Properties of Fuzzy Implication Operators

Kyung Whan Oh and Wyllis Bandler
Department of Computer Science Florida State University
Tallahassee, Florida

ABSTRACT

In this paper we discuss both forward implication and backward implication, and the difference between them is defined. We introduce some properties of fuzzy implication operators, and show the expectation, the variance, and the distribution of each fuzzy implication operator, assuming that the two propositions in a given compound proposition are independent of each other and the truth values of the propositions are uniformly distributed on the interval [0, 1].

KEYWORDS: ***Fuzzy implication; forward implication; backward implication; fuzzy logic; expectation; variance; distribution; contour***

INTRODUCTION

If fuzzy implication is to be used as the basis of inference, then the truth value of $p \rightarrow q$ needs to be defined. In fuzzy expert systems, the problem is, given values for $p \rightarrow q$ and p, to find a consistent value for q. Bandler and Kohout [1] explain some modes for inference in fuzzy expert systems related to fuzzy implication operators. In this paper, we show the distributions of fuzzy implication values using graphical contour lines and find the expectation and variance of each implication operator. These are, of course, different for the different fuzzy implication operators, and we emphasize also that we can use these results for the interpretation of the truth degree $p \rightarrow q$ or as a measure of confidence for the conclusion q in fuzzy expert systems. Bandler and Kohout [2] have applied fuzzy implication operators to see nonsymmetrical dependencies and implication among the variables in the analysis of clinical data through the theory of fuzzy relational products. When we use operators 5, 5.5, and 6 of Definition 2 (see next section) the difference between the forward implication $\rightarrow$ and the backward implication $\leftarrow$ is simply the difference between values of the

Address correspondence to Dr. Wyllis Bandler, Department of Computer Science, Florida State University, Tallahassee, FL 32306.

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two propositions in the compound proposition, and this result, with the expectation and variance of fuzzy implication operators, can be understood in the analysis of the nonsymmetrical relation. In the design of expert systems, management of uncertainty is a basic and important issue, and it is related to a computation analysis of uncertainty from the premises to the conclusion (see Zadeh [3]). Fuzzy logic underlying approximate reasoning is an approach to the management of uncertainty and requires the use of fuzzy implication operators. In fuzzy expert systems, the combination of distinct incomplete pieces of conclusions may be performed with the evidence on each conclusion. Fuzzy implication in the inference processes can provide the evidence for belief or disbelief in the conclusion. We note that approximate reasoning can be done with the Dempster-Shafer theory [4] using fuzzy implication operators. Many authors have worked on fuzzy implication operators and their applications; see Baldwin [5, 6], Dubois and Prade [7], Weber [8], and Willmott [9].

ANALYTICAL VIEW OF FUZZY IMPLICATION OPERATORS

In classical two-valued logic, one wishes a truth-functional connection, which evaluates the logical formulas of two or more propositions (e.g., "p and q," "p or q," and "if p then q"), and their truth values are either true or false. Multiple-valued logic is required in the theory of fuzzy sets and relations. One wishes to manipulate the degrees of truth that attach to fuzzy statements. The following discussion is related to implication and introduces some properties of the fuzzy implication operators listed by Bandler and Kohout [10] (see Definition 2).

Before we discuss multiple-valued implication, let us look at the standard Boolean operators on the set $B = \{0, 1\}$.

DEFINITION 1 Let p and q be propositions, and let $v(p)$ and $v(q)$ be the truth values of p and q, respectively.

1. Conjunction: $v(p \text{ and } q) = \min(v(p), v(q))$
2. Disjunction: $v(p \text{ and } q) = \max(v(p), v(q))$
3. Negation: $v(\text{not } p) = 1 - v(p)$
4. Implication: $v(p \rightarrow q)$, given by

		$v(q)$ 0	1
$v(p)$	0	1	1
	1	0	1

Since Zadeh introduced fuzzy sets and suggested using min $(v(p), v(q))$,

$\max(v(p), v(q))$, and $1 - v(p)$ for conjunction, disjunction, and negation, respectively, in the fuzzy situation, many authors have proposed other possibilities for these operators [11].

