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Zermelo in the mirror of the Baer correspondence, 1930–1931

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Abstract

Around 1931 Zermelo had an extended correspondence with the young Reinhold Baer concerning the edition of Cantor's collected works. Some of the letters also deal with Skolem's paradox and Gödel's first incompleteness theorem. Whereas Zermelo's letters are lost, most of Baer's letters are contained in the Zermelo *Nachlass*. Besides giving insight into Zermelo's reaction to Skolem's and Gödel's results, the letters also demonstrate Baer's clear understanding of the behavior of models of set theory and of the relevance of Gödel's first incompleteness theorem.

Zusammenfassung

Um 1931 korrespondiert Zermelo intensiv mit dem jungen Reinhold Baer wegen der Herausgabe der gesammelten Werke Cantors; doch auch das Skolemsche Paradoxon und Gödels erster Unvollständigkeitssatz spielen eine Rolle. Während Zermelos Briefe verloren sind, befinden sich Baers Briefe großenteils in Zermelos Nachlass. Sie erlauben Einblicke in Zermelos Reaktion auf die Ergebnisse von Gödel und Skolem. Zugleich bezeugen sie, wie klar Baer in mengentheoretischen Modellen denken konnte und die Bedeutung des ersten Gödelschen Unvollständigkeitssatzes erkannt hat.

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1. Introduction

Early in 1930, having been provided with the galley proofs of [Skolem, 1930],[1] Zermelo learnt about [Skolem, 1923]: If the notion of definiteness in the separation axiom is made precise by first-order definability, then the axioms of set theory—if consistent at all—admit a countable model. The fact that such a countable model satisfies the sentence claiming the existence of uncountably many sets became

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[1] See Ebbinghaus [2001, 6 and 10] for details.

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known as Skolem's paradox. Up to this time, Zermelo, believing in an ideationally conceived[2] Cantorian universe of sets, had aimed at a finitary axiom system describing this universe as adequately as possible. He therefore was worried when faced with Skolem's results that pointed to unsurmountable difficulties in this respect. A little bit later he also got acquainted with Gödel's first incompleteness theorem that points to a likewise principal limitation of formalized mathematics. This result went against his conviction that there are no undecidable statements in mathematics.

As extensively discussed,[3] Zermelo located the reason for these insufficiencies in the limitation to finitary systems and, henceforth, intensively worked on the conception of infinitary languages and an infinitary logic as the only appropriate framework for doing mathematics. At the same time he fought against the Skolem paradox with the ultimate goal of refuting the existence of countable models of set theory.

The Zermelo *Nachlass* contains about two dozen letters that Reinhold Baer wrote to Zermelo between 1930 and 1932 and that are mainly concerned with the edition of Cantor's collected works [Cantor, 1932].[4] Some of the letters address the results of Gödel and Skolem. As the Baer *Nachlass* no longer exists, the counterparts are lost.[5] However, to some extent the Baer letters allow Zermelo's thoughts to be reconstructed, thus giving us further information about the important period around 1930 when Zermelo shifted from finitary to infinitary conceptions.

Baer and Zermelo had gotten to know each other when Baer held a position in the mathematics department of the University of Freiburg (Germany) from 1926 to 1928 as an assistant of the algebraist Alfred Loewy. Besides aiming at his *Habilitation* in algebra, Baer was open to set theory and the foundations of mathematics. Three set-theoretic publications, among them [Baer, 1929], give evidence to these interests. After his *Habilitation* in 1928 he became a *Privatdozent* at the University of Halle, the university of Georg Cantor. Because of the anti-Jewish decrees of the Nazis he left Germany in the spring of 1933. A two-year stay at the University of Manchester was followed by stays at the Institute for Advanced Study in Princeton and at the University of North Carolina at Chapel Hill. In 1938 he became a professor of mathematics at the University of Illinois at Urbana. In 1956 he returned to Germany to accept a professorship at the University of Frankfurt, where he founded a school in group theory whose most influential work was concerned with soluble and nilpotent groups. Baer died in 1979 at the age of 77. Further details may be found in the obituary [Gruenberg, 1981].

During his appointment in Halle Baer maintained contact with Zermelo. His letters show that he was fully aware of the problems and perspectives coming with Gödel's and Skolem's work. He thus belongs to the small group of people that instantly recognized what had happened here.

In the following we comment on Baer's letters as far as they are concerned with Gödel and Skolem [Baer, 1930/1931], treating Skolem first. In Appendix we provide the original texts. We refer to the letters by their date.

[2] Ideell gesetzt and "existing only in the sense of a Platonic idea" [Zermelo, 1932b, 1].
[3] See, for example, van Dalen and Ebbinghaus [2000], Ebbinghaus [2001], Taylor [1993], Taylor [2002].
[4] In the preface, Zermelo acknowledges Baer's essential support.
[5] [Zermelo, 1931] is an exception.

2. Skolem: The existence of countable models of set theory

In [1929] Zermelo had given a more elaborate version of his notion of definiteness, as the original one from [1908], due to its vagueness, had been subject to intensive criticism. In an answer to this paper, Skolem [1930] points out that—up to second-order quantification—he had given the same definition already in [1923], thereby informing Zermelo also about the existence of countable models of first-order set theory. As mentioned above, this information caused a shift in Zermelo's thinking. Whereas his [1929] still yields a finitary version of definiteness and, hence, a finitary axiom system of set theory, he now will become convinced that only infinitary languages together with an infinitary logic are really able to capture the richness of mathematics. Moreover, he will aim at a refutation of the Skolem paradox by searching for a proof that no countable model of set theory could exist. As described by van Dalen and Ebbinghaus [2000], in 1937 he wrote up a flawed proof of this kind; the reason for the mistake—thinking of the powerset in a countable model of set theory as being closed under *arbitrary* unions of elements—may be localized both in the difficulties he had with argumentations *inside* a model of set theory and in his epistemological convictions. It seems that the impetus of these convictions was so strong that he never checked the formal correctness of Skolem's [1923],[6] but instantly aimed at overcoming the deficiency of finitary approaches to mathematics. The Baer correspondence, starting when Zermelo had just been informed about [Skolem, 1930], will give valuable informations concerning Zermelo's first reactions.

