A note on monotonically metacompact spaces

Harold R. Bennett [a], Klaas Pieter Hart [b], David J. Lutzer [c],*

[a] *Texas Tech University, Lubbock, TX 79409, United States*
[b] *TU Delft, Delft, Netherlands*
[c] *College of William and Mary, Williamsburg, VA 23187, United States*

A B S T R A C T

We show that any metacompact Moore space is monotonically metacompact and use that result to characterize monotone metacompactness in certain generalized ordered (GO) spaces. We show, for example, that a generalized ordered space with a σ-closed-discrete dense subset is metrizable if and only if it is monotonically (countably) metacompact, that a monotonically (countably) metacompact GO-space is hereditarily paracompact, and that a locally countably compact GO-space is metrizable if and only if it is monotonically (countably) metacompact. We give an example of a non-metrizable LOTS that is monotonically metacompact, thereby answering a question posed by S.G. Popvassilev. We also give consistent examples showing that if there is a Souslin line, then there is one Souslin line that is monotonically countable metacompact, and another Souslin line that is not monotonically countably metacompact.

© 2009 Elsevier B.V. All rights reserved.

1. Introduction

For two collections of sets $\mathcal{U}$ and $\mathcal{V}$ we write $\mathcal{U} \prec \mathcal{V}$ to mean that for each $U \in \mathcal{U}$ there is some $V \in \mathcal{V}$ with $U \subseteq V$. Clearly $\mathcal{U} \prec \mathcal{V}$ implies $\bigcup \mathcal{U} \subseteq \bigcup \mathcal{V}$ but it might happen that $\bigcup \mathcal{U} \neq \bigcup \mathcal{V}$.

A space X is (*countably*) *metacompact* if each (countable) open cover of X has a point-finite open refinement that also covers X. Popvassilev [15] defined that a space is *monotonically* (*countably*) *metacompact* if there is a function r that associates with each (countable) open cover $\mathcal{U}$ of X an open point-finite refinement $r(\mathcal{U})$ that covers X, where r has the property that if $\mathcal{U}$ and $\mathcal{V}$ are open covers with $\mathcal{U} \prec \mathcal{V}$ then $r(\mathcal{U}) \prec r(\mathcal{V})$. The function r is called a *monotone* (*countable*) *metacompactness operator* for X. *Warning: The literature contains other, non-equivalent definitions of monotone countable metacompactness*[1] *and throughout this paper we study the monotone metacompactness property of Popvassilev.*

---

* Corresponding author.
  E-mail address: lutzer@math.wm.edu (D.J. Lutzer).

[1] The referee of this paper pointed out that Good, Knight, and Stares [11] have given another definition of a monotone countable metacompactness property. According to Ying and Good [17], that property is equivalent to Hodel's β-space property [13]. Consequently, the Good–Knight–Stares and Popvassilev definitions of monotone countable metacompactness are *not* equivalent, and neither implies the other. On the one hand, the space $[0, \omega_1]$ with is usual topology is certainly a β-space but in the light of Proposition 3.4 of this paper, $[0, \omega_1)$ is not monotonically countably metacompact in Popvassilev's sense. On the other hand, the example machine $Bush(S, T)$ in [5] constructs a family of linearly ordered spaces that are always monotonically metacompact in the sense of Popvassilev, but never β-spaces.

Our first main result shows that monotone metacompactness is a property of many generalized metric spaces. As a corollary to a more technical Proposition 3.1 we will show:

Theorem 1.1. *Any metrizable space, and any metacompact Moore space, is monotonically metacompact.*

Recall that a *GO-space* is a triple $(X, \tau, <)$ where $<$ is a linear ordering of the set X and τ is a Hausdorff topology on X that has a base of convex subsets of $(X, <)$, possibly including some singletons. For any GO-space $(X, \tau, <)$, we have $\lambda \subseteq \tau$, where λ is the usual open interval topology of the ordering $<$. If $\tau = \lambda$, then $(X, \lambda, <)$ is called a *linearly ordered topological space* (LOTS) and for any GO-space $(X, \tau, <)$, $(X, \lambda, <)$ is called the *underlying LOTS for the GO-space.*

Our later results will deal with GO-spaces $(X, \tau, <)$ that have dense sets that are σ-closed-discrete, and with GO-spaces $(X, \tau, <)$ whose underlying LOTS $(X, \lambda, <)$ has a dense subset that is σ-closed-discrete in (X, λ). Of particular interest are separable GO-spaces and GO-spaces whose underlying LOTS is separable. The best-known examples of this type are the Sorgenfrey and Michael lines, and GO-spaces constructed from the Alexandroff double arrow, i.e., the set $X = \mathbb{R} \times \{0, 1\}$ with the lexicographic ordering.

B.J Ball [3] has shown that any LOTS is *countably paracompact* (= any countable open cover has a locally finite refinement), and hence countably metacompact, and that result was extended to GO-spaces in [14]. Adding "monotonicity" to the countable metacompactness property is a significant strengthening, as our results will show. For example, in Propositions 3.4 and 3.6, we prove:

Theorem 1.2. *Suppose $(X, \tau, <)$ is a GO-space that is monotonically countably metacompact. Then (X, τ) is hereditarily paracompact.*

Theorem 1.3. *Suppose $(X, \tau, <)$ is a compact LOTS or, more generally, a locally countably compact GO-space. If (X, τ) is monotonically countably metacompact, then X is metrizable.*

In certain cases we can characterize which GO-spaces are monotonically (countably) metacompact. To state our result, we need some special notation. For any GO-space $(X, \tau, <)$, let

$$I_\tau := \left\{ x \in X: \{x\} \in \tau \right\},$$
$$R_\tau := \left\{ x \in X - I_\tau: [x \to) \in \tau \right\},$$
$$L_\tau := \left\{ x \in X - I_\tau: (\leftarrow, x] \in \tau \right\}, \quad \text{and}$$
$$E_\tau := X - (I_\tau \cup R_\tau \cup L_\tau).$$

Warning: The above definitions for sets I_τ, R_τ, L_τ, and E_τ are slightly different from definitions given in other papers on GO-spaces. In some other papers, the set of right-looking points is defined as $\{x \in X: [x, \to) \in \tau - \lambda\}$, but theorems in this paper require that R_τ must be defined as above.

Theorem 1.4. *Let $(X, \tau, <)$ be a GO-space whose underlying LOTS $(X, \lambda, <)$ has a σ-closed-discrete dense subset. Then the following are equivalent*:

(a) *(X, τ) is monotonically metacompact*;
(b) *(X, τ) is monotonically countably metacompact*;
(c) *the set $R_\tau \cup L_\tau$ is σ-closed-discrete in (X, τ)*;
(d) *the set $R_\tau \cup L_\tau$ is σ-closed-discrete in (X, λ).*

Theorem 1.4 applies to GO-spaces constructed on the usual real line and shows that the Michael line is monotonically countably metacompact, while the Sorgenfrey line is not. In addition, it shows that the Alexandroff double arrow is not monotonically metacompact.

