, respectively). Furthermore, there are four points mutually collinear to any two noncollinear points. Thus this is an \(\text{SPG}(2, n - 3, 2, 4)\).

The *collinearity graph* of any partial linear space is a graph with vertices the points of the partial linear space, and with two points adjacent if and only if they are collinear. The noncollinearity graph is the complement of the collinearity graph.

Information about generalized quadrangles can be found in [12], about semipartial geometries in [6], and about all other allusions to geometric notions in [7, 11].

### 3. Main results

Throughout this section \(G\) will refer to an \((n, k, \lambda, \mu)\) strongly regular graph.

**Lemma 3.1.** For any \(u\) in \(V(G)\), if \(G_2(u)\) is disconnected, then it contains no edges and \(G\) is a complete multipartite graph.

**Proof.** If \(v\) and \(w\) are two vertices in different components of \(G_2(u)\), then they are adjacent to the same \(\mu\) vertices in \(G_1(u)\). Hence if \(G_2(u)\) is disconnected, every vertex in \(G_2(u)\) is adjacent to the same \(\mu\) vertices in \(G_1(u)\), and thus every
vertex in \( G_2(u) \) is adjacent to all \( k \) vertices in \( G_1(u) \). The complement of \( G \) is a union of complete graphs and so \( G \) is a complete multipartite graph.  

From now on we shall assume that \( G \) is not a complete multipartite graph.

Define an equivalence relation \( \equiv \) on the vertices of \( G_2(u) \) by the following rule: \( v \equiv w \) if and only if \( v \) and \( w \) have \( \mu \) common neighbours in \( G_1(u) \). This equivalence relation has the properties

\[
d_{G_2(u)}(v, w) > 2 \Rightarrow v \equiv w
\]

and

\[
d_{G_2(u)}(v, w) = 2 \Rightarrow v \not\equiv w.
\]

**Lemma 3.2.** If \( u \) is in \( V(G) \) then \( G_2(u) \) has diameter at most 3.

**Proof.** Suppose that \( G_2(u) \) contains a geodetic path of length four, say \( v_0v_1v_2v_3v_4 \). Then \( v_0 \equiv v_3 \), \( v_0 \equiv v_4 \) and \( v_4 \equiv v_1 \), so therefore \( v_1 \equiv v_3 \), which is a contradiction as they are at distance two in \( G_2(u) \). \( \square \)

**Lemma 3.3.** For any \( u \) in \( V(G) \), if \( G_2(u) \) is distance regular of diameter three then it is antipodal.

**Proof.** Assume that \( G_2(u) \) is primitive. Then consider the graph \( H \) with vertices \( V(G_2(u)) \) and where two vertices are adjacent if their distance (in \( G_2(u) \)) is three. As \( G_2(u) \) is primitive, this graph is connected, and every vertex of \( H \) is in the same equivalence class of \( \equiv \). Thus every vertex of \( G_2(u) \) is adjacent to all \( k \) vertices in \( G_1(u) \) and so \( G \) is complete multipartite.

Now suppose \( G_2(u) \) is a bipartite graph \( B \), such that for one vertex \( v \in V(B) \) there are two distinct vertices \( w \) and \( x \) in \( B_3(v) \). Now as \( B \) is bipartite, \( w \) and \( x \) have distance two in \( B \). However they are both at distance three (in \( B \)) from \( v \). Therefore \( v \equiv w \), \( v \equiv x \) and thus \( w \equiv x \) which is a contradiction. So \( B_3(v) \) contains no more than one vertex and hence \( B \) is antipodal (and in fact is a double cover). \( \square \)

Thus, if \( G_2(u) \) is a distance-regular graph of diameter three, then it is antipodal. The next theorem shows that the parameters of \( G \) determine the intersection array of \( G_2(u) \).

**Theorem 3.4.** Let \( G \) be an \((n, k, \lambda, \mu)\) strongly regular graph such that for \( u \in V(G) \), \( G_2(u) \) is an antipodal distance regular graph of diameter three. Then \( G_2(u) \) has the intersection array \( \{k, m, c_2, 1, c_2, k \} \) where

\[
\tilde{k} = k - \mu, \quad m = \frac{k(k - \lambda - 1)}{(k + 1)\mu} + 1 \quad \text{and} \quad c_2 = \frac{\mu + \tilde{k} - \lambda - 1}{m + 1}.
\]
Proof. We set $a_1 = \tilde{k} - mc_2 - 1$ (that is, the number of vertices mutually adjacent in $G_2(u)$ to any pair of adjacent vertices in $G_2(u)$). Then the following identities hold

1. $k(k - \lambda - 1)/\mu = (m + 1)(\tilde{k} + 1)$;
2. $\tilde{k} = k - \mu$;
3. $\lambda - a_1 = \mu - c_2$;
4. $\tilde{k}(\lambda - a_1) + \tilde{k}m(\mu - c_2) + m\mu = \mu(k - \lambda - 2)$.