The earliest letter in the *Nachlass* (May 27, 1930) answers a letter of Zermelo that apparently had contained the news about Skolem's [1930] together with an engaged reaction. Having wished Zermelo good luck in his *war* against Skolem, Baer pleads for a de-escalation:

> Some objective remarks: Fraenkel, von Neumann, etc. have an axiom system that, because of what it says about "definite," allows for "absolutely" countable models. Actually, this is not a disaster; for the usual set theoretic inferences are not hurt by this fact; can it not be seen as an advantage of these systems that they provide more possibilities as the system you treat in your new paper?[7]

Having put forward the question of how many "reasonable" versions of definiteness might exist for Zermelo's system and adopting the pragmatic point of view by which Skolem [1923] had justified his first-order version of definiteness, he continues:

> To be called reasonable, first of all a notion of definiteness should ensure the usual set theoretical conclusions. Only in the second place would I request that cardinalities should be invariant under transitions to larger domains of sets.—Anyway, there is one fact that should be observed: a subset of a set belonging to some domain of sets may exist even if there is no definite function according to the axiom of separation which defines it; it only is not forced to exist in this case.
>
> Anyhow, I am very interested in your answer to Skolem; did I understand you correctly: you want to prove that a domain of sets which satisfies the axiom of choice cannot be countable in any domain of sets including it? For this purpose you really must have a very sharp axiom of separation! I am very curious to what extent this holds.

The last paragraph shows that Zermelo really aimed at a quick answer to Skolem as he had indicated in [1930, Footnote 1] and that already very early he tried to refute the existence of countable models. So

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6 According to Kreisel [1980, 210] "he had a [...] staunch realist *Weltanschauung*: so much so that he simply refused to look at the tainted subjects."

7 [Zermelo, 1930a]; here Zermelo considers ZF set theory without restricting separation or replacement; he builds up the von Neumann hierarchy, where the initial level may be some domain of urelements. The "possibilities," i.e., the models of ZF set theory Baer refers to, are just the levels of these hierarchies that belong to strongly inaccessible cardinals.

Skolem's arguments did not convince him. At a first glance it is not clear whether he doubts Skolem's proof of the Löwenheim–Skolem theorem or the applicability of the theorem to set theory. Up to a possible exception discussed in Footnote 10, there seems to be no passage in the *Nachlass* where Zermelo clearly utters doubts of the first kind, whereas doubts of the latter kind are abundant and, therefore, might have been the by far dominant ones. The middle part of the last quotation reveals their nature: Baer's remark exactly fits the criticism which Zermelo brings forward in [1931?, 2] to the effect that first-order definiteness concerned only the *definition* of certain subsets, but not their objective *existence* in the totality of *all* subsets; it thus "cannot be exploited for an arbitrary restriction of the domain of sets and for a nonsensical relativization of the notions of subset and cardinality, in this way reinterpreting or diluting the notion of the continuum." Hence, we may assume, his refutation of the existence of countable models of set theory was to exhibit the uncountability of the continuum or, equivalently, of the power set of countably infinite sets. When trying to do so, the properties needed would have to go beyond Skolem's first-order axioms; in Baer's words they would have to make use of "a very sharp axiom of separation." From Baer's next letter, less than a week later, we can reconstruct certain features which show that Zermelo planned to proceed in the way just described. The letter (June 2, 1930) consists of five "remarks to the Löwenheim–Skolem paradox." The essence is contained in the first two remarks.

Given a model M of set theory and a countable set S of M, Zermelo apparently, using a well-ordering of the power set S' of S, had constructed a partial enumeration of S' of a length uncountable in M with the intention of concluding from this that S' had to be uncountable. Baer has doubts, and he gives an exact analysis of the reason:

> 1. There is a point in your proof of the "invariance of cardinalities" which still does not satisfy me. [...] Let β be an ordinal number from M which in M belongs to a number class that is higher than the second number class.[8] Then you still have to show that in any model extending M the ordinal β belongs to a number class that is higher than the second one. This seems to be the real problem. For the following reason I believe that a positive solution will be improbable:
>
> The idea of a proof would be as follows: let β be the first ordinal in [a model] M' that belongs to the third number class and let M'' be a model including M'[9] in which β belongs to the second number class. Then from the set B of all ordinal numbers $< \beta$ (B is a set in both M' and M'') one can pick a subsequence $\beta_1, \beta_2, \ldots, \beta_n, \ldots$ of type ω such that $\lim_{n\to\omega} \beta_n = \beta$, $\beta_n < \beta$. This subsequence $\{\beta_n\}$ is in M'' a subset of B; but I do not know whether $\{\beta_1, \beta_2, \ldots, \beta_n, \ldots\}$ must be a set in M', too, and in general, this might not be true. (Footnote: The axiom of replacement also does not help, because the formation of the set $\{\beta_n\}$ in M'' and the mapping of the β_n to the integers need not be a function which is admissible in M'.)