The proof of Theorem 1.4, combined with M.J. Faber's metrization theorem for GO-spaces [10], will show:

Corollary 1.5. *Let $(X, \tau, <)$ be a GO-space with a σ-closed-discrete dense subset. Then the following are equivalent*:

(a) *(X, τ) is monotonically metacompact*;
(b) *(X, τ) is monotonically countably metacompact*;
(c) *(X, τ) is metrizable.*

Our next example, based on the Michael line (see Example 4.1 for details), shows that the hypothesis in Corollary 1.5 concerning the existence of a σ-closed-discrete dense subset cannot be removed, and at the same time, it answers a question posed in a recent paper [15] by S.G. Popvassilev, namely "Must a monotonically metacompact LOTS be metrizable"?

Example 1.6. There is a LOTS that is monotonically metacompact but not metrizable.

Theorem 1.4 can also be used to show that in certain GO-spaces, monotonic (countable) metacompactness is a hereditary property.

Corollary 1.7. *Suppose $(X, \tau, <)$ is a GO-space whose underlying LOTS $(X, \lambda, <)$ has a σ-closed-discrete dense set. Let $Y \subseteq X$. Then the subspace $(Y, \tau_Y, <)$ is monotonically (countably) metacompact.*

However, we do not know whether monotone (countable) metacompactness is a hereditary property in other kinds of GO-spaces.

To what extent can our results be extended? It is known that any GO-space with a σ-closed-discrete dense subset is *perfect* (= every closed set is a G_δ-set in the space) and one might ask, for example, whether Corollary 1.5 could be proved for perfect GO-spaces. We will show that the answer is "Consistently no" by looking at a *Souslin line* (= a LOTS that satisfies the countable chain condition but is not separable). Souslin lines are hereditarily Lindelöf and therefore perfect. Whether such spaces exist is undecidable in ZFC [16]. Whether there is a perfect LOTS that does not have a σ-closed-discrete dense subset is an old problem of Maarten Maurice that is undecidable in ZFC, at least for spaces of weight ω_1, and is intimately related to the Souslin problem [4]. In our paper's final section, we show:

Example 1.8. If there is a Souslin line, then some Souslin lines are monotonically countably metacompact, while other Souslin lines are not.

Throughout this paper, $\mathbb{R}$, $\mathbb{P}$, and $\mathbb{Q}$ denote the sets of real, irrational, rational numbers respectively, and $\mathbb{Z}$ is the set of integers. The authors want to thank the referee for a series of helpful remarks that improved the current paper and suggested directions for further investigation (see Question 4.14).

2. Preliminary results

We must carefully distinguish between subsets of a space that are *relatively discrete* (= their subspace topology is the discrete topology) and subsets that are *closed-discrete* (= every point of the space has a neighborhood containing at most one point of the given subset). Clearly, a set is closed-discrete if it is both relatively discrete and closed. In general, we will need to distinguish between subsets that are σ-*relatively-discrete* (= countable unions of relatively discrete subsets) and those that are σ-*closed-discrete* (= countable unions of closed-discrete subspaces). However, the two concepts are equivalent in perfect spaces as our next lemma shows. The lemma is well-known and easily proved, and applies to any topological space, not just to GO-spaces.

Lemma 2.1. *If (X, τ) is a perfect topological space, then any relatively discrete subset is σ-closed-discrete. Hence, any σ-relatively discrete subset of a perfect space (X, τ) is σ-closed-discrete in (X, τ).*

Proof. The proof is a standard argument but we include it for completeness. Let D be a relatively discrete subset of X and for each $x \in D$ let $U(x)$ be an open set with $U(x) \cap D = \{x\}$. Write the open set $V := \bigcup \{U(x): x \in D\}$ as $V = \bigcup \{F(n): n \geqslant 1\}$ where each $F(n)$ is closed in X. Then the set $D(n) := D \cap F(n)$ is closed and discrete and $D = \bigcup \{D(n): n \geqslant 1\}$. $\square$

Lemma 2.2. *The existence of a σ-closed-discrete dense set in a GO-space $(X, \tau, <)$ is a hereditary property and implies that (X, τ) is perfect* [7].

By way of contrast, the existence of a σ-relatively discrete dense set in a GO-space $(X, \tau, <)$ is not enough to make (X, τ) perfect and is not a hereditary property. For example, in the Michael line M, the set of irrationals is a relatively discrete dense set, but M is not perfect. Example 5.3 in [7] describes a GO-space that has a σ-relatively-discrete dense set, and has a subspace that does not.

Lemma 2.3. *Let E be a closed-discrete subset of a GO-space $(X, \tau, <)$ and let $S \subseteq X$. Suppose that for each $x \in S$ there is some $e(x) \in E$ with $x < e(x)$ and such that the collection $\mathcal{C} := \{[x, e(x)]: x \in S\}$ is a pairwise disjoint collection. Then the collection $\mathcal{C}$ is discrete in (X, τ) and the set S is a closed-discrete subset of (X, τ).*

Proof. Let $y \in X$ and let U be a convex neighborhood of y that contains at most one point of E. Suppose $[x_i, e(x_i)]$ are four distinct members of $\mathcal{C}$ and for contradiction suppose that U meets all four sets. Without loss of generality, we may assume $x_1 < x_2 < x_3 < x_4$. Because the collection $\mathcal{C}$ is pairwise disjoint, we must have $x_1 \leqslant e(x_1) < x_2 \leqslant e(x_2) < x_3 \leqslant e(x_3) < x_4 \leqslant e(x_4)$. Then convexity of U, plus that fact that U meets both $[x_1, e(x_1)]$ and $[x_4, e(x_4)]$, shows that both $e(x_2)$ and $e(x_3)$ belong to U and to E, and that is impossible. Therefore the open set U meets at most three members of $\mathcal{C}$. Because $\mathcal{C}$ is

pairwise disjoint, that is enough to show that $\mathcal{C}$ is a discrete collection. Because S contains exactly one point from each member of a discrete collection, the set S is closed and discrete. $\quad\square$

Lemma 2.4. *Suppose $(X, \tau, <)$ is a GO-space and that the underlying LOTS (X, λ) has a σ-closed-discrete dense set $D = \bigcup\{D(n):$ $n \geqslant 1\}$. Suppose $S \subseteq X$ is σ-relatively discrete in the subspace topology τ_S and that no point of S is isolated in τ. Then S is σ-closed-discrete in (X, λ) and therefore also in (X, τ).*

Proof. It is enough to prove the lemma in case S is relatively discrete in (X, τ). For each $x \in S$ let $U(x)$ be a convex τ-neighborhood of x with $U(x) \cap S = \{x\}$. Because $S \cap I_\tau = \emptyset$, we have $S \subseteq R_\tau \cup E_\tau \cup L_\tau$. Let $S_1 := S \cap (R_\tau \cup E_\tau)$. For each $x \in S_1$ there is some $y(x) > x$ with $[x, y(x)) \subseteq U(x)$. Because $x \in R_\tau \cup E_\tau$, the set $(x, y(x))$ is not empty. Then there is some integer $N(x)$ and some $d(x) \in D(N(x)) \cap (x, y(x))$. Replacing $y(x)$ by some point of $(x, y(x))$ if necessary, we may assume that $d(x)$ is the only point of $(x, y(x)) \cap D(N(x))$. By Lemma 2.3, for each $k \geqslant 1$ the collection $\{[x, d(x)]: N(x) = k\}$ is a discrete collection in (X, λ), and $S_1(k) := \{x \in S_1: N(x) = k\}$ is a closed-discrete subset of (X, λ) for each k. But $S_1 = \bigcup\{S_1(k): k \geqslant 1\}$. Points of $S \cap L_\tau$ are treated analogously. Hence S is σ-closed-discrete in (X, λ) and hence also in (X, τ). $\quad\square$