Identities (1) and (2) follow directly by considering the number of vertices in $G_2(u)$ and the valency of $G_2(u)$ respectively. To prove (3), consider two fibres $F_1$ and $F_2$ of $G_2(u)$. As the vertices in each fibre are mutually at distance three, they all have the same $\mu$ common neighbours in $G_1(u)$. Therefore any two vertices, one from $F_1$ and one from $F_2$, have the same number of common neighbours in $G_1(u)$. We can find vertices $v \in F_1$, $w \in F_2$, $x \in F_2$ such that $v$ is adjacent to $w$ but not adjacent to $x$. As $v$ is adjacent to $w$ they have $a_1$ common neighbours in $G_2(u)$ and hence $\lambda - a_1$ in $G_1(u)$, whereas $v$ and $x$ have $c_2$ common neighbours in $G_2(u)$ and hence $\mu - c_2$ in $G_1(u)$. To prove (4) we will fix a vertex $v \in G_2(u)$ and count triples $(v, x, y)$ such that $x \in G_1(u)$, $v \not= y \in G_2(u)$ and $v$ and $y$ are adjacent to $x$ (in $G$). There are $\mu$ choices for a vertex $x$ adjacent to $y$; each of these vertices are adjacent to $u$, $v$ and to $\lambda$ vertices in $G_1(u)$ thus leaving $k - \lambda - 2$ choices for $y$. Thus there are $\mu(k - \lambda - 2)$ such triples. Alternatively, there are $\tilde{k}$ choices for $y$ at distance one in $G_2(u)$ from $v$ each yielding $\lambda - a_1$ choices for $x$, $\tilde{k}m$ choices for $y$ at distance two in $G_2(u)$ from $v$ each yielding $\mu - c_2$ choices for $x$ and $m$ choices for $y$ at distance three in $G_2(u)$ from $v$ each yielding $\mu$ choices for $x$. Thus (4) is proved. Manipulation of the four identities above gives the stated result. \(\Box\)

We have shown that if the second neighbourhood of any vertex is an antipodal distance-regular graph then we know its parameters. If we assume that the second neighbourhood of every vertex is an antipodal distance-regular graph of diameter three we can say more.

**Theorem 3.5.** Let $G$ be a strongly regular graph such that $G_2(u)$ is a distance-regular graph with intersection array \{$k, mc_2, 1; 1, c_2, \tilde{k}$\} for every vertex $u$ in $V(G)$. Then $G$ is the noncollinearity graph of a semipartial geometry $\text{SPG}(m + 1, \tilde{k}, m + 1, (m + 1)(\tilde{k} + 1 - c_2))$.

**Proof.** Consider the incidence structure $\mathcal{I}$ defined as follows. The points of $\mathcal{I}$ are the vertices of the graph. For any two vertices $u$ and $v$ at distance 2 in $G$, define the line $[uv]$ to be the set of vertices consisting of $u$ and the fibre containing $v$ in $G_2(u)$. It is straightforward to show that this is well-defined. As each fibre contains $m + 1$ points, each line contains $m + 2$ points, and as each fibre of $G_2(x)$ yields one line through $x$, there are $\tilde{k} + 1$ lines through each point. Notice that adjacent vertices in $G$ cannot be collinear in $\mathcal{I}$.
Now consider a line $[uv]$ and a point $w$ not on it. First we suppose that $w \in G_1(u)$. Then $w$ is adjacent to $u$ and is either adjacent to every vertex in the fibre containing $u$ or to none of the vertices in the fibre containing $u$. Therefore it is collinear with either $0$ or $m$ of the points on $[uv]$. Now suppose that $w \in G_2(u)$. Then it is not adjacent to $u$, and as it is in a different fibre to $v$ it is adjacent to precisely one vertex in that fibre. Hence it is collinear with the remaining $m + 1$ points on $[uv]$.