Baer's analysis hits just the point where Zermelo's argument was doomed to fail. But cautiously he remarks that this consideration only makes it plausible that there is a serious problem and that it does not show the Skolem paradox to be justified. On the other hand, he emphasizes that Skolem's deduction of the paradox was correct; for with respect to a possible gap in Skolem's argumentation Zermelo may have addressed, he says:

> 2. Nor do I believe that your objection against Skolem's proof can be elaborated in such a way that one might be successful in finding a gap; for as far as I know, in the proofs for the existence of absolutely countable models of set theory difficulties always stem from

8 The first number class consists of the finite ordinals; the second number class consists of the countably infinite ordinals, among them the first infinite ordinal ω; the third number class consists of the ordinals of the first uncountable cardinality $\aleph_1$, among them the smallest uncountable ordinal ω_1; the next number class consists of the ordinals of power $\aleph_2$ with the smallest ordinal ω_2; etc.

9 The modern reader should assume that M' is a transitive submodel of M''; then the ordinals in M' form an initial segment of the ordinals in M''.

the axiom of separation, whereas the axiom of choice and the axiom of replacement are harmless, because for any given set they only claim the existence of a single set with a certain property; hence, in general they do not affect the cardinality of the model.

By the way, despite all my endeavor, I have not been able to find a sound objection against the applicability of the surely true theorem of Löwenheim[10] to the usual axioms of set theory.

But whatever I have said, it is my opinion, not my knowledge.

Once more less than a week later, Baer writes a new letter (June 7, 1930) "to torment you a little bit with Skolemism in order to let all good things come to an end."[11] It also has two important parts. The first one deals with the adequacy of Skolem's procedure, thereby again expressing faith in the correctness of his arguments:

One has to distinguish between two questions (I see from your card that you do the same):

(1) Are the assumptions of the Löwenheim–Skolem theorem satisfied with the usual axiom systems of set theory?

(2) Can one exhibit a countable model of these axiom systems in case they are consistent?

A positive answer to (1) yields a positive answer to (2), and a negative answer to (2) yields a negative answer to (1), but a negative answer to (1) does not yield much with respect to (2).

I had great reservations about whether (1) applies to Fraenkel's axiomatization; however, it appears to me that Skolem's argumentations[12] [...] are conclusive; however, one easiliy can slip with these delicate things.

The second part of the present letter is devoted to the absoluteness arguments discussed in the preceding one. Apparently Zermelo had responded to Baer's objections, neglecting Baer's technical analysis and varying his procedure. So—in contrast to Baer—he must still have been convinced that usual set theory could not be performed in a first-order framework—his answer to both (1) and (2) was a clear "no." As we know from his flawed refutation of the existence of countable models of set theory from 1937, also in the future he would not change his mind. Baer's counterarguments—those against the original procedure by which he starts and those against the new variant—are a further fine example of an early discussion of absoluteness:

Concerning (2), I have not really understood your argumentation (I mean your last card). Let M_1 and M_2 be two models, M_1 contained[13] in M_2, M_1 allowing well-orderings and diagonalization,[14] M_1 countable in M_2; then the initial number ω_1^1 of the third number class in M_1 belongs to the second number class in M_2 as does any initial number ω_ν^1 from M_1; on the other hand, the initial number ω_1^2 of the 3rd number class in M_2 does not exist in M_1.

[10] The text does not make it possible to decide whether Zermelo doubts the Löwenheim–Skolem *theorem* or its *application*. As argued above, doubts of the first kind are rare. In fact, Baer's emphasis of the "sure truth" of the theorem seems to be the only passage in the *Nachlass* that might point to such doubts. The next letter will give evidence that Zermelo might have accepted the theorem.

[11] Literally translated: "in order to let your trees not grow into heaven."

[12] In Skolem [1929], where it is proved that Fraenkel's axiomatization [1922] can be given in the first-order language of set theory.

[13] Again as a transitive submodel. As M_1 is countable in M_2 and the ordinals of M_1 form an initial segment of the ordinals in M_2, the ordinals of M_1 are less than the first ordinal in M_2 which is uncountable in M_2, i.e., in Baer's notation, less than ω_1^2.

[14] Probably Zermelo made use of diagonalizations in order to define the partial enumeration of the power set S' of the countable set S from M_1 at successors of limit ordinals. The procedure might have been similar to that lying at the heart of Cantor's proof of the uncountability of the continuum: Given $S = \{s_0, s_1, \ldots\}$ and a countable subset $\{s'_0, s'_1, \ldots\}$ of S' in M_1 which consists of the elements of S' enumerated so far, one can get a new subset s' of S in M_1 by letting $s_i \in s'$ iff $s_i \notin s'_i$ for $i \in \omega$.

Baer now argues that Zermelo's enumeration of S' cannot be extended beyond ω_1^1, since diagonalization only applies to sets which are countable in M_1 and ω_1^1 is uncountable in M_1—a fact that "you put into my mouth." Therefore, he continues, Zermelo is right in shifting the problem to that of the relativity of ordinal number classes. However, his criticism reveals that also with this variant Zermelo fails in arguing strictly in the model concerned:

> You want to deduce a contradiction from the fact that all ω_ν^1, the initial numbers of M_1, in M_2 belong to the second number class, i.e., from
>
> $$\omega_0 \leqslant \omega_\nu^1 < \omega_1^2 = \text{initial number of the 3rd number class in } M_2.$$
>
> For this purpose you form $\omega_{\omega_1}^1$; but that is impossible; for the "absolute" ω_1 is not at your disposal; you only may form $\omega^1_{\omega_1^1}$, but that does not seem to lead to a contradiction. You cannot even claim the existence of $\omega^1_{\omega_1^2}$ which, by the way, would yield the contradiction, too. Rather, there will exist only ω_ν^1 satisfying $0 \leqslant \nu < \omega_1^2$ if you cannot ensure the existence of larger ω_ν^1 by other means.