3. Main theorems

Proposition 3.1. *Suppose X is a metacompact Moore space. There is a function r such that for each collection $\mathcal{U}$ of open subsets of X, $r(\mathcal{U})$ is a collection of open subsets of X satisfying*:

(a) $r(\mathcal{U})$ *is point-finite*;
(b) $r(\mathcal{U}) \prec \mathcal{U}$;
(c) $\bigcup r(\mathcal{U}) = \bigcup \mathcal{U}$;
(d) *if $G, H \in r(\mathcal{U})$ have $G \subseteq H$ then $G = H$; and*
(e) *if $\mathcal{V}$ is an open collection with $\mathcal{U} \prec \mathcal{V}$, then $r(\mathcal{U}) \prec r(\mathcal{V})$.*

Proof. Suppose X is a metacompact Moore space. Let $\langle \mathcal{G}(n) \rangle$ be a development for X where each $\mathcal{G}(n)$ is a point-finite open cover of X. We may assume that $\mathcal{G}(n+1) \prec \mathcal{G}(n)$. In addition, for each n, every point of X belongs to a maximal member of the point-finite collection $\mathcal{G}(n)$ so we may assume that each member of each $\mathcal{G}(n)$ is maximal in $\mathcal{G}(n)$, i.e., if $G, H \in \mathcal{G}(n)$ are distinct, then neither $G \subseteq H$ nor $H \subseteq G$.

Let $\mathcal{U}$ be a collection of open subsets of X. Define $\mathcal{U}(1) = \{G \in \mathcal{G}(1): G \prec \mathcal{U}\}$, where we write $G \prec \mathcal{U}$ to mean that G is a subset of some member of $\mathcal{U}$. For $n \geqslant 1$, let

$$\mathcal{U}(n+1) := \left\{ G \in \mathcal{G}(n+1): G \prec \mathcal{U} \text{ and } G \not\prec \bigcup\{\mathcal{U}(i): 1 \leqslant i \leqslant n\} \right\}.$$

Then $r(\mathcal{U}) := \bigcup\{\mathcal{U}(n): n \geqslant 1\}$ is a collection of open sets in X that refines $\mathcal{U}$ and has $\bigcup r(\mathcal{U}) = \bigcup \mathcal{U}$.

Next we show that $r(\mathcal{U})$ is point-finite. Fix $p \in X$ with $p \in \bigcup \mathcal{U}$. Find the first n such that $p \in \bigcup \mathcal{U}(n)$. Then we have some $G_n \in \mathcal{G}(n)$ with $p \in G_n$ where $G_n \prec \mathcal{U}$. Find $m \geqslant n+1$ so that $\text{St}(p, \mathcal{G}(m)) \subseteq G_n$ and note that if $k \geqslant m$ and $p \in G \in \mathcal{G}(k)$, then $G \subseteq \text{St}(p, \mathcal{G}(m)) \subseteq G_n$ so that $G \prec \bigcup\{\mathcal{U}(j): j \leqslant m\}$ and therefore $G \notin \mathcal{U}(k)$. Consequently,

$$\left\{ G \in r(\mathcal{U}): p \in G \right\} \subseteq \bigcup\{\mathcal{G}(j): j \leqslant m\}$$

and the latter collection is point-finite. Hence $r(\mathcal{U})$ is also point-finite.

To prove (d), suppose distinct $G, H \in r(\mathcal{U})$ have $G \subset H$. Find integers m, n with $G \in \mathcal{G}(m)$ and $H \in \mathcal{G}(n)$. Then $m \neq n$ because each member of $\mathcal{G}(n)$ is maximal. We cannot have $m > n$ because no member of $\mathcal{G}(m)$ was chosen for $r(\mathcal{U})$ if it was contained in a previously chosen member of $r(\mathcal{U})$. So we must have $m < n$. Because $H \in \mathcal{G}(n) \prec \mathcal{G}(m)$ there is some $G' \in \mathcal{G}(m)$ with $H \subseteq G'$. But then $G \subset H \subseteq G'$, which shows that the element $G \in \mathcal{G}(m)$ is not maximal in $\mathcal{G}(m)$, and that is impossible. Hence (d) holds.

To verify (e), suppose $\mathcal{U} \prec \mathcal{V}$. Clearly $\mathcal{U}(1) \prec \mathcal{V}(1)$. Suppose $n \geqslant 1$ and that $\bigcup\{\mathcal{U}(i): i \leqslant n\} \prec \bigcup\{\mathcal{V}(i): i \leqslant n\}$. Let $G \in \mathcal{U}(n+1)$. Then $G \prec \mathcal{U}$ so that $G \prec \mathcal{V}$. If $G \prec \bigcup\{\mathcal{V}(i): i \leqslant n\}$ then $G \prec r(\mathcal{V})$, and otherwise $G \in r(\mathcal{V})$. Hence (e) holds. $\quad\square$

An immediate corollary of the previous result is Theorem 1.1.

Corollary 3.2. *Any metacompact Moore space, and any metric space, is monotonically metacompact.*

Corollary 3.3. *Suppose (X, μ) is a metrizable or metacompact Moore space and $S \subseteq X$. Let μ^S be the topology on X having $\mu \cup \{\{x\}: x \in S\}$ as a base. Then (X, μ^S) is monotonically metacompact.*

Proof. By Proposition 3.1 we know that the space (X, μ) has a monotone metacompactness operator r that acts on collections of μ-open sets, even if they do not cover X. Let $\mathcal{U}$ be any open cover of (X, μ^S). Define $\mathcal{U}_\mu := \{\text{Int}_\mu(U): U \in \mathcal{U}\}$ and

note that $X - S \subseteq \bigcup \mathcal{U}_\mu$. Find the point-finite μ-open refinement $r(\mathcal{U}_\mu)$ and define $s(\mathcal{U}) := r(\mathcal{U}_\mu) \cup \{\{x\}: x \in S\}$. Then the collection $s(\mathcal{U})$ is point-finite in X, covers all of X, and refines $\mathcal{U}$. Further, if $\mathcal{U} \prec \mathcal{V}$ then $s(\mathcal{U}) \prec s(\mathcal{V})$ as required. □

Experience has shown that adding "monotonicity" to a covering property makes the property much stronger. The best example of this is Gary Gruenhage's proof that a monotonically compact[2] Hausdorff space must be metrizable [12]. As noted in the Introduction, every GO-space is countably metacompact. Our next result (which is Theorem 1.2 of the Introduction) shows that adding monotonicity to countable metacompactness makes the property much stronger.

Proposition 3.4. *Suppose* $(X, \tau, <)$ *is a GO-space that is monotonically countably metacompact. Then* (X, τ) *is hereditarily paracompact.*

Proof. If (X, τ) is not hereditarily paracompact, then by [9] there is an uncountable regular cardinal κ and a stationary subset $S \subseteq [0, \kappa)$ that embeds in X under a mapping that is strictly increasing, or strictly decreasing. Consider the case where the mapping is strictly increasing, the other case being analogous. Then we may view S as a subset of X and know that the ordering $<_S$ inherited from $(X, <)$ is the same as the ordering of S as a subspace of κ. This allows us to write such things as "if $\alpha \in S$, then $\alpha^+ \in X$", where α^+ is the first element of S that lies above α, and "$\{(\leftarrow, \alpha^+), (\alpha, \rightarrow)\}$ is an open cover of X".