Ten fuzzy implication operators are defined in Definition 2. All such operators have truth values in the closed real interval [0, 1]. A fuzzy implication operator, $\rightarrow$, is a binary operation from $[0, 1] \times [0, 1]$ into [0, 1], which is a generalization of Boolean implication; that is, the values assigned in the crisp "corners," where the values $v(p)$ of p and $v(q)$ of q are 0 (false) or 1 (true), must accord with those of classical Boolean logic.

DEFINITION 2 Let $a = v(p)$ and $b = v(q)$, where p and q are propositions. Let $r = v(p \rightarrow q)$.

1. Standard sharp

$$r = \begin{cases} 1 & \text{if } a < 1 \text{ or } b = 1 \\ 0 & \text{otherwise} \end{cases}$$

2. Standard strict

$$r = \begin{cases} 1 & \text{if } a \leq b \\ 0 & \text{otherwise} \end{cases}$$

3. Standard star

$$r = \begin{cases} 1 & \text{if } a \leq b \\ b & \text{otherwise} \end{cases}$$

4. Gaines 43

$$r = \min(b/a, 1), \qquad \text{where } 0/0 = 1$$

4.5 Modified Gaines 43

$$r = \min\left(1, \frac{b}{a}, \frac{1-a}{1-b}\right), \qquad \text{where if } b = 1 \text{ then } r = 1$$

5. Lukasiewicz

$$r = \min(1 - a + b, 1)$$

5.5 Kleene-Dienes-Lukasiewicz

$$r = 1 - a + ab$$

6. Kleene-Dienes

$$r = \max(b, 1 - a)$$

7. Early Zadeh

$$r = \max(\min(a, b), 1 - a)$$

8. Willmott

$$r = \min(\max(1-a,\ b),\ \max(a,\ 1-b,\ \min(b,\ 1-a)))$$

The operators are listed in order of increasing fuzziness and fall into three groups. Operators 1 and 2 are crisp-valued. Operators 3 through 5, while truly fuzzy, have more than half of their values (for $a \leq b$) equal to 1. Finally, operators 5.5 through 8 yield values of 1 only for $a = 0$ or $b = 1$.

Operator 1 is too severe to find much favor. Operator 4 was introduced by Goguen [12], but Gaines [13] noticed that this implication bears a formal resemblance to conditional probability since, using Zadeh's definition for $v(p$ and $q)$, $v(p \to q) = v(p$ and $q)/v(p)$, while the conditional probability is given by $P(q$ given $p) = P(p$ and $q)/P(p)$.

Sometimes, we may suspect that two propositions p and q are related, but we do not know a priori whether it makes more sense to consider $p \to q$ or $q \to p$. Therefore, we compare the truth of $p \to q$ to that of $q \to p$. Let

$$d = v(p \to q) - v(q \to p),$$

so $d \to 0$ if $p \to q$ is more true than $q \to p$. We next present formulas for d for each of the 10 operators in Definition 2. In order to establish these formulas, we use the following lemma.

LEMMA 1 Let x, y, and z be real numbers. Then the following equalities hold:

1. $x - \min(y, z) = \max(x - y, x - z)$
2. $x - \max(y, z) = \min(x - y, x - z)$
3. $\min(x, y) - z = \min(x - z, y - z)$
4. $\max(x, y) - z = \max(x - z, y - z)$

THEOREM 1 Let $a = v(p)$, $b = v(q)$, and $a, b \in [0, 1]$, and let $p \to q$, $p \leftarrow q$ be the fuzzy implication from p to q and from q to p, respectively. Let $r1 = v(p \to q)$, $r2 = v(p \leftarrow q)$, and $d = r1 - r2$. Then for all three implication operators 5, 5.5, and 6 of Definition 2, $d = b - a$.

Proof For operator 5,

$$\begin{aligned} d &= \min(1,\ 1-a+b) - \min(1,\ 1-b+a) \\ &= \max(\min(1,\ 1-a+b) - 1,\ \min(1,\ 1-a+b) - (1-b+a)) \\ &= \max(\min(0,\ b-a),\ \min(b-a,\ 2(b-a)) \end{aligned}$$

CASE 1. $(b > a) d = \max(0, b - a) = b - a$
CASE 2. $(b \leq a) d = \max(b - a, 2(b - a)) = b - a$
Therefore, $d = b - a$.