In the last letter concerned with Skolem (July 12, 1930), Baer refers to Zermelo's "new reason" to fight against Skolemism—as it seems, a fight for a designated set-theoretic universe.[15] Whereas hitherto he has shown understanding for Zermelo's convictions and a pragmatism in the sense of Skolem, he now distances himself further from Zermelo:[16]

> But Cantorian set theory per se does not exist, whereas there might exist various models of set theory; as Zermelo has shown in a not yet published paper,[17] in such models the series of ordinal numbers may have different "length." Actually, what S[kolem] claims is only this: If there is a model at all, then there also is a countable one; up to now the existence of effectively uncountable models has not been shown, in any form, however hypothetical it might be, maybe apart from Mr. Becker[18] or Brouwer who can see this by urintuition or something of that kind.—I would appreciate if somewhere in the near future we would have the occasion to "settle" all these things orally.

Such a conversation did not take place. As we know, less than two years later Baer left Germany and only returned after Zermelo's death.

3. Gödel: The first incompleteness theorem

In September 1931, at the annual meeting of the German mathematical association, both Gödel and Zermelo delivered talks. Gödel represented his incompleteness results, while Zermelo described a system of infinitary languages that he had developed along the lines of his program to overcome the deficiencies

[15] As described in Ebbinghaus [2001], between 1930 and 1931 Zermelo worked out an earlier idea that aimed at a designated universe of set theory, the universe of sets that are definable up to isomorphisms. He will claim that this model exactly coincides with what Cantor had in mind. As Baer explicitly comments on it only in a letter from January 1931, it is not clear whether "Cantorian set theory *per se*" in the following passage already refers to it.

[16] In the following letters concerning Gödel, the equilibrium will come back.

[17] In Zermelo [1930b].

[18] Oskar Becker (1889–1964) was a philosopher who obtained his *Habiliation* with Husserl in 1922 at the University of Freiburg and became a professor at the University of Bonn in 1931. He was sympathetic toward Brouwer's intuitionism as opposed to Hilbertian formalism; cf. [Becker, 1927]. During his time in Freiburg Becker showed interest in Zermelo's foundational work. He supported Zermelo in preparing the edition of Cantor's philosophical papers (Part IV of [Cantor, 1932]).

of finitary mathematics, a program that was originally aimed against Skolem, but now was also aimed against Gödel. A picture of the resulting debate is given in Dawson [1997, 76].[19] There is the question how early and to what extent Zermelo was informed about Gödel's results and their scope. The published extended abstract [Zermelo, 1932a] contains a programmatic introduction arguing against both Gödel and Skolem because of their finitary approaches; however, it was written only after the meeting (cf. [Zermelo, 1931]).

Apparently, Gödel's [1931] was brought to Zermelo's attention by Baer. On May 13, 1931 Baer writes:

> The main result of Gödel's paper, about which I informed you in my last letter,[20] can pointedly and, hence, not fully correctly, be formulated as follows: If *A* is a countable, consistent logical system, for example arithmetic, then there is a sentence in *A* which is not decidable in *A* (but may be decidable in extended systems). Hurrah, also logicians have discovered diagonalization!—By the way, you can really profit greatly from this "new gentleman"; he is strongly interested in philosophy and the foundations of mathematics, moreover in functional analysis (let him give a talk about this, it is very amusing), summation procedures, etc.

Zermelo's answer may have been sceptical with respect to the relevance of Gödel's result. Baer seems to share the scepticism, but immediately cuts back by emphasizing the importance of incompleteness (May 24, 1931):

> Many thanks for your cards. I believe that we share our opinion about Goedel. But actually I think that his work is rather rewarding; for after all it proves that a logic of the strength of, say, *Principia mathematica* is insufficient as a base for mathematics to the extent we would like to have. There are two possible consequences:
> (1) one surrenders to "classical" logic (as Skolem does) and amputates mathematics;
> (2) one declares, as you do, that "classical" logic is insufficient.
>
> Standpoint (1) is more comfortable; for it gets by on "God's present" of the natural numbers, whereas standpoint (2) has to swallow the total series of transfinite ordinal numbers, as they are necessary to build up set theory to the extent you have performed and as without this greater "present of God" one cannot get beyond what Skolem etc. reach—but it is strange that up to now, apart from the continuum problem, the "poorhouse mathematics" of Skolem etc. suffices to represent all what happens in the "mathematics of the rich."—In any case, Gödel can be credited for having made clear how far one can get with standpoint (1); it is a matter of faith, whether one now decides in favor of (1) or (2). I suspect that logicians will decide in favor of (1), while mathematicians rather will decide in favor of (2).

Baer's remark that Zermelo would have declared classical logic to be insufficient, together with the context in which it is given, may be considered evidence that Zermelo's conviction about the necessity of an alternative logic (that was to be an infinitary one) arose from his resistance against Skolem and was strengthened by his resistance against Gödel.[21]

The following letters do not continue this discussion. On September 9, 1931 Baer informs Zermelo that he will not come to the meeting of the German mathematical association. After the meeting, on October 26, 1931, having been informed by Zermelo "concerning Gödel," he expresses interest in Gödel's first letter [Grattan-Guinness, 1979] to Zermelo. In (November 3, 1931) he comments on Zermelo's answer to Gödel [Dawson, 1985], tending toward Zermelo's point of view:

[19] The discussion was continued in a series of letters; see Grattan-Guinness [1979] and Dawson [1985].