Finally, we consider two noncollinear points $u$ and $v$ (therefore adjacent in $G$), and calculate the number at distance two in $G$ from both. Now $G_2(u)$ contains $\frac{k(k - \lambda - 1)}{\mu}$ vertices of which $k - \lambda - 1$ are adjacent to $v$. Using the identities given in Theorem 3.4, we have

$$\frac{k(k - \lambda - 1)}{\mu} - (k - \lambda - 1) = (m + 1)(k + 1 - c_2).$$

Therefore, our incidence structure satisfies all the conditions of a semipartial geometry with the stated parameters. $\Box$

We notice that the semipartial geometry found in Theorem 3.5 has $\alpha$ equal to $s$. This condition is sufficient for the converse to hold with two minor exceptions. Let $S$ be an SPG($s, t, s, p$). Then if $\mu = s(t + 1)$ the noncollinearity graph of $S$ is a union of complete graphs (and hence there are no second neighbourhoods), and if $s = 1$ then the second neighbourhood of any vertex is a complete graph (a 1-fold cover). Apart from these cases, we have the following result.

**Theorem 3.6.** Let $G$ be the noncollinearity graph of an SPG($s, t, s, p$) with $s > 1$ and $\mu \neq s(t + 1)$. Then $G$ is a strongly regular graph such that the second neighbourhood of every vertex is an antipodal distance-regular graph of diameter three.

**Proof.** Debroey and Thas [6] show that the collinearity graph of an SPG($s, t, s, p$) is a strongly regular graph of valency $s(t + 1)$, such that any two non-adjacent vertices have $\mu$ common neighbours. As $\mu \neq s(t + 1)$ the noncollinearity graph is connected and hence strongly regular. Given any point $p$ we shall use the conditions given in Lemma 1.1 to demonstrate that $G_2(p)$ is an antipodal distance-regular graph of diameter three. The vertices of $G_2(p)$ are precisely the points collinear with $p$—that is the points on the lines through $p$. The points other than $p$ on any one line form an independent set of size $s$ in $G_2(p)$ — the fibres of $G_2(p)$.

Given any two lines $l$ and $m$ through $p$, any point on $l$ is collinear with all but one of the points on $m$ (as it is collinear with $p$ it must be collinear with $s$ of the points on $m$). Therefore in $G$ any vertex in one fibre is adjacent to precisely one vertex in every other fibre, and hence we have matchings between the fibres.
Let \( I \) be any line through \( p \). Any point not collinear with \( p \) is collinear to either no points on \( I \) or every point on \( I \) except \( p \) (corresponding to the vertices in \( G_1(p) \) being adjacent to every vertex in a fibre or none of the vertices in a fibre respectively). Let \( m \) be any further line through \( p \), \( a \) be any point on \( I \) and \( b \) be the unique point on \( m \) not collinear with \( a \). There are \( \mu \) points mutually collinear with \( a \) and \( b \), and clearly they must all lie on lines through \( a \). If \( b \) is collinear with one point on a line through \( a \) then it is collinear with all \( s \) points other than \( a \) on that line. Thus, the \( \mu \) points mutually collinear to \( a \) and \( b \) lie on \( \mu/s \) lines through \( a \) (including of course the line \( ap \) joining \( a \) and \( p \)). Now \( p \) is collinear with every point on the line \( ap \) and with every point except one on the remaining \( \mu/s - 1 \) lines. Thus there is one point on each of \( \mu/s - 1 \) lines that is mutually collinear with \( a \) and \( b \) and not collinear with \( p \) (or equal to \( p \)). Thus if \( v \) and \( w \) are vertices in different fibres of \( G_2(p) \), then there are \( \mu/s - 1 \) vertices in \( G_1(p) \) that are not adjacent to either \( v \) or \( w \). (Recall here that a vertex in \( G_1(p) \) is either adjacent to all or none of the vertices of a fibre.)

As \( G \) is strongly regular every fibre has the same number of neighbours in \( G_1(p) \) and by the above argument, every pair of fibres have the same number of common non-neighbours in \( G_1(p) \). Therefore every pair of fibres has the same number of common neighbours in \( G_1(p) \). So every pair of vertices in different fibres of \( G_2(p) \) have the same number of common neighbours in \( G_1(p) \) and hence as \( G \) is strongly regular the number of common neighbours they have in \( G_2(p) \) depends only on whether the two vertices are adjacent or otherwise. Therefore \( G_2(p) \) is an antipodal distance-regular graph of diameter three. \( \square \)

Semipartial geometries with \( \alpha = s \) have been classified by Hall [10, Theorem 4].