For operator 5.5,

$$d = (1-a+ab) - (1-b+ab) = b-a$$

For operator 6,

$$d = \max(1-a,\ b) - \max(1-b,\ a)$$
$$= \max(1-a-\max(1-b,\ a),\ b-\max(1-b,\ a))$$
$$= \max(\min(b-a,\ 1-2a),\ \min(2b-1,\ b-a))$$

CASE 1. $(b \le 1-a) \Rightarrow (b-a \le 1-2a)$ and $(b-a \ge 2b-1)$

$$d = \max(b-a,\ 2b-1) = b-a$$

CASE 2. $(b > 1-a) \Rightarrow (b-a > 1-2a)$ and $(b-a < 2b-1)$

$$d = \max(1-2a,\ b-a) = b-a$$

Therefore, $d = b - a$. ■

THEOREM 2 Let $r1 = v(p \rightarrow q)$ and $r2 = v(p \leftarrow q)$. Let $a = v(p)$ and $b = v(q)$. Let $d = r1 - r2$. Then, for all fuzzy implication operators of Definition 2, the following relations hold:

1. $r1 - r2 < 0 \Rightarrow b - a < 0$
2. $r1 - r2 > 0 \Rightarrow b - a > 0$
3. $b - a > 0 \Rightarrow r1 - r2 \ge 0$
4. $b - a < 0 \Rightarrow r1 - r2 \le 0$

Proof For operator 1,

(i) $0 \le a < 1, 0 \le b < 1: d = 0$
(ii) $a = 1, b < 1: d = -1$
(iii) $a < 1, b = 1: d = 1$
(iv) $a = 1, b = 1: d = 0$

For operator 2,

(i) $a < b: d = 1$
(ii) $a = b: d = 0$
(iii) $a > b: d = -1$

For operator 3,

(i) $a < b: d = 1 - a > 0$
(ii) $a = b: d = 0$
(iii) $a > b: d = b - 1 < 0$

For operator 4,

(i) $a < b: d = 1 - a/b > 0$
(ii) $a = b: d = 0$
(iii) $a > b: d = b/a - 1 < 0$

For operator 4.5,

(i) $a < b$: $r1 = 1$, $r2 = \min(a/b, (1 - b)/(1 - a)) < 1$
Therefore, $d = r1 - r2 > 0$.
(ii) $a = b$: $d = 0$
(iii) $a > b$: $r1 = \min(b/a, (1 - a)/(1 - b)) < 1$, $r2 = 1$
Therefore, $d = r1 - r2 < 0$.

For operators 5, 5.5, 6, see Theorem 1.
For operator 7,

(i) $a < b, a < 1/2$: $d = \min(1 - 2a, b - a) > 0$
(ii) $a < b, a \geq 1/2$: $d = 0$
(iii) $a = b$: $d = 0$
(iv) $a > b, 1 > b \geq 1/2$: $d = 0$
(v) $a > b, b < 1/2$: $d = \max(2b - 1, b - a) < 0$