[20] The letter is not contained in the *Nachlass*.

[21] There is just one earlier witness, aiming only at Skolem: In the second appendix to [1930b], Zermelo lists five projects for further research; in an explanation of the last project on the relationship between mathematics and *Anschauung* he states that mathematical science only starts by treating intuitively (anschaulich) given material in an "infinitistic-logical" (infinitistisch-logisch) manner. Concerning the question whether there might be earlier witnesses independent of Skolem and Gödel and going back as early as 1921, see Ebbinghaus [2001].

> I think that your answer to Gödel hits the point. It appears to me (but I have not checked things to a degree to give a final judgement) that formally Gödel's proof will be correct; as regards content it shows that he and the people whose point of view he follows, are wrong.—In his new *History of Logic*[22] H. Scholz very nicely writes that the logicians in Vienna[23] would be very pleasant and competent, but that we would like to obey their dictatorship just as little as any other one.

The exchange of letters ends in February 1932, when Cantor's collected works had been edited.

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Appendix

May 27, 1930

> [. . .] Zu Ihrem Krieg gegen Skolem etc. Heil und Sieg und fette Beute. Der "Aufsatz" von Skolem ist ja inzwischen erschienen, na, einen übermässig intelligenten Eindruck macht er ja nicht.
>
> Ein paar sachliche Bemerkungen dazu: Fraenkel, v. Neumann etc. haben ein Axiomensystem, das auf Grund der Aussagen über "definit" auch Erfüllungen zulässt, die "absolut" abzählbar sind. Das ist doch an sich kein Unglück; denn die üblichen mengentheoretischen Schlussweisen werden dadurch nicht betroffen; ist es nicht fast ein Vorzug dieser Systeme, dass sie noch mehr Möglichkeiten Spielraum geben als das von Ihnen in Ihrer neuen Arbeit behandelte? Gerade im Sinne der Tendenzen, die Sie in dieser neuen Arbeit propagieren. All Ihre Mengenbereiche genügen auch den Fraenkelschen etc. Axiomen, nicht aber umgekehrt. Es ist doch vielleicht noch eine interessante Frage, wieviele "vernünftige" Möglichkeiten es gibt, die Definitheit zu erklären—bei fester Ordnungszahl und Charakteristik (ich hoffe, Ihre Bezeichnungen richtig wiedergegeben zu haben). Dabei ist ein Begriff der Definitheit in erster Linie dann als vernünftig zu bezeichnen, wenn er die üblichen mengentheoretischen Schlüsse sichert; in zweiter Linie erst würde ich die Forderung stellen, dass die Mächtigkeiten invariant gegenüber Übergängen zu grösseren Mengenbereichen sein sollen.—Eines muss man doch bei allem beachten: eine Teilmenge einer Menge eines Mengenbereichs kann auch existieren, wenn es keine gemäss Aussonderungsaxiom definite Funktion gibt, die sie definiert; sie muss es dann nur nicht.—
>
> Jedenfalls interessiert mich Ihre Erwiderung gegen Skolem sehr; habe ich Sie richtig verstanden: Sie wollen beweisen, dass ein Mengenbereich, in dem das Auswahlaxiom erfüllt ist, sich in keinem umfassenden Mengenbereich als abzählbar erweisen kann? Dazu müssen Sie doch wohl ein sehr scharfes Aussonderungsaxiom haben! Ich bin sehr gespannt, wie allgemein das gilt! [. . .]

June 2, 1930

> [. . .] Zum Löwenheim–Skolemschen Paradoxon folgende Bemerkungen:
>
> 1. Ihr Beweis von der "Invarianz der Mächtigkeiten" hat mich an einer Stelle noch nicht befriedigt. Sie gehen so vor: Sei M ein Modell einer Mengenlehre, m eine in M abzählbare Menge, m' die Potenzmenge von m in M. Dann lässt sich m' wohlordnen und, wenn v irgendeine Teilmenge von m' ist, die in M enthalten und in M abzählbar ist, so gibt es ein in der Wohlordnung von m' erstes Element von m', das nicht in v vorkommt [say, m_v]. Dann bilden Sie die Mengenfolge v_α durch
>
> $$v_0 = \{m\}, \quad v_{\alpha+1} = v_\alpha + \{m_{v_\alpha}\}, \quad \lim_{\beta<\alpha} v_\beta = v_\alpha,$$

[22] It is [Scholz, 1931]; cf. pages 64sq.

[23] The members of the Vienna circle.

und diese v_α sind definiert für alle Ordinalzahlen α, die in M zur zweiten Zahlklasse gehören. Ist jetzt β eine Ordinalzahl aus M, die in M zu einer höheren als der zweiten Zahlklasse gehört, so haben Sie also noch zu zeigen, dass β auch in jedem M umfassenden Modell zu einer höheren als der zweiten Zahlklasse gehört. Das scheint mir das eigentliche Problem zu sein, dessen positive Lösung ich aus folgendem Grunde für unwahrscheinlich halte:

Der Beweisansatz dürfte etwa so sein: sei β in M' die erste Ordinalzahl, die zur dritten Zahlklasse gehört und sei M'' ein M' umfassendes Modell, in dem β zur zweiten Zahlklasse gehört; dann lässt sich aus der Menge B aller Ordinalzahlen $< \beta$ (B ist in M' und in M'' Menge) eine Teilfolge $\beta_1, \beta_2, \dots, \beta_n, \dots$ vom Typus ω herausgreifen, so dass $\lim_{n\to\omega} \beta_n = \beta$, $\beta_n < \beta$ ist. Diese Teilfolge $\{\beta_n\}$ ist in M'' Teilmenge von B; ob aber $\{\beta_1, \beta_2, \dots, \beta_n, \dots\}$ auch in M' Menge ist, weiss ich nicht, und das dürfte wohl im allgemeinen auch nicht zu bejahen sein. Footnote: Hier hilft auch das Ersetzbarkeitsaxiom nicht, da die Bildung der Menge $\{\beta_n\}$ in M'' und die Zuordnung der β_n zu den ganzen Zahlen noch keine in M' zulässige Funktion zu sein braucht.