**Theorem 3.7.** The finite partial linear space \((\mathcal{P}, \mathcal{L})\) is a semipartial geometry with \( s = \alpha \) if and only if it is (isomorphic to) one of the following:

1. A transversal design \( \text{TD}(s + 1, t + 1) \);
2. A strongly regular graph with no triangles;
3. \( \mathcal{P} \) is the point set of a Moore graph of diameter two and valency \( q + 1 \) and every line of \( \mathcal{L} \) is the set of \( q + 1 \) points adjacent to some vertex;
4. \( \mathcal{P} \) is the set of all unordered pairs from some finite set \( X \) and a line of \( \mathcal{L} \) is the set of three unordered pairs contained in a three element subset of \( X \);
5. \( \mathcal{P} \) is the set of points outside a nonsingular quadric in a finite odd-dimensional projective space over \( \text{GF}(2) \) and \( \mathcal{L} \) is the set of lines with no points on the quadric;
6. \( \mathcal{P} \) is the point set of a finite dimensional projective space over \( \text{GF}(q) \) and \( \mathcal{L} \) is the set of lines that are not totally isotropic with respect to a fixed nondegenerate symplectic form. (A line is totally isotropic if the restriction of the form to the line is the zero form.)
So, let us consider the noncollinearity graphs for each of these cases in turn.

(1) A transversal design TD(s + 1, t + 1) has s + 1 copies of \( K_{t+1} \) as noncollinearity graph.

(2) A strongly regular graph \( G \) with no triangles is a semipartial geometry with \( s = \alpha = 1 \) — the noncollinearity graph associated with this is merely the complement of \( G \) and its second neighbourhood is complete, and therefore of diameter one.

By Theorem 3.6, the noncollinearity graphs of the remaining four cases are all strongly regular graphs such that the second neighbourhood of every vertex is an antipodal distance-regular graph of diameter three. As the parameters of such a graph are of the form \( \{k, mc_z, 1; 1, c_z, k\} \) they are determined by the three values \( k, m \) and \( c_z \). It is straightforward to calculate the parameters of the noncollinearity graph of an SPG(\( s, t, s, c_z \)). From these and the identities in Theorem 3.4, the parameters of the second neighbourhood of any vertex are \( k = t, m = s - 1 \) and \( c_z = (st + s - \mu)/s \).

(3) The noncollinearity graph is the original Moore graph from which the semipartial geometry is defined (and in fact this is just an alternative description of the semipartial geometry that we associate with a Moore graph). Therefore we recover one of the original examples.

(4) The noncollinearity graph is the complement of the line graph of \( K_{q \times 1} \) — our third example.

(5) There are two nonsingular quadrics in PG(\( 2n - 1, 2 \)), namely the hyperbolic quadric with Witt index \( n - 1 \) and the elliptic quadric with Witt index \( n - 2 \). Let \( \epsilon = +1 \) for the hyperbolic quadric and \( \epsilon = -1 \) for the elliptic quadric. Then the quadric yields an SPG(\( 2, 2^{2n-3} - \epsilon 2^{n-2} - 1, 2, 2^{2n-3} - \epsilon 2^{n-1} \)) and the corresponding second neighbourhood is a 2-fold cover of \( K_{2^{2n-3} - \epsilon 2^{n-2}} \) with \( c_z = 2^{2n-4} \). Covers with these parameters can be found in [15] (in the guise of regular 2-graphs), but their description is sufficiently complicated that we have not yet determined whether these are the same graphs.

(6) The semipartial geometry associated with PG(\( 2n + 1, q \)) is an SPG (\( q, q^{2n} - 1, q, q^{2n+1} - q^{2n} \)) and its noncollinearity graph is a strongly regular graph where the second neighbourhood of every vertex is distance-regular with parameters \( k = q^{2n} - 1, m = q - 1 \), \( c_z = q^{2n-1} \) — that is, a \( q \)-fold cover of \( K_{q^{2n}} \). Covers with these parameters have been constructed by Somma [14] and it is not hard to show that the second neighbourhoods are isomorphic to these graphs. In particular consider PG(\( 3, q \)) and a nondegenerate symplectic form. The points of PG(\( 3, q \)) and the totally isotropic lines form the generalised quadrangle \( W(q) \) of order \( q \). Therefore the collinearity graph of \( W(q) \) is precisely the noncollinearity graph of the semipartial geometry defined by (6), and we have thus recovered our first example.

A question that we have not addressed in this paper is that of determining when a strongly regular graph may have a cover as the second neighbourhood of
one vertex, but not of all the vertices. We present one example to show that this situation can occur.

If \( \mathcal{O} \) is an oval in \( \text{PG}(2, q) \) that is not a conic (and thus \( q \) is necessarily even) then there is a generalized quadrangle of order \( q \), denoted \( T_2(\mathcal{O}) \), that has a regular point (see [12]). However, as \( T_2(\mathcal{O}) \) is not isomorphic to \( W(q) \), the collinearity graph of \( T_2(\mathcal{O}) \) cannot have antipodal distance regular graphs of diameter three as the second neighbourhood of every vertex.

References