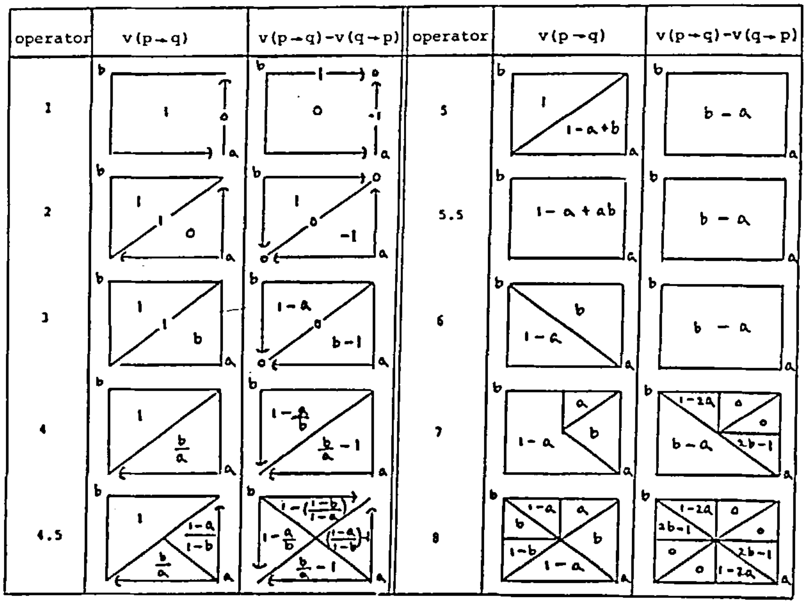
For operator 8,

(i) $b \leq 1 - a, a < b, b > 1/2$: $d = 2b - 1 > 0$
(ii) $b \leq 1 - a, a < b, a \leq 1/2, b \leq 1/2$: $d = 0$
(iii) $b \leq 1 - a, a = b$: $d = 0$
(iv) $b \leq 1 - a, a > b, a \leq 1/2, b < 1/2$: $d = 0$
(v) $b \leq 1 - a, a > b, a > 1/2, b \geq 1/2$: $d = b - a < 0$
(vi) $b \leq 1 - a, a > b, a > 1/2, b < 1/2$: $d = 1 - 2a < 0$
(vii) $b > 1 - a, a \leqslant b, a < 1/2, b \geq 1/2$: $d = 1 - 2a > 0$
(viii) $b > 1 - a, a < b, a < 1/2, b < 1/2$: $d = b - a > 0$
(ix) $b > 1 - a, a < b, a \geq 1/2, b > 1/2$: $d = 0$
(x) $b > 1 - a, a = b$: $d = 0$
(xi) $b > 1 - a, a > b, a < 1/2, b < 1/2$: $d = b - a < 0$
(xii) $b > 1 - a, a > b, a \geq 1/2, b < 1/2$: $d = 2b - 1 < 0$
(xiii) $b > 1 - a, a > b, a > 1/2, b \geq 1/2$: $d = 0$ ■

Theorems 1 and 2 can be illustrated by the difference diagrams of the fuzzy implication operators (see Figure 1).

For each operator of Figure 1, we show a graph of $v(p \to q)$ for various combinations of $v(p)$ and $v(q)$. The abscissa and ordinate are the $v(p)$ and $v(q)$ axes, respectively, on the closed interval [0, 1]. Some of the operators change their functional form across the lines $b = a$, $b = 1 - a$, $a = 1/2$, and/or $b = 1/2$. The difference, $d = v(p \to q) - v(p \leftarrow q)$, compares the two directions of implication. If $d > 0$, then $p \to q$ is more true than $q \to p$.

We summarize the observations of the behavior of the graphs in Figure 1, in Table 1. These observations lead to the following conclusion: When operators 2 through 6 are used, $p \to q$ is more true than $q \to p$ iff q is more true than p, based on statement 4.



—→ means the open set.

Figure 1.

STATISTICAL VIEW OF FUZZY IMPLICATION OPERATORS

In this section, we consider the expected value of a fuzzy implication, its variance, and the distribution of the implication values, assuming that the propositions p and q are independent of each other and the truth values $v(p)$ and $v(q)$ are uniformly distributed across the interval [0, 1]. Let $a = v(p)$ and $b =$

Table 1.

Statement	Applicability to Operators
1. $d = 0$ if $a = b$.	All
2. $d = 0$ iff $a = b$.	2-6
3.1. $d \geq 0$ if $a < b$.	All
3.2. $d \leq 0$ if $a > b$.	All
4.1. $d > 0$ iff $a < b$.	2-6
4.2. $d < 0$ iff $a > b$.	2-6
5. $v(p \to q)$ and d are discontinuous approaching the line $a = b$.	2, 3
6. $v(p \to q)$ and d are discontinuous at one or more corner points.	1-4.5
7. $d = b - a$.	5, 5.5, 6
8. $v(p \to q$ and d are everywhere continuous and obey statements 1-4.	5, 5.5, 6

$v(q)$. Then the value of the implication $I = v(p \rightarrow q)$ is some function a and b, that is, $I = I(a, b)$.