Suppose there is a monotone countable metacompactness operator r on X. Let S^d be the set of limit points of S in X that belong to S. Then S^d is also stationary in κ. For each $\alpha \in S^d$ consider the open cover $\mathcal{U}(\alpha) = \{(\leftarrow, \alpha^+), (\alpha, \rightarrow)\}$ of X and find $r(\mathcal{U}(\alpha))$. Choose $O(\alpha) \in r(\mathcal{U}(\alpha))$ with $\alpha \in O(\alpha)$. Then $O(\alpha) \subseteq (\leftarrow, \alpha^+)$ and there is some $f(\alpha) \in S$ with $f(\alpha) < \alpha$ such that $[f(\alpha), \alpha] \subseteq O(\alpha)$. The Pressing Down Lemma provides some $\beta \in S$ such that the set $T := \{\alpha \in S^d: f(\alpha) = \beta\}$ is stationary. Choose a strictly increasing sequence $\alpha(1) < \alpha(2) < \cdots$ in T with the property that $\alpha(n)^+ < \alpha(n+1)$ and let $\mathcal{V} = \bigcup\{\mathcal{U}(\alpha(n)): n \geqslant 1\}$. Then $\mathcal{V}$ is a countable open cover of X so that $r(\mathcal{V})$ is defined. For each i, the cover $r(\mathcal{U}(\alpha(i)))$ refines $r(\mathcal{V})$ so there is some $W(i) \in r(\mathcal{V})$ with $O(\alpha(i)) \subseteq W(i)$. Note that for each i we have

$$\beta \in [\beta, \alpha(i)] = [f(\alpha(i)), \alpha(i)] \subseteq O(\alpha(i)) \subseteq W(i),$$

because $\alpha(i) \in T \subseteq S^d$ gives $f(\alpha(i)) = \beta$. We will show that there are infinitely many distinct sets in the collection $\{W(i): i \geqslant 1\}$ and that will contradict point-finiteness of $r(\mathcal{V})$.

Let $j_1 = 1$ and consider $\alpha(j_1) \in O(\alpha(j_1)) \subseteq W(j_1)$. Because $W(j_1) \in r(\mathcal{V})$ which refines $\mathcal{V} = \bigcup\{\mathcal{U}(\alpha(i)): i \geqslant 1\}$ there is some $i_1 \geqslant 1$ for which either $W(j_1) \subseteq (\leftarrow, a(i_1)^+)$ or else $W(j_1) \subseteq (\alpha(i_1), \rightarrow)$. The second alternative cannot happen because $W(j_1)$ contains $\beta < \alpha(i_1)$ while $(\alpha(i_1), \rightarrow)$ does not, so we have $W(j_1) \subseteq (\leftarrow, \alpha(i_1)^+)$. Let $j_2 = i_1 + 1$. Note that $\alpha(i_1)^+ < \alpha(j_2)$ and consider $\alpha(j_2) \in W(j_2)$. Because $\alpha(j_2) \notin W(j_1)$ we know that $W(j_2) \neq W(j_1)$. Because $W(j_2) \in r(\mathcal{V})$ and $r(\mathcal{V})$ refines $\mathcal{V}$, there is some i_2 such that either $W(j_2) \subseteq (\leftarrow, \alpha(i_2)^+)$ or else $W(j_2) \subseteq (\alpha(i_2), \rightarrow)$. The second alternative cannot occur because $\beta \in [\beta, \alpha(j_2)] \subseteq O(\alpha(j_2)) \subseteq W(j_2)$ while $\beta \notin (\alpha(i_2), \rightarrow)$, so $W(j_2) \subseteq (\leftarrow, \alpha(i_2)^+)$. Let $j_3 = i_2 + 1$ and consider $W(j_3)$. Because $\alpha(j_3) \in W(j_3)$ and $\alpha(j_3) \notin W(j_1) \cup W(j_2)$ we see that the sets $W(j_1)$, $W(j_2)$, and $W(j_3)$ are distinct. This recursion continues, producing an infinite sequence of distinct members $W(j_k)$ of $r(\mathcal{V})$, with $\beta \in W(j_k)$ for each k, and that is impossible because $r(\mathcal{V})$ is point-finite. □

There is a generalization of Proposition 3.4 that might be of interest. A deep result of Balogh and Rudin [2] shows that a monotonically normal space is paracompact if and only if it does not contain a closed subspace that is a topological copy of a stationary subset of a regular uncountable cardinal. Therefore, because monotone countable metacompactness is a closed-hereditary property, the proof given for the previous theorem actually shows that a monotonically normal space that is monotonically countably metacompact must be paracompact.

Popvassilev proved in [15] that neither of the ordinal spaces $[0, \omega_1)$ and $[0, \omega_1]$ is monotonically countably metacompact. Our Proposition 3.4 gives another proof of that result.

One might wonder whether, among subspaces of ordinals, the hypothesis of monotone metacompactness would give a conclusion even stronger than hereditary paracompactness. Our next example shows that one cannot obtain metrizability from monotone metacompactness. Because our next example is a LOTS under a different order, it solves Popvassilev's question from [15], but it fails to be first-countable. A first-countable example is given in Example 4.1, below.

Example 3.5. There is a subspace $X \subseteq [0, \omega_1]$ such that X is monotonically metacompact but not metrizable.

Proof. Let $X := \{\alpha \in [0, \omega_1): \alpha$ is not a limit ordinal$\} \cup \{\omega_1\}$ topologized as a subspace of $[0, \omega_1]$. Then in its subspace topology, X is a GO-space. (Note that, under a different ordering, X is actually a LOTS.) Let $\mathcal{U}$ be any open cover of X. Let β be the first ordinal such that $(\beta, \omega_1] \cap X$ is a subset of some member of $\mathcal{U}$ and define $r(\mathcal{U}) := \{(\beta, \omega_1] \cap X\} \cup \{\{\gamma\}: \gamma \leqslant \beta, \gamma \in X\}$. Then r is a monotone metacompactness operator for X, and yet X is not metrizable. □

Our next result proves Theorem 1.3 of the Introduction.

---

[2] X is *monotonically compact* if for every open cover $\mathcal{U}$ of X, there is a finite open refinement $r(\mathcal{U})$ such that if $\mathcal{U} \prec \mathcal{V}$ then $r(\mathcal{U}) \prec r(\mathcal{V})$.

Proposition 3.6. *Suppose* $(X, \lambda, <)$ *is a compact LOTS. Then X is monotonically countably metacompact if and only if (X, λ) is metrizable.*

Proof. If X is metrizable, then X is monotonically metacompact by Corollary 3.2. To prove the other half, suppose (X, λ) is compact and monotonically countably metacompact. We will show that X is *monotonically countably compact* ($=$ every countable open cover $\mathcal{U}$ has a finite open refinement $r(\mathcal{U})$ in such a way that if $\mathcal{U}$ and $\mathcal{V}$ are countable open covers with $\mathcal{U}$ refining $\mathcal{V}$ then $r(\mathcal{U})$ refines $r(\mathcal{V})$), and then we will invoke Popvassilev's theorem that any monotonically countably compact LOTS is metrizable [15].