Damit ist aber nur plausibel gemacht, dass die "Invarianz der Mächtigkeiten" sich auf diesem Weg kaum wird zeigen lassen; daraus soll natürlich nicht folgen, dass das Löwenheim–Skolemsche Paradoxon zu Recht besteht, bzw. Ihre Vermutung falsch ist.

2. Ich glaube auch nicht, dass Ihr Einwand gegen Skolems Beweis sich so wird ausbauen lassen, dass es gelingt, eine Lücke aufzuweisen; denn soweit ich die Beweise für die Existenz absolut abzählbarer Modelle der Mengenlehre kenne, macht bei ihnen immer nur das Aussonderungsaxiom Schwierigkeiten, während das Auswahlaxiom und das Ersetzbarkeitsaxiom deshalb harmlos sind, weil sie zu einer beliebigen Menge immer nur die Existenz einer einzigen Menge gewisser Eigenschaft fordern, also die Mächtigkeit des Modells i.A. nicht treffen.

Mir ist es übrigens nicht gelungen, trotzdem ich mir grosse Mühe gegeben habe, einen stichhaltigen Einwand gegen die Anwendbarkeit des wohl sicher richtigen Löwenheimschen Satzes auf die üblichen Axiomensysteme der Mengenlehre zu finden.

Aber das sind natürlich nur Meinungen, kein Wissen. [...]

June 7, 1930

[...] Sozusagen, damit Ihre Bäume nicht in den Himmel wachsen, möchte ich Sie noch ein wenig mit der Skolemitis "ärgern". Man muss da zwei Fragen unterscheiden (aus Ihrer Karte entnehme ich, dass Sie es auch tun):

1. sind die Voraussetzungen des Löwenheim–Skolemschen Satzes bei den üblichen Axiomensystemen der Mengenlehre erfüllt?

2. lässt sich, falls diese Ax.-syst. der Mengenlehre widerspruchsfrei sind, ein abzählbares Modell angeben?

Aus einer positiven Antwort auf 1. folgt auch die positive Antwort auf 2., aus einer negativen Antwort auf 2. auch die negative Antwort auf 1.; aus einer negativen Antwort auf 1. folgt aber nicht viel bzgl. 2.

ad 1. habe ich sehr starke Bedenken gehabt, ob es bei der Fraenkelschen Axiomatik geht; jedoch schien es mir, als ob die Skolemschen Argumentationen (vgl. Th. Skolem: Über einige Grundlagenfragen der Mathematik; Skrifter utgift af det Norske Videnskaps Akademi i Oslo I. Mat.-Naturv. Klasse 1929 No. 4 §2 S. 9ff.) zwingend wären; aber bei diesen heiklen Dingen ist ein Versehen leicht möglich.

ad 2. Ihre Argumentation habe ich noch nicht recht verstanden (ich meine Ihre letzte Karte). Seien M_1 und M_2 zwei Modelle, M_1 in M_2 enthalten, in M_1 Wohlordnung und Diagonalverfahren ausführbar, M_1 in M_2 abzählbar; dann gehört die Anfangszahl ω_1^1 der dritten Zahlklasse in M_1 in M_2 zur zweiten Zahlklasse, überhaupt jede Anfangszahl ω_ν^1 aus M_1; dagegen ist die Anfangszahl ω_1^2 der 3. Zahlklasse in M_2 in M_1 gar nicht vorhanden.

Die Construktion Ihrer Mengen v_α aus M_1 vermittels Diagonalverfahren und Wohlordnung lässt sich dann für alle $\alpha \leqslant \omega_1^1$ ausführen; dagegen können Sie auf $v_{\omega_1^1}$ das Diagonalverfahren nicht mehr anwenden, da, wie Sie mir ganz richtig in den Mund legten, ja $v_{\omega_1^1}$ in M_1 nicht mehr abzählbar ist, das Diagonalverfahren aber nur auf in M_1 abzählbare Teilmengen der fraglichen Potenzmenge anwendbar ist. $v_{\omega_1^1+1}$ lässt sich also mit Ihrer Methode nicht mehr bilden.