Because a and b are assumed to be uniformly and independently distributed across [0, 1], the expected value of the implication is

$$E(I) = \iint_R I(a, b)\, da\, db$$

and its variance is

$$\mathrm{Var}(I) = E[(I - E(I))^2] = \iint_R (I(a, b) - E(I))^2\, da\, db$$

$$= E[I^2] - E[I]^2$$

where $R = \{(x, y): 0 \leq x \leq 1, 0 \leq y \leq 1\}$

Table 2 lists $E(I)$ and $\mathrm{Var}(I)$ under these assumptions for the 10 operators in Definition 2. This table provides a benchmark for what to expect for an implication value and the typical spread in values, assuming that the two propositions are completely unrelated.

For illustration, we show the computation process for operators 7 and 8. For operator 7, $I = \max(\min(x, y), 1 - x)$, let

$$R1 = \{(x, y): x \leq y, x < 1/2\}$$

$$R2 = \{(x, y): x \leq y, x \geq 1/2\}$$

$$R3 = \{(x, y): x > y, x + y \geq 1\}$$

$$R4 = \{(x, y): x > y, x + y < 1\}$$

Table 2.

Operator	Expectation $E(I)$	Variance $\mathrm{Var}(I)$
1	1	0
2	1/2 = 0.5	1/4 = 0.25
3	2/3 = 0.667	5/36 = 0.1389
4	3/4 = 0.75	5/48 = 0.1042
4.5	ln 2 = 0.693	$2 - 2 \ln 2 - (\ln 2)^2$ = 0.1333
5	5/6 = 0.833	1/18 = 0.0556
5.5	3/4 = 0.75	7/144 = 0.0486
6	2/3 = 0.667	1/18 = 0.0556
7	5/8 = 0.625	3/64 = 0.0469
8	7/12 = 0.583	5/144 = 0.0347