Let $\mathcal{U}$ be any countable open cover of X. Monotone countable metacompactness gives a point-finite refinement $r(\mathcal{U})$ whose members are convex subsets of X. Replace $r(\mathcal{U})$ by the subcollection $s(\mathcal{U}) := \{V \in r(\mathcal{U}): V \text{ is maximal in } r(\mathcal{U})\}$. If r is a monotone metacompactness operator, then so is s. Notice that no member of $s(\mathcal{U})$ is contained in any other member of $s(\mathcal{U})$.

We claim that $s(\mathcal{U})$ is finite. If not, choose infinitely many distinct sets $V_i \in s(\mathcal{U})$. At most one contains the left end-point of X, for otherwise one member of $s(\mathcal{U})$ would be contained in another member of $s(\mathcal{U})$ and that is impossible, and at most one contains the right endpoint of X. Discarding those two, we may assume that each $V_i = (a_i, b_i)$ for some $a_i, b_i \in X$. For each i, V_i is the only member of $s(\mathcal{U})$ having a_i as its left endpoint, because each member of $s(\mathcal{U})$ is maximal in $s(\mathcal{U})$, so that $a_i \neq a_j$ whenever $i \neq j$. Passing to a subsequence if necessary, we may assume that $\langle a_i \rangle$ is a strictly monotone sequence. Consider the case where $\langle a_i \rangle$ is increasing. Suppose $i < j$. If $b_j \leqslant b_i$ then we would have $a_i < a_j < b_j \leqslant b_i$ and that is impossible because no member of $s(\mathcal{U})$ can contain another. Hence $\langle b_i \rangle$ is also monotone increasing. Because X is compact, $p := \sup\langle a_i \rangle$ and $q := \sup\langle b_i \rangle$ are points of X and $p \leqslant q$. If $p < q$ then infinitely many members of the point-finite collection $s(\mathcal{U})$ contain p, so we have $p = q$. Choose any $V \in s(\mathcal{U})$ that contains p. Then V contains infinitely many of the sets $V_i = (a_i, b_i)$ and that is impossible because no member of $s(\mathcal{U})$ can contain another member of $s(\mathcal{U})$. □

Corollary 3.7. *Suppose* $(X, \tau, <)$ *is a locally countably compact GO-space. Then X is monotonically* (*countably*) *metacompact if and only if (X, τ) is metrizable.*

Proof. If (X, τ) is metrizable, then it is monotonically metacompact by Proposition 3.1 and therefore also monotonically countably metacompact. Next, suppose $(X, \tau, <)$ is monotonically (countably) metacompact. By Proposition 3.4, (X, τ) is hereditarily paracompact, so that (X, τ) is locally compact. Because of Proposition 3.6, (X, τ) is locally metrizable. Because (X, τ) is paracompact, we see that (X, τ) is metrizable. □

Recall the special subsets R_τ and L_τ defined for a GO-space $(X, \tau, <)$ in the Introduction. Theorems of M.J. Faber [10] and Jan van Wouwe [18] show that one key to metrization theory for a GO-space $(X, \tau, <)$ is the hypothesis that $R_\tau \cup L_\tau$ is a σ-closed-discrete subset of (X, τ). We will show that this same hypothesis plays a central role in the study of monotone metacompactness in GO-spaces. We will need the extra hypothesis that the underlying LOTS $(X, \lambda, <)$ of the given GO-space has a σ-closed-discrete dense subset. Examples in the final section of our paper will show that this extra hypothesis is needed.

The remaining results in this section deal with GO-spaces that have σ-closed-discrete dense subsets, and GO-spaces whose underlying LOTS have σ-closed-discrete dense subsets. We will combine them to prove Theorem 1.4 and Corollary 1.5 of the Introduction. Upon first reading of these results, it might be helpful for the reader to replace the hypothesis "σ-closed-discrete dense set" by "separable".

Proposition 3.8. *Suppose* $(X, \tau, <)$ *is a GO-space for which the underlying LOTS $(X, \lambda, <)$ has a σ-closed-discrete dense set. If (X, τ) is monotonically countably metacompact, then $R_\tau \cup L_\tau$ is σ-closed-discrete as a subspace of (X, τ) and as a subspace of (X, λ).*

Proof. From the definition of R_τ (in the Introduction), no point of R_τ is isolated in (X, τ) so that Lemma 2.4 guarantees that R_τ is σ-closed-discrete in (X, τ) if and only if R_τ is σ-closed-discrete in (X, λ). The same assertion holds for L_τ.

Let $D := \bigcup\{D(n): n \geqslant 1\}$ be a σ-closed-discrete dense set in the underlying LOTS (X, λ). Let r be a monotone countable metacompactness operator for (X, τ). We will show that R_τ is σ-closed-discrete in (X, τ). Because $R_\tau \cap D$ is σ-closed-discrete in (X, λ), it is enough to show that the set $R' := R_\tau - D$ is σ-closed-discrete in (X, λ).

For each $p \in R'$, let $\mathcal{U}(p) := \{(\leftarrow, p), [p, \rightarrow)\}$ and find $r(\mathcal{U}(p))$. Choose any $O(p) \in r(\mathcal{U}(p))$ that contains p and note that $O(p) \subseteq [p, \rightarrow)$ because $r(\mathcal{U}(p))$ refines $\mathcal{U}(p)$. Then there is some $y(p) > p$ such that $[p, y(p)) \subseteq O(p)$. Because $p \in R_\tau$, the set $(p, y(p))$ must be infinite. Let $N(p)$ be the first integer k such that $(p, y(p)) \cap D(k) \neq \emptyset$. Decreasing $y(p)$ if necessary, we may assume that $|[p, y(p)) \cap D(N(p))| = 1$ and we choose the unique $d(p) \in (p, y(p)) \cap D(N(p))$.

For each k and each $d \in D(k)$, let $R(d, k) := \{p \in R': N(p) = k, d(p) = d\}$. Note that $R(d, k) \subseteq X - D(k)$ and it is easy to see that $R(d, k)$ is a subset of the unique convex component of $X - D(k)$ that has d as its supremum.

It will be enough to show that each set $R(d, k)$ is a relatively discrete subspace of (X, τ), because then Lemma 2.4 guarantees that each set $R'(k) := \bigcup\{R(d, k): d \in D(k)\}$ is relatively discrete in (X, τ) and therefore (by Lemma 2.4) is σ-closed-discrete in (X, λ). Consequently we will know that the set $R' = \bigcup\{R'(k): k \geqslant 1\}$ is also σ-closed-discrete in (X, λ) and hence also in (X, τ), as claimed.

Fix any set $R(d_0, k_0)$. If $R(d_0, k_0)$ contains no strictly decreasing sequence, then $R(d_0, k_0)$ is well-ordered by the given ordering of X and we can write $R(d_0, k_0) = \{p(\alpha): \ \alpha < \beta\}$. Because each $p(\alpha) \in R' \subseteq R_\tau$, we see that each set $[p(\alpha), p(\alpha + 1)) \in \tau$ and $[p(\alpha), p(\alpha + 1)) \cap R(d_0, k_0) = \{p(\alpha)\}$, showing that $R(d_0, k_0)$ is discrete as a subspace of (X, τ), as claimed. Now consider the case where there is some strictly decreasing sequence $p(0) > p(1) > \cdots$ in $R(d_0, k_0)$. (We will show that this case cannot occur.) From above, $[p(j), d_0) \subseteq O(p(j)) \in r(\mathcal{U}(p(j)))$. Let $\mathcal{V} = \bigcup \{\mathcal{U}(p(j)): \ j \geqslant 1\}$. Note that $\mathcal{V}$ is a countable open cover of X, so $r(\mathcal{V})$ exists. Each $\mathcal{U}(p(j))$ refines $\mathcal{V}$ so that $r(\mathcal{U}(p(i)))$ refines $r(\mathcal{V})$. Consequently we can choose a set $W(j) \in r(\mathcal{V})$ with

$$\big[p(j), d_0\big) \subseteq O\big(p(j)\big) \subseteq W(j).$$

Note that $p(0) \in W(j)$ for each $j \geqslant 1$.

