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Constitutive modeling of isotropic hyperelastic materials in an exponential framework using a self-contained approach

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ABSTRACT

In this paper, an exponential framework for strain energy density functions of elastomers and soft biological tissues is proposed. Based on this framework and using a self-contained approach that is different from a guesswork or combination viewpoint, a set strain energy density functions in terms of the first and second strain invariants is rebuilt. Among the constructed options for strain energy density, a new exponential and mathematically justified model is examined. This model benefits from the existence of second strain invariant, simplicity, stability of parameters, and the state of being accurate. This model can capture strain softening, strain hardening and is able to differentiate between various deformation-state dependent responses of elastomers and soft tissues undergoing finite deformation. The model has two material parameters and the mathematical formulation is simple to render the possibility of numerical implementations. In order to investigate the appropriateness of the proposed model in comparison to other hyperelastic models, several experimental data for incompressible isotropic materials (elastomers) such as VHB 4905 (polyacrylate rubber), two various silicone rubbers, synthetic rubber neoprene, two different natural rubbers, b186 rubber (a carbon black-filled rubber), Yeoh vulcanizate rubber, and finally porcine liver tissue (a very soft biological tissue) are examined. The results demonstrate that the proposed model provides an acceptable prediction of the behavior of elastomers and soft tissues under large deformation for different applied loading states.

1. Introduction

The most outstanding property of elastomers is their ability to undergo large deformation under relatively small stress. Over the last few decades, there have been considerable applications of them at real life scenarios as well as model developments based on different assumptions to predict the non-linear behavior of such materials. As example of these large stretches, rubber boot of an automotive CV joint that is simultaneously seeing large deformations and different loads and temperatures, large deformation analysis of dielectric membranes (Goulbourne et al., 2007), elastomeric polymer light-emitting devices and displays (Liang et al., 2013), stretchable organic solar cells (Lipomi et al., 2011), flexible and stretchable electrodes for dielectric elastomer actuators (Rosset and Shea, 2013), high strain rate response of rubber membrane (Albrecht and Ravi-Chandar, 2013), inflation of tubular elastomeric balloons (Mao et al., 2014) all rely on the remarkable large-stretch properties of elastomeric materials. Experimental evidences suggest that there exist three regions for engineering stress-stretch curves of rubbers: a large softening region with monotone increasing is followed by a moderate region and then often, it experiences an abrupt upturn under large strains. The computed results show rather abrupt changes in the slope of the curves at biological tissues than rubbers.

As Fig. 1 depicts, the behavior of elastomeric materials differs at different states of deformation. Another significant outcome of this simulation is that, a hyperelastic model could be handled well in uniaxial mode but might fail at another deformation. Considering this complexity in analysis for a real construction in the finite element method demands an accurate mechanical model. Accordingly, one often adopts a strategy that seeks to use the most accurate mechanical model as well as the simplest acceptable model that contributes a reasonable approximation of a deformation, usually leads to a weighted model with large number of parameters.

A considerable number of investigations have been conducted over the last decades on the elastomers and their large strain behaviors. Different constitutive models have been proposed to

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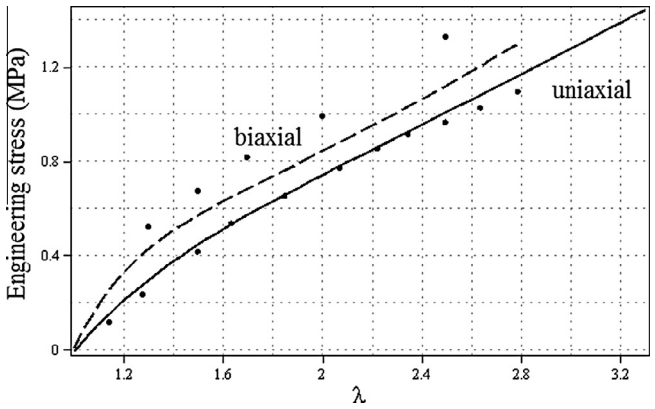

Fig. 1. Material behavior at different deformations, for example: uniaxial and biaxial deformations. The Arruda–Boyce model (1993) can follow experimental data accurately at uniaxial deformation but this model is not sufficient for another mode of deformation for gum rubber. Data is taken from James et al. (1975).

predict these characteristics which some require complex material investigation in order to derive material parameters. In the framework of the theory of isotropic hyperelasticity, strain energy functions may be divided in two overall categories: statistical models and phenomenological models.