Then

$$E(I) = \iint_R I\,dx\,dy$$

$$= \iint_{R1} (1-x)\,dx\,dy + \iint_{R2} x\,dx\,dy$$

$$+ \iint_{R3} y\,dx\,dy + \iint_{R4} (1-x)\,dx\,dy$$

$$= 7/24 + 1/12 + 1/8 + 1/8 = 5/8$$

and

$$\mathrm{Var}(I) = \iint_R I^2\,dx\,dy - E(I)^2$$

$$= \iint_{R1} (1-x)^2\,dx\,dy + \iint_{R2} x^2\,dx\,dy$$

$$+ \iint_{R3} y^2\,dx\,dy + \iint_{R4} (1-x)^2\,dx\,dy - E(I)^2$$

$$= 15/64 + 11/192 + 7/96 + 7/96 - 25/64 = 3/64$$

For operator 8, $I = \min(\max(1-x, y), \max(x, 1-y, \min(y, 1-x)))$. Let

$$R1 = \{(x, y)\colon 0 \le x \le 1-y,\ 1/2 \le y \le 1\}$$

$$R2 = \{(x, y)\colon 0 \le x \le 1/2,\ x \le y \le 1/2\}$$

$$R3 = \{(x, Y)\colon y \le x \le 1-y,\ 0 \le y \le 1/2\}$$

$$R4 = \{(x, y)\colon 1/2 \le x \le 1,\ 1-x \le y \le x\}$$

$$R5 = \{(x, y)\colon 1/2 \le x \le 1,\ x \le y \le 1\}$$

$$R6 = \{(x, y)\colon 0 \le x \le 1/2,\ 1-x < y \le 1\}$$

Then

$$E(I) = \iint_R I\,dx\,dy$$

$$= \iint_{R1} y\,dx\,dy + \iint_{R2} (1-y)\,dx\,dy$$

$$+ \iint_{R3} (1-x)\,dx\,dy + \iint_{R4} y\,dx\,dy$$

$$+ \iint_{R5} x\,dx\,dy + \iint_{R6} (1-x)\,dx\,dy$$

$$= 1/12 + 1/12 + 1/8 + 1/8 + 1/12 + 1/12 = 7/12$$

and

$$\begin{aligned}\mathrm{Var}(I) &= \iint_R I^2\,dx\,dy - E^2(I)\\ &= \iint_{R1} y^2\,dx\,dy + \iint_{R2} (1-y)^2\,dx\,dy\\ &\quad + \iint_{R3} (1-x)^2\,dx\,dy + \iint_{R4} y^2\,dx\,dy\\ &\quad + \iint_{R5} x^2\,dx\,dy + \iint_{R6} (1-x)^2\,dx\,dy - E(I)^2\\ &= 11/192 + 11/192 + 7/96 + 7/96 + 11/192 - 49/144 = 5/144\end{aligned}$$

Let c be a fuzzy implication value, and let $F(c)$ denote the cumulative distribution function:

$$F(c) = \mathrm{Prob}\{I \le c\}$$

Figure 2 is derived from Figure 1 and is a contour plot of the same implication values. We find the cumulative distribution function through the area on the ab plane with the implication values less than or equal to c, and the areas are computed with Figure 2. Figure 3 describes the distribution of c.

We can also consider the probability density function of each fuzzy implication operator by finding the derivative of the distribution function $F(c)$, that is

$$F'(c) = \frac{dF(c)}{dc}$$

For example, in the cases of operators 4.5 and 7, we find $F'(c)$.

For operator 4.5,

$$F'(c) = \begin{cases} 1/(c+1)^2 & \text{if } c<1\\ 1/2 & \text{if } c=1\end{cases}$$

For operator 7,

$$F'(c) = \begin{cases} 2c & \text{if } c<1/2\\ -2c+3 & \text{if } c\ge 1/2\end{cases}$$

CONCLUSIONS

In this paper we have investigated some properties of fuzzy implication operators. The difference between the forward implication $\rightarrow$ and the backward implication $\leftarrow$, using operators 5, 5.5, and 6, is simply the difference between values of the two propositions in the compound proposition.