To complete the proof, we will show that there are infinitely many distinct sets in the collection $\{W(j): \ j \geqslant 1\}$ and that will contradict point-finiteness of $r(\mathcal{V})$ at $p(0)$.

Consider the set $W(1)$. Because $W(1) \in r(\mathcal{V})$ and $r(\mathcal{V})$ refines $\mathcal{V}$, there is some $p(i_1)$ such that $W(1)$ is contained in some member of $\mathcal{U}(p(i_1))$, so that either $W(1) \subseteq (\leftarrow, p(i_1))$ or $W(1) \subseteq [p(i_1), \rightarrow)$. The first option cannot occur because $W(1)$ contains the non-empty set $(p(1), d_0)$ while $(\leftarrow, p(i_1))$ does not. Therefore $W(1) \subseteq [p(i_1), \rightarrow)$. Consider $j_2 = i_1 + 1$ and the associated set $W(j_2)$. Because $W(j_2)$ contains $p(j_2)$ while $W(1)$ does not, $W(j_2) \neq W(1)$. Repeating this argument with $p(j_2)$ and $W(j_2)$ in place of $p(1)$ and $W(1)$ we find $p(i_2)$ with $W(j_2) \subseteq [p(i_2), \rightarrow)$. Let $j_3 = i_2 + 1$. This recursion continues, producing the required infinite sequence of distinct elements of $r(\mathcal{V})$ all of which contain $p(0)$, something that is impossible because $r(\mathcal{V})$ is point-finite. That completes the proof that R_τ is σ-relatively discrete in (X, τ).

The proof that L_τ is σ-relatively discrete is analogous, with "reverse well-ordering" in place of "well-ordering" when considering the set $L(d_0, k_0)$ (the analog of $R(d_0, k_0)$ in the above argument). □

It is possible that a GO-space $(X, \tau, <)$ has a σ-closed-discrete dense set even if the underlying LOTS does not. For example, if δ is the discrete topology on the set $X := [0, \omega_1]$ with the usual ordering $<$, then the GO-space $(X, \delta, <)$ has a σ-closed-discrete dense subset even though the underlying LOTS, which is the ordinal space $[0, \omega_1]$, does not. However, a slight modification of the proof of Proposition 3.8 gives:

Corollary 3.9. *Suppose the GO-space $(X, \tau, <)$ has a σ-closed-discrete dense subset. If (X, τ) is (countably) monotonically metacompact then $R_\tau \cup L_\tau$ is σ-closed-discrete in (X, τ).*

Proof. Let $D := \bigcup \{D(k): \ k \geqslant 1\}$ be a σ-closed-discrete dense subset of (X, τ). Then in (X, τ), every closed set is a G_δ-set so that every relatively discrete subset of (X, τ) is σ-closed-discrete in (X, τ) by Lemma 2.1.

Use the notation in the proof of Proposition 3.8. We show that each set $R(d, k)$ is a discrete subspace of (X, τ) which makes each set $R(k) := \bigcup \{R(d, k): \ d \in D(k)\}$ would also be a discrete subspace of (X, τ), and hence σ-closed-discrete in (X, τ), so that $R = \bigcup \{R(k): \ k \geqslant 1\} \cup (R \cap D)$ is also be σ-closed-discrete in (X, τ).

Fix (d_0, k_0) and consider the set $R(d_0, k_0)$. As in the proof of (3.8), the set $R(d_0, k_0)$ cannot contain any strictly decreasing sequence, so that it is well-ordered by the ordering $<$ of X and, just as in (3.8), must be relatively discrete, as required. □

Faber's metrization theorem, Theorem 3.1 in [10], will be the key to our next result. We change some of Faber's notation to avoid conflicts with the notation used in this paper.

Theorem 3.10. *Suppose $(X, \tau, <)$ is a GO-space and $Y \subseteq X$. Then the subspace (Y, τ_Y) is metrizable if and only if*

(a) *(Y, τ_Y) has a dense set D that is the union of countably many subsets of Y, each being closed-discrete in (Y, τ_Y); and*
(b) *the sets $\{y \in Y: \ [y, \rightarrow) \cap Y \in \tau_Y\}$ and $\{y \in Y: \ (\leftarrow, y] \cap Y \in \tau_Y\}$ are both σ-closed-discrete in the subspace (Y, τ_Y).*

Proposition 3.11. *Suppose that $(X, \tau, <)$ is a GO-space and that the underlying LOTS $(X, \lambda, <)$ has a σ-closed-discrete dense set. If the set $R_\tau \cup L_\tau$ is a σ-discrete subspace of (X, τ), then (X, τ) is monotonically metacompact.*

Proof. Let D be a σ-closed-discrete subset of (X, λ) that is dense in (X, λ). In the light of Lemma 2.3, the set $R_\tau \cup L_\tau$ is σ-closed-discrete in (X, λ) and hence also in (X, τ).

Let μ be the topology on X having the collection

$$\lambda \cup \big\{[x, y): \ x \in R, \ x < y\big\} \cup \big\{(x, y]: \ x < y \in L\big\}$$

as a base. Then μ is a GO-topology on X and Faber's metrization theorem shows that (X, μ) is metrizable. Also, we see that $\lambda \subseteq \mu \subseteq \tau$. Now let $S := \{x \in X: \ \{x\} \in \tau\}$. Then, in the notation of Corollary 3.3, $\tau = \mu^S$ showing that (X, τ) is monotonically metacompact. □

Proof of Theorem 1.4 and Corollary 1.5. We can combine Propositions 3.8 and 3.11 to give a proof of Theorem 1.4 of the Introduction. As noted at the beginning of the proof of Proposition 3.11, (c) and (d) are equivalent. Clearly (a) implies (b)

in that theorem, and Proposition 3.8 shows that (b) implies (c). Proposition 3.11 shows that (c) implies (a). The proof of Corollary 1.5 is similar. Clearly (a) implies (b) in Corollary 1.5 and if (X, τ) is monotonically countably metacompact, then Corollary 3.9 shows that $R_\tau \cup L_\tau$ is σ-discrete. In Corollary 1.5, (X, τ) has a σ-closed-discrete dense subset, so that the set I_τ is also σ-closed-discrete. Hence (X, τ) is metrizable, by Theorem 3.10. $\square$

We already proved in Proposition 3.4 that monotone (countable) metacompactness has certain hereditary consequences. A natural question is whether monotone (countable) metacompactness is itself a hereditary property among GO-spaces. We can give an affirmative answer for GO-spaces whose underlying LOTS has a dense σ-closed discrete set. We begin with a lemma.