Statistical mechanics models or chain models such as Arruda–Boyce 8-chain model (1993) appear to provide the most predictive model of the larger strain behavior under different states of deformation (Boyce and Arruda, 2000). Chain models take advantage of this useful property that they comprise minimum number of parameters, which are related to molecular quantities. It is nonetheless useful to mention that the four parameters Meissner and Matejka model (2003) is not related to molecular quantities, the four parameters Edwards and Vilgis model (1986) related to the molecular quantities and its form is so complicated, two number of five parameters of Miehe et al. model (2004) is not related to molecular features and Arruda–Boyce model (1993) fails at small deformation. Moreover, it is worth to mention two physically-based models in finite elasticity of elastomers which have been developed and applied to fit similar experimental data. The first model is by Drozdov and Gottlieb (2006) which coincides with the Ogden law by a special choice of adjustable parameters, and the second model is by Drozdov and Christiansen (2006) that is a constrained chain model with four parameters. Although chain models obtain many advantages for behavior modeling of the rubbers, it is not suitable for taking some important observed phenomena such as irreversible deformations and inelastic volumetric expansion (Gernay et al., 2013). As mentioned previously, chain models have a complicated structure, so that in some cases they are not amenable to provide a closed-form solution. Maybe this is the reason that Horgan and Saccomandi (2003, 1999) and others have a tendency to use more simple strain energy functions for analytical closed-form solution to boundary-value problems instead of complicated chain models. Moreover, chain models, due to complicated forms, are not a desirable for numerical solutions.

The phenomenological models are supported by a mathematical relationship and treat the problem from viewpoint of continuum mechanics. In this viewpoint, strain energy function may be modeled as invariant-base and principal stretch-base. For example, Mooney (1940) published an invariant-base phenomenological model in terms of principal invariants of the right Cauchy–Green strain tensor, I_1 and I_2. Later, Treloar (1943a,b) proposed a so-called Neo-Hookean material model in terms of I_1 with only one material parameter; although, these two models are restricted to the small deformations. Rivlin (1948) introduced a generalized model, also called polynomial hyperelastic model in terms of strain invariants. Following this structure, several investigators attempted to consider strain invariants in their models in the framework of polynomial hyperelastic model. As examples of these models Yeoh (1993), Isihara et al. (1951), Biderman (1958), James et al. (1975), Lopez-Pamies (2010), Attard and Hunt (2004) and Hartmann and Neff (2003) altogether are polynomial forms of strain energy and encompass high order strain invariants. Tschoegl (1971) emphasized that keeping of higher order terms in the generalized Mooney–Rivlin polynomial function leads to a better correlation with test data for both filled and unfilled rubbers. Although high-order polynomial models, in terms of strain invariants, fit the almost any hyperelastic experimental curve, but they can also introduce difficulties during numerical solution (Meunier et al., 2007, 2008). Hence, researchers prefer to use a complete yet as simple as possible constitutive model for numerical and analytical solutions. The two-parameter Gent model (1996) is a first invariant-based and has some attractive features, so that Boyce (1996) compared this model with the 8-chain model and realized that they are almost identical to structure and qualities of test results. Also, Pucci-Saccomandi model (2002) and Yeoh-Fleming model (1997) combine the concept of Gent. The other strain energy functions include Gent and Thomas (1958), Hart-Smith (1966), Carroll (2011), Khajehsaeid et al. (2013) and Nunes (2011) for modeling the nonlinear elastic response of elastomers. Another brilliant stretch-base strain energy function has been proposed by Ogden (1972). It delivers a good agreement with Treloar's experimental data for extension of unfilled natural rubber.