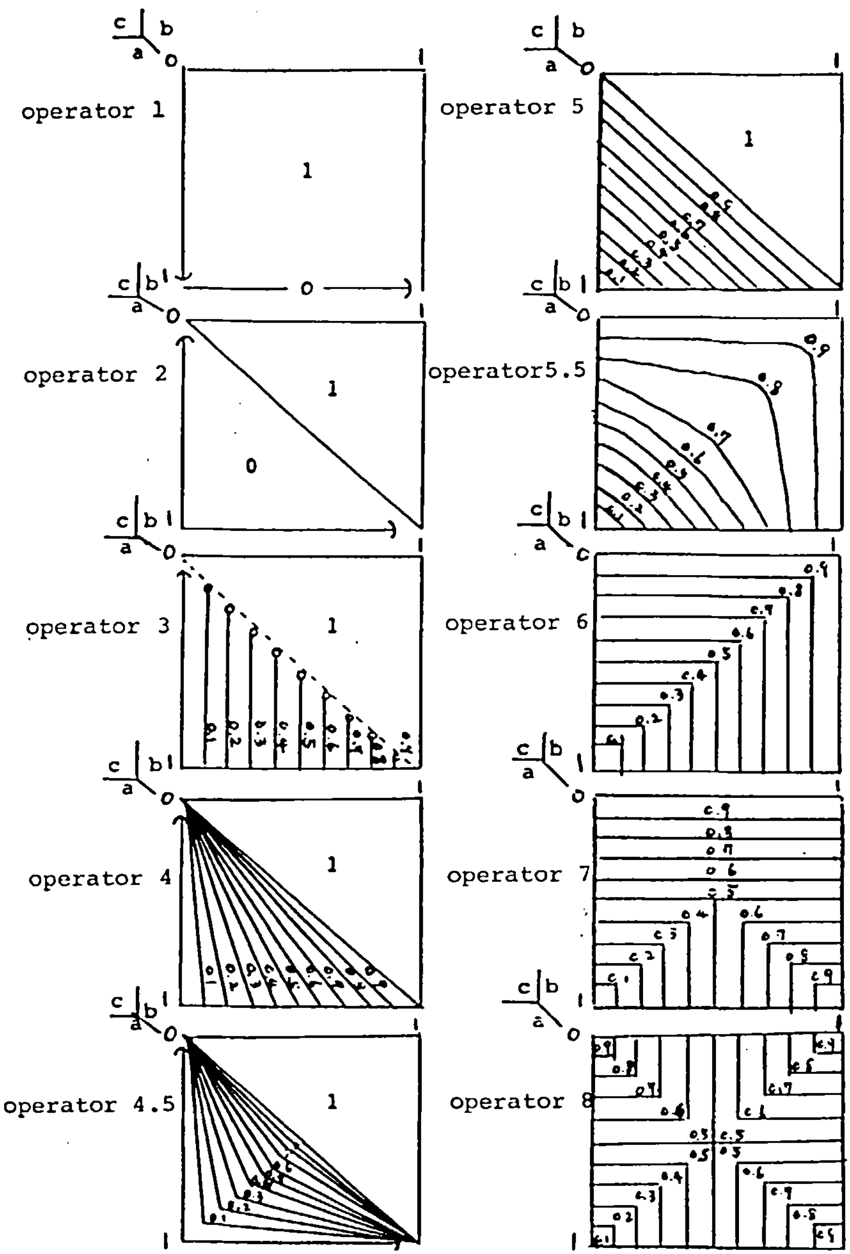

operator 1
operator 2
operator 3
operator 4
operator 4.5
operator 5
operator5.5
operator 6
operator 7
operator 8

Figure 2.

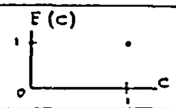
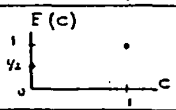
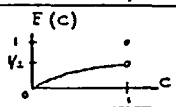
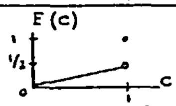
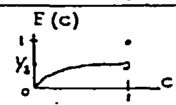
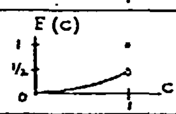
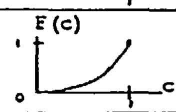
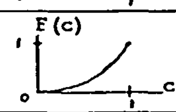
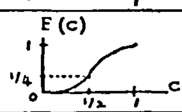
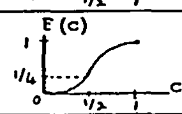
operator		cumulative distribution function
1		$F(c) = 1 \quad \text{if } c = 1$
2		$F(c) = \begin{cases} 1/2 & \text{if } c = 0 \\ 1 & \text{if } c = 1 \end{cases}$
3		$F(c) = \begin{cases} -1/2\, c^2 + c & \text{if } c < 1 \\ 1 & \text{if } c = 1 \end{cases}$
4		$F(c) = \begin{cases} c/2 & \text{if } c < 1 \\ 1 & \text{if } c = 1 \end{cases}$
4.5		$F(c) = \begin{cases} c/(c+1) & \text{if } c < 1 \\ 1 & \text{if } c = 1 \end{cases}$
5		$F(c) = \begin{cases} 1/2\, c^2 & \text{if } c < 1 \\ 1 & \text{if } C = 1 \end{cases}$
5.5		$F(c) = c + (1 - c) \ln (1 - c)$
6		$F(c) = c^2$
7		$F(c) = \begin{cases} c^2 & \text{if } c < 1/2 \\ -(c - 3/2)^2 + 5/4 & \text{if } c \geq 1/2 \end{cases}$
8		$F(c) = \begin{cases} c^2 & \text{if } c < 1/2 \\ -3\,(c - 1)^2 + 1 & \text{if } c \geq 1/2 \end{cases}$

Figure 3.

The expectation and variance of I is different for the different fuzzy implication operators. If $v(p)$ and $v(q)$ are uniformly distributed and independent of each other, then I is not uniformly distributed for any of the 10 kinds of fuzzy implication operators.

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