Nun verschieben Sie ganz richtig, wie ich es ja auch in meinem vorigen Brief angedeutet habe, das Problem auf das der Relativität der Ordinalzahlklassen und wollen aus der Tatsache, dass alle ω_ν^1, die Anfangszahlen aus M_1, in M_2 zur zweiten Zahlklasse gehören, d.h. aus

$$\omega_0 \leqslant \omega_\nu^1 < \omega_1^2 = \text{Anfangszahl der 3. Zahlklasse in } M_2$$

einen Widerspruch herleiten. Hierzu bilden Sie $\omega_{\omega_1}^1$; das geht aber nicht; denn das "absolute" ω_1 steht Ihnen ja nicht zur Verfügung; sie können nur $\omega_{\omega_1^1}^1$ bilden, was zu keinem Widerspruch Anlass zu geben scheint. Nicht einmal die Existenz von $\omega_{\omega_1^2}^1$ können Sie behaupten, die übrigens auch schon den gesuchten Widerspruch lieferte; vielmehr werden nur solche ω_ν^1 existieren, die $0 \leqslant \nu < \omega_1^2$ erfüllen, es sei denn, dass Sie die Existenz grösserer ω_ν^1 auf andere Weise sichern könnten, was den Widerspruch lieferte; hierzu können Sie aber das Diagonalverfahren nicht heranziehen; denn das liefert nur die Existenz von Ordinalzahlen α, die $\alpha \leqslant \omega_\nu^1$ für geeignetes ν erfüllen, im wesentlichen sogar nur $\alpha \leqslant \omega_1^1$ und bei successiver Anwendung kommt man zu $\omega_{\omega_1^1}^1, \omega_{\omega_{\omega_1^1}^1}^1, \dots$; aber die

Reihe der so gebildeten Anfangszahlen α_ν kann man wegen Burali–Fortis Antinomie nur für bereits konstruierte ν, bzw. solche ν, deren Existenz bereits anderweitig bewiesen ist, weiterbauen; das ganze Verfahren kann man weiter iterieren; da man aber keinen limes über alle möglichen Verfahren bilden kann, wird man über abzählbares kaum hinaus kommen. [...]

July 12, 1930

Ihren neuen wahren Grund zur Bekämpfung der Skolemitis werden Sie wohl inzwischen selbst widerlegt haben. Cantorsche Mengenlehre schlechthin gibt es doch nicht; dagegen gibt es vielleicht allerlei Modelle der Mengenlehre; und in denen können, wie Zermelo in einer noch nicht veröffentlichten Arbeit gezeigt hat, die Reihen der Ordinalzahlen verschieden "lang" sein. Was S[kolem] behauptet, ist doch nur dies: Gibt es überhaupt ein Modell, so auch ein abzählbares; die Existenz effektiv überabzählbarer Modelle ist noch in keiner noch so hypothetischen Form bewiesen worden, abgesehen vielleicht von Herrn Becker oder Brouwer, die das [mit] Urintuition oder dergl. einsehen können.—Ich würde mich freuen, wenn wir bald einmal Gelegenheit hätten, dies alles mündlich zu "bereinigen". [...]

May 13, 1931

[...] Das Hauptergebnis der Goedelschen Arbeit, auf die ich Sie in meinem letzten Brief hinwies, kann man pointiert und also nicht ganz korrekt so formulieren: ist A ein abzählbares, widerspruchsloses logisches System, z.B. die Arithmetik, so gibt es in A einen Satz, der in A nicht entscheidbar ist (wohl aber in umfassenderen Systemen). Hurra, die Logiker haben auch schon das Diagonalverfahren entdeckt!—Von dem "neuen Herrn" können Sie übrigens wirklich viel haben; er ist sehr stark philosophisch und an Grundlagen interessiert, weiter Funktionalanalysis (lassen Sie ihn darüber vortragen, das ist sehr amüsant), Summationsverfahren u.s.w.

May 24, 1931

[...] vielen Dank für Ihre interessanten Karten. Ich glaube, dass wir betr. Goedel ziemlich einer Meinung sind. Nur finde ich die Arbeit eigentlich doch recht lohnend; denn sie beweist schliesslich, dass sich auf einer Logik im Umfang der *principia mathematica* oder dergl. keine Mathematik in dem von uns gewünschten Umfang aufbauen lässt. Daraus kann man zwei Consequenzen ziehen: 1. man capituliert vor der "klassischen" Logik (wie Skolem) und amputiert die Mathematik; 2. man erklärt dann, wie Sie es tun, die "klassische" Logik für unzulänglich.—Der Standpunkt 1. ist bequemer; denn er kommt mit dem "Gottesgeschenk" der natürlichen Zahlen aus, während 2. die ganze Reihe der transfiniten Ordinalzahlen schlucken muss, da diese dann unumgängliche Voraussetzung beim Aufbau der Mengenlehre in dem von Ihnen durchgeführten Umfange sind und man ohne dieses grössere "Gottesgeschenk" eben nicht über das hinauskommt, was Skolem etc. erreichen.—Merkwürdig ist nur, dass bisher, abgesehen vom Continuumproblem, die "Armenhausmathematik" der Skolem etc. völlig ausreichte, um alles wiederzugeben, was in der "Mathematik der Reichen" geschieht.—Goedel hat jedenfalls das Verdienst, genau klargestellt zu haben, wieweit man mit dem Standpunkt 1. kommt; ob man sich nun für 1. oder 2. entscheidet, ist Glaubenssache. Ich vermute, dass sich die Logistiker für 1., die Mathematiker mehr für 2. entscheiden werden. [...]

November 3, 1931

[...] Ihre Antwort an Goedel finde ich ganz treffend; mir scheint es so (doch habe ich die Dinge zu wenig genau geprüft, um ein bindendes Urteil auszusprechen): formal wird G's Beweis richtig sein; inhaltlich zeigt er, dass er und die Leute, auf deren Standpunkt er sich gestellt hat, Unrecht haben.—Sehr hübsch schreibt H. Scholz in einer kürzlich erschienenen Geschichte der Logik, dass die Wiener Logistiker sehr nett und tüchtig seien, dass wir uns aber ihrer Diktatur ebensowenig wie einer anderen fügen wollten. [...]