There exists several comprehensive comparison and excellent review of the development of phenomenological and statistical mechanics treatment of rubber elasticity at different literatures, e.g., Elıas-Zúñiga and Beatty (2002), Boyce and Arruda (2000), Marckmann and Verron (2006), Steinmann et al. (2012), Martins et al. (2006) and Vahapoglu and Karadeniz (2006). As a result of these investigations, a large number of well-known models are not reliable on the entire ranges of strain and different modes of deformation, simultaneously. Furthermore, they might be failed to be well matched with different materials.

In this work, we propose a new framework for strain energy density function of elastomers and soft tissues (Section 2). In Section 2, based on this framework we construct a set strain energy density functions in terms of the first and second strain invariants. Among this set, we select a simple exponential mathematical model with two material parameters for the behavior modeling of a wide range of rubber-like materials and biological tissues at different states of deformation. In Section 3, the calculation method of the material parameters is presented. In Section 4, we fit the proposed model to different test data to demonstrate the model's performance in describing rubber-like and biological materials. As a result of these comparisons, the model is able to capture mechanical behavior of such materials. In Section 5, the significance of different terms in the proposed model is discussed, comprehensively.

2. Model development

The general motion of a continuum is described by $\mathbf{x} = \chi(\mathbf{X}, t)$, where $\mathbf{X}$ and $\mathbf{x}$ denote the position vectors of material particle in its reference configuration and current configuration at time t, respectively. The deformation gradient is shown by $\mathbf{F} = \partial\mathbf{x}/\partial\mathbf{X}$. Since $\det(\boldsymbol{F}) > 0$, the polar decomposition theorem states that F is uniquely decomposed as

$$\mathbf{F} = \mathbf{RU} = \mathbf{VR} \tag{1}$$

where $\mathbf{U}$ and $\mathbf{V}$ are the right and left stretch tensors, respectively. $\mathbf{U}$ and $\mathbf{V}$ are positive definite symmetric tensors and $\mathbf{R}$ is a proper

orthogonal rotation tensor, which represents the rotation of the eigenvectors of $\mathbf{U}$, $\boldsymbol{N}_i$, to the eigenvectors of $\mathbf{V}$, $\boldsymbol{n}_i$.

$$\boldsymbol{n}_i = \boldsymbol{R}\boldsymbol{N}_i \tag{2}$$

Let λ_i ($i = 1,2,3$) be eigenvalues of the stretch tensors. Indeed, λ_i are the principal stretches of the deformation. The invariants of the right Cauchy–Green strain tensor are

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \quad I_2 = \lambda_1^2\lambda_2^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2\lambda_3^2, \quad I_3 = \lambda_1^2\lambda_2^2\lambda_3^2 \tag{3}$$

For a strain energy density function of the form $W(I_1,I_2)$ in terms of strain invariants, the constitutive equation for Cauchy stress of an incompressible, isotropic, non-linear elastic material can be found by

$$\boldsymbol{\sigma} = -p\mathbf{I} + 2\frac{\partial W}{\partial I_1}\mathbf{B} - 2\frac{\partial W}{\partial I_2}\mathbf{B}^{-1} \tag{4}$$

where p is the indeterminate scalar arising from the constraint of incompressibility and $\boldsymbol{B}$ denotes the left Cauchy–Green strain tensor and is shown by $\mathbf{B} = \mathbf{F}\mathbf{F}^T$. In this case, the strain invariants become $I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$, $I_2 = \lambda_1^{-2} + \lambda_2^{-2} + \lambda_3^{-2}$ and $I_3 = 1$.