Lemma 3.12. *Suppose* $(X, \tau, <)$ *is a GO-space whose underlying LOTS* $(X, \lambda, <)$ *has a σ-closed-discrete dense set. Let $S \subseteq X$ and let τ^S be the topology on X for which $\tau \cup \{\{x\}: x \in S\}$ is a base. If $(X, \tau, <)$ is monotonically (countably) metacompact, then so is the GO-space $(X, \tau^S, <)$.*

Proof. As in the Introduction, let $R(\tau) := \{x \in X - I(\tau): [x, \rightarrow) \in \tau\}$. From Proposition 3.8 we know that $R(\tau)$ is σ-closed-discrete in (X, τ) and in (X, λ). Because $\tau \subseteq \tau^S$, we know that $R(\tau)$ is also σ-closed-discrete in (X, τ^S). In order to apply Theorem 1.4 to the GO-space $(X, \tau^S, <)$ we must show that the set $R(\tau^S) := \{x \in X - I(\tau^S): [x, \rightarrow) \in \tau^S\}$ is σ-closed-discrete in (X, τ^S). But that is automatic because $R(\tau^S) \subseteq R(\tau)$. Similarly, the set $L(\tau^S)$ is σ-closed-discrete in (X, τ^S). Now Theorem 1.4 applies to show that the GO-space $(X, \tau^S, <)$ is monotonically (countably) metacompact. $\square$

Proposition 3.13. *Suppose* $(X, \tau, <)$ *is a GO-space whose underlying LOTS* $(X, \lambda, <)$ *has a σ-closed-discrete dense set and suppose that (X, τ) is monotonically (countably) metacompact. Then for every $Y \subseteq X$, the subspace (Y, τ_Y) of (X, τ) is also monotonically (countably) metacompact.*

Proof. Let $Y \subseteq X$. Let $S = X - Y$ and create the topology τ^S as in Corollary 3.3. By Lemma 3.12 we know that (X, τ^S) is monotonically (countably) metacompact. Note that $(\tau^S)_Y = \tau_Y$, i.e., that (Y, τ_Y) is a subspace of (X, τ^S). In fact, (Y, τ_Y) is a *closed* subspace of the monotonically (countably) metacompact space (X, τ^S) and therefore inherits monotone (countable) metacompactness. $\square$

4. Examples and questions

Suppose $(X, <)$ is a linearly ordered set and $Y \subset X$. We say that a set $S \subset Y$ is *relatively convex in Y* if a point b of Y has $b \in S$ whenever $a \leqslant b \leqslant c$ for points $a, c \in S$. For any subset $T \subseteq Y$ we let $C(T) = \bigcup \{[u, v]: u \leqslant v, \ u, v \in T\}$ and we refer to $C(T)$ as the *convex hull of T in X*. Note that $C(T) \cap Y = T$ provided T is a relatively convex subset of Y.

Example 4.1. There is a first-countable monotonically metacompact LOTS that is not metrizable.

Proof. Let $X := (\mathbb{R} \times \{0\}) \cup (\mathbb{P} \times \mathbb{Z})$ with the lexicographic ordering and the open interval topology λ of that ordering. This LOTS (X, λ) contains the Michael line and is therefore non-metrizable.

Let $Y = \mathbb{R} \times \{0\}$ and let μ be the subspace topology that Y inherits from (X, λ). Then $(Y, \mu, <)$ is the Michael line so that Proposition 3.11 gives a monotone metacompactness operator r_Y for (Y, μ). We may assume that r_Y always produces collections of sets that are relatively convex in Y. Let $\mathcal{U}$ be any open cover of X. Without loss of generality, we may assume that members of $\mathcal{U}$ are convex open subsets of X. Let $\mathcal{U}_Y := \{U \cap Y: U \in \mathcal{U}\}$. Find $r_Y(\mathcal{U}_Y)$. For any $S \in r_Y(\mathcal{U}_Y)$ let $C(S)$ be the convex hull of S in X. Because S cannot contain a rational endpoint of itself, it is easy to check that each $C(S)$ is open in X. The collection $r_1(\mathcal{U}) := \{C(S): S \in r_Y(\mathcal{U}_Y)\}$ refines $\mathcal{U}$, covers Y, and is point-finite in X. To complete the proof, let $r(\mathcal{U}) := r_1(\mathcal{U}) \cup \{\{(x, n)\} \in X: n \neq 0\}$. $\square$

With a little more care, we can construct a non-metrizable LOTS that is monotonically metacompact and monotonically Lindelöf. We thank Dennis Burke for pointing out the next example.

Example 4.2. There is a non-metrizable LOTS X having a monotone metacompactness operator R with the additional property that for each open cover $\mathcal{U}$ of X, $R(\mathcal{U})$ is countable. Hence R is also a monotone Lindelöf operator in the sense of [6].

Proof. Let $B \subseteq \mathbb{R}$ be a Bernstein set, i.e., a subset of $\mathbb{R}$ such that neither B nor $C := \mathbb{R} - B$ contains an uncountable compact subset of $\mathbb{R}$. Let $X = (\mathbb{R} \times \{0\}) \cup (C \times \mathbb{Z})$ have the open interval topology λ of the lexicographic order. Let $Y = \mathbb{R} \times \{0\}$ and let μ be the usual open interval topology on the set Y. Note that μ is *not* the subspace topology that Y inherits from (X, λ).

Let $\mathcal{U}$ be an open cover of X by convex sets. Let $a(\mathcal{U}) := \{\text{Int}_\mu(U \cap Y): U \in \mathcal{U}\}$. Then $a(\mathcal{U})$ is a collection of open sets in the metric space (Y, μ) that covers B. According to Proposition 3.1 there is a monotone operator b such that $b(\mathcal{U})$ refines $a(\mathcal{U})$, has $\bigcup b(\mathcal{U}) = \bigcup a(\mathcal{U})$, and is a point-finite collection of relatively convex open intervals in Y. Because (Y, μ)

is separable, $b(\mathcal{U})$ must be countable. Because $b(\mathcal{U})$ covers the Bernstein set $B \times \{0\}$, the set $Y - \bigcup b(\mathcal{U})$ is countable. For any $S \in b(\mathcal{U})$ let $C(S)$ be the convex hull of S in X. Let

$$R(\mathcal{U}) := \big\{C(S)\colon S \in b(\mathcal{U})\big\} \cup \Big\{\big\{(x,n)\big\}\colon (x,n) \in X \text{ and } (x,0) \in Y - \bigcup b(\mathcal{U})\Big\}.$$

Then $R(\mathcal{U})$ is the required countable, open, point-finite refinement of $\mathcal{U}$. $\quad\square$

Although most of our results use the existence of σ-closed-discrete dense sets, they can sometimes be applied in more general contexts.

Example 4.3. The lexicographic square $X = [0,1] \times [0,1]$ with the open interval topology of the lexicographic order is not monotonically (countably) metacompact.

Proof. The space X contains the Alexandroff double interval $Y := [0,1] \times \{0,1\}$ as a closed subspace. Proposition 3.11 shows that Y is not monotonically countably metacompact. Hence neither is X. Alternatively, apply Proposition 3.6. $\quad\square$

Our results characterize monotone metacompactness in GO-spaces whose underlying LOTS has a σ-closed-discrete dense set. Souslin lines are historically important examples of LOTS that are perfect but do not have any σ-closed-discrete dense subsets. Whether or not Souslin lines exist is undecidable in ZFC [16]. As our next two examples show, in any model of ZFC that contains Souslin lines, some Souslin lines will be monotonically metacompact, and others will not be.