References

Baer, R., 1929. Zur Axiomatik der Kardinalzahlenarithmetik. Math. Z. 29, 381–396.

Baer, R., 1930/1931. Letters to E. Zermelo, May 27, 1930–November 3, 1931. Zermelo *Nachlass*, Universitätsbibliothek Freiburg, part of C129/4.

Becker, O., 1927. Mathematische Existenz. Zur Logik und Ontologie mathematischer Phänomene. Jahrbuch für Philosophie und Phänomenologische Forschung 8, 1–370. Also separate, second ed., Max Niemeyer, Tübingen, 1973.
Cantor, G., 1932. Gesammelte Abhandlungen mathematischen und philosophischen Inhalts, Zermeo, E. (Ed.). Springer-Verlag, Berlin. Reprint 1980.
van Dalen, D., Ebbinghaus, H.-D., 2000. Zermelo and the Skolem paradox. Bull. Symbolic Logic 6, 145–161.
Dawson, J.W. Jr., 1985. Completing the Gödel–Zermelo correspondence. Historia Math. 12, 66–70.
Dawson, J.W. Jr., 1997. Logical Dilemmas. The Life and Work of Kurt Gödel. A.K. Peters, Wellesley, MA.
Ebbinghaus, H.-D., 2001. Zermelo: Definiteness and the universe of definable sets. Manuscript. Institut für mathematische Logik, University of Freiburg. Hist. Philos. Logic, in press.
Fraenkel, A.A., 1922. Der Begriff "definit" und die Unabhängigkeit des Auswahlaxioms. In: Preussische Akademie der Wissenschaften, Sitzungsberichte, Physikalisch-mathematische Klasse, pp. 253–257.
Gödel, K., 1931. Über formal unentscheidbare Sätze der *Principia mathematica* und verwandter Systeme I. Monatsh. Math. Phys. 38, 173–198.
Grattan-Guinness, I., 1979. In memoriam Kurt Gödel: His 1931 correspondence with Zermelo on his incompletability theorem. Historia Math. 6, 294–304.
Gruenberg, K.W., 1981. Reinhold Baer. Bull. London Math. Soc. 13, 339–361.
Kreisel, G., 1980. Kurt Gödel. Biographical Memoirs of Fellows of the Royal Society 26, 149–224.
Moore, G.H., 1980. Beyond first-order logic: The historical interplay between mathematical logic and axiomatic set theory. Hist. Philos. Logic 1, 95–137.
Scholz, H., 1931. Geschichte der Logik. Junker und Dünnhaupt, Berlin.
Skolem, T., 1923. Einige Bemerkungen zur axiomatischen Begründung der Mengenlehre. In: Wissenschaftliche Vorträge gehalten auf dem fünften Kongress der Skandinavischen Mathematiker in Helsingfors vom 4. bis 7. Juli 1922. Akademiska Bokhandeln, Helsinki, pp. 217–232.
Skolem, T., 1929. Über einige Grundlagenfragen der Mathematik. Norske Videnskaps-Akademi i Oslo. Matematik-Naturvidenskabelig Klasse 4, 1–49.
Skolem, T., 1930. Einige Bemerkungen zu der Abhandlung von E. Zermelo: "Über die Definitheit in der Axiomatik". Fund. Math. 15, 337–341.
Taylor, R.G., 1993. Zermelo, reductionism, and the philosophy of mathematics. Notre Dame J. Formal Logic 34, 539–563.
Taylor, R.G., 2002. Zermelo's Cantorian theory of axiomatic systems. Bull. Symbolic Logic 8, 478–515.
Zermelo, E., 1908. Untersuchungen über die Grundlagen der Mengenlehre I. Math. Ann. 65, 261–281.
Zermelo, E., 1929. Über den Begriff der Definitheit in der Axiomatik. Fund. Math. 14, 339–344.
Zermelo, E., 1930a. Über Grenzzahlen und Mengenbereiche. Neue Untersuchungen über die Grundlagen der Mengenlehre. Fund. Math. 16, 29–47.
Zermelo, E., 1930b. Bericht an die Notgemeinschaft der Deutschen Wissenschaft über meine Forschungen betreffend die Grundlagen der Mathematik. Manuscript, 5 pages and two one page appendices; December 3, 1930. Zermelo *Nachlass*, Universitätsbibliothek Freiburg, part of C129/140; reproduced in Moore [1980].
Zermelo, E., 1931. Letter to Reinhold Baer. German with English translation. In: Weingartner, P., Schmetterer, L. (Eds.), Gödel Remembered. Papers from the 1983 Gödel Symposium Held in Salzburg, July 10–12, 1983. Bibliopolis, Naples, pp. 43–48.
Zermelo, E., 1931?. Sätze über geschlossenen Bereiche. Typewritten manuscript, 2 pages. Zermelo *Nachlass*, Universitätsbibliothek Freiburg, part of C129/236.
Zermelo, E., 1932a. Über Stufen der Quantifikation und die Logik des Unendlichen. Jahresber. Deutsch. Math.-Verein. 41, 85–88.
Zermelo, E., 1932b. Mengenlehre 1932. Typewritten manuscript, 5 pages. Zermelo *Nachlass*, Universitätsbibliothek Freiburg, part of C129/290. Partly reprinted in Ebbinghaus [2001, 20–22].

Further reading

Kreisel, G., 1981. Kurt Gödel. Biographical Memoirs of Fellows of the Royal Society 27, 697.
Kreisel, G., 1982. Kurt Gödel. Biographical Memoirs of Fellows of the Royal Society 28, 719.