Example 4.4. If there is a Souslin line, then there is a Souslin line that is monotonically metacompact.

Proof. If there is a Souslin line, then there is a Souslin tree $(T, \preccurlyeq)$, i.e., a tree with uncountably many levels in which all each levels and all anti-chains are countable. Then there is a Souslin tree T with the properties that each $t \in T$ has ω-many immediate successors [16]. As in [16], a *node of the tree* is the set of all members of the tree with exactly the same set of predecessors. Each node is an anti-chain, so each node is countable. Choose an ordering for each node that makes it look like the set of all integers.

Let $B(T)$ be the branch space of T. Then $B(T)$ is linearly ordered by the "first-difference ordering". Put the open interval topology of that linear ordering on $B(T)$. Because of the special node-orderings chosen above, for each $t \in T$ the set $[t] := \{b \in B(T)\colon t \in b\}$ is a convex open set in the branch space and the set of all possible $[t]$ is a basis for $B(T)$.

Let $\mathcal{U}$ be any open covering of $B(T)$. Given $b \in U \in \mathcal{U}$ there is some $t \in T$ with $b \in [t] \subseteq U$. We will say that t is $\mathcal{U}$-minimal if $[t]$ is contained in some member of $\mathcal{U}$ and if no predecessor of t in T has this property. Let $r(\mathcal{U}) := \{[t]\colon t \in T$ and t is $\mathcal{U}$-minimal$\}$. Then $r(\mathcal{U})$ is an open cover of $B(T)$ that refines $\mathcal{U}$. Note that if $[t_1],[t_2] \in r(\mathcal{U})$ and $b \in B(T)$ has $b \in [t_1] \cap [t_2]$, then $t_1, t_2 \in b$. Because any two members of the branch b are comparable in T, we have either $t_1 \prec t_2$ or vice-versa, and both are impossible by $\mathcal{U}$-minimality of t_1 and t_2. $\quad\square$

Remark 4.5. Note that the argument in the previous example shows that any non-Archimedean space is monotonically metacompact.

Example 4.6. If there is a Souslin line, then there is a Souslin line that is not monotonically metacompact.

Proof. We give two examples, one connected and one totally disconnected. If there is a Souslin line, then there is a compact connected Souslin line S_1 and a compact totally disconnected Souslin line $S_2 = S_1 \times \{0,1\}$ with the lexicographic ordering. Because S_1 and S_2 are compact LOTS that are not metrizable, Proposition 3.6 shows that neither is monotonically metacompact. $\quad\square$

Question 4.7. Characterize monotone (countable) metacompactness in GO-spaces, without making assumptions about the existence of special dense sets (as in Proposition 3.8 and Corollary 1.5).

Question 4.8. Must the GO-space $(X, \tau, <)$ be monotonically (countably) metacompact if its subspace $Y := X - I_\tau$ is monotonically (countably) metacompact?

Question 4.9. If $(X, \tau, <)$ is a monotonically (countably) metacompact GO-space and $Y \subseteq X$, must the subspace (Y, τ_Y) be monotonically (countably) metacompact? In other words, is monotone (countable) metacompactness a hereditary property among GO-spaces? By Proposition 1.7, the answer is "Yes" in case the underlying LOTS of $(X, \tau, <)$ has a σ-closed-discrete dense set, but in general this question remains open.

Question 4.10. If $(X, \tau, <)$ is a monotonically (countably) metacompact GO-space and $S \subseteq X$, is the GO-space $(X, \tau^S, <)$ also monotonically (countably) metacompact? (See Lemma 3.12 for the definition of τ^S.) Note that the proof of Proposition 1.7 shows that an affirmative answer to this question would give an affirmative answer to the previous question.

Question 4.11. Suppose X is a compact Hausdorff space that is monotonically metacompact. Is X metrizable? (Compare Gruenhage's theorem that a monotonically compact Hausdorff space is metrizable [12].)

In Proposition 3.1 we showed that among Moore spaces, metacompactness and monotone metacompactness are equivalent properties. This suggests investigating the role of monotone metacompactness in other generalized metric spaces.

Question 4.12. Must a metacompact quasi-developable space X be monotonically metacompact? What if X is hereditarily metacompact (so that each level of the quasi-development may be assumed to be point-finite)? What if X has a σ-disjoint base?

Question 4.13. Which stratifiable spaces [8] are monotonically metacompact? Two particularly interesting examples of stratifiable spaces are due to McAuley and Ceder (see [1] for a description of these spaces). Are these spaces monotonically metacompact?

Question 4.14. The referee suggested that we ask which of our results can be proved for the class of monotonically normal spaces (which is wider than the class of GO-spaces). We have already remarked that a monotonically normal space that is monotonically countably metacompact must be paracompact. We do not know whether such a space is *hereditarily* paracompact. In addition, we do not know whether a compact monotonically normal space must be metrizable whenever it is monotonically countably metacompact.

References

[1] M. Amono, T. Mizokami, On the position of ceder space and McAuley space, Topology Proc. 32 (2008) 89–100.
[2] Z. Balogh, M. Rudin, Monotone normality, Topology Appl. 47 (1992) 115–127.
[3] B.J. Ball, Countable paracompactness in linearly ordered spaces, Proc. Amer. Math. Soc. 5 (1954) 190–192.
[4] H. Bennett, D. Lutzer, On a question of Maarten Maurice, Topology Appl. 153 (2006) 1631–1638.
[5] H. Bennett, D. Lutzer, The Big Bush machine, preprint.
[6] H. Bennett, D. Lutzer, M. Matveev, The monotone Lindelöf property and separability in ordered spaces, Topology Appl. 151 (2005) 180–186.
[7] H. Bennett, D. Lutzer, S. Purisch, On dense subspaces of generalized ordered spaces, Topology Appl. 93 (1999) 191–205.
[8] C. Borges, On stratifiable spaces, Pacific J. Math. 17 (1966) 1–16.
[9] R. Engelking, D. Lutzer, Paracompactness in ordered spaces, Fund. Math. 94 (1971) 49–58.
[10] M.J. Faber, Metrizability in Generalized Ordered Spaces, Math. Centre Tracts, vol. 53, Mathematisch Centrum, Amsterdam, 1974.
[11] C. Good, R. Knight, I. Stares, Monotone countable paracompactness, Topology Appl. 101 (2000) 281–298.
[12] G. Gruenhage, Monotonically compact Hausdorff spaces are metrizable, Questions Answers Gen. Topology 27 (2009) 57–59.
[13] R. Hodel, Spaces defined by sequences of open covers which guarantee that certain sequences have cluster points, Duke Math. J. 39 (1972) 253–263.
[14] D. Lutzer, On generalized ordered spaces, Dissertationes Math. 89 (1972), 42 pp.
[15] S.G. Popvassilev, $\omega_1 + 1$ is not monotonically countably compact, Questions Answers Gen. Topology 27 (2009).
[16] S. Todorcevic, Trees and linearly ordered sets, in: K. Kunen, J. Vaughan (Eds.), Handbook of Set Theoretic Topology, North-Holland, Amsterdam, 1984, pp. 235–293.
[17] G. Ying, C. Good, A note on monotonic countable paracompactness, Comment. Math. Univ. Carolin. 42 (2001) 771–778.
[18] J.M. van Wouwe, GO-Spaces and Generalizations of Metrizability, Math. Centre Tracts, vol. 104, Mathematisch Centrum, Amsterdam, 1979.