A strain energy function is constructed by applying the governing postulates. Ogden (1997) and Treloar (1975) have presented a detailed discussion about the restrictions on the form of the strain energy. The postulates they have made include the following items:

1. The strain energy density function must be non-negative for all deformations.
2. The strain energy density function must be invariant under coordinate transformations.
3. The strain energy density must be a function of either the stretch or the strain invariants. Also because of isotropy, the strain energy density function is symmetrical with respect to the principal stretches λ_1, λ_2 and λ_3.
4. The strain energy density function must have a zero value at the undeformed state $\lambda_1 = \lambda_2 = \lambda_3 = 1$.
5. The strain energy density function must be minimum at the undeformed state $\lambda_1 = \lambda_2 = \lambda_3 = 1$. This guarantees that the material is stress free at the undeformed state.
6. The strain energy density function must approach positive infinity at the singularity case ($\lambda_1 = 0$ or $\lambda_2 = 0$ or $\lambda_3 = 0$) and for very large deformations ($\lambda_1 = \infty$ or $\lambda_2 = \infty$ or $\lambda_3 = \infty$).

Hence,

$$\lim_{\lambda_i\to 0} W = \infty, \quad \lim_{\lambda_i\to\infty} W = \infty, \quad \lim_{\lambda_i\to 0}\frac{\partial W}{\partial \lambda_i} = -\infty, \quad \lim_{\lambda_i\to\infty}\frac{\partial W}{\partial \lambda_i} = \infty \tag{5}$$

Darijani and naghdabadi (2010) proposed 16 strain energy density functions by considering linear combination of the mathematical functions including polynomial, power law, logarithmic and exponential forms. They demonstrated that the strain energy density functions including exponential terms (even with a few parameters), in comparison with the other functions, can precisely fit the test data for both compressible and particularly incompressible materials. They presented the strain energy function in exponential form as follows

$$\begin{aligned} W(\lambda_i) = &\sum_{k=1}^{\infty} A_k[\exp(m_k(\lambda_1 - 1)) + \exp(m_k(\lambda_2 - 1)) \\ &+\exp(m_k(\lambda_3 - 1)) - 3] + \sum_{k=1}^{\infty} B_k\left[\exp(n_k(\lambda_1^{-1} - 1))\right. \\ &\left.+\exp\left(n_k(\lambda_2^{-1} - 1)\right) + \exp\left(n_k(\lambda_3^{-1} - 1)\right) - 3\right] \end{aligned} \tag{6}$$

However, this function contains a finite number of terms and is a somewhat weighted model. Applying this model into finite element analysis and mathematical computations for investigation of material response may lead to a complicated calculation. It can be stated that there is a challenge among researchers for introducing a strain energy density which can capture the material behavior and be as simple as a possible mathematical function having few parameters. Having the comparatively high ability of exponential forms of strain energy density in capturing of mechanical behavior encouraged us to propose a new general framework that is a generalization of Eq. (6) as follows

$$\begin{aligned} W = &\sum_{k=1}^{\infty} A_k[\exp(f(\lambda_1,\lambda_2,\lambda_3)) - 1] \\ &+ \sum_{k=1}^{\infty} B_k[\exp(g(\lambda_1,\lambda_2,\lambda_3)) - 1] \end{aligned} \tag{7}$$

where the coefficients A_k and B_k are the material parameters and their unit of measurement is Pascal. Based on the mathematical properties of exponential functions specially their growth rate, it seems that they can model the mechanical behavior of the elastomers and biological tissues in the case of undergoing severe strain-stiffening in the stress-stretch response.

The functions f and g should be presented in a way that they satisfy the fundamental postulates on the form of the strain energy density function. Because of isotropy, the strain energy density function is symmetrical with respect to the principal stretches λ_1, λ_2 and λ_3. Hence, $f(\lambda_1,\lambda_2,\lambda_3)$ and $g(\lambda_1,\lambda_2,\lambda_3)$ must be symmetric functions. The other mentioned governing postulates on strain energy density can be satisfied via the restrictions imposed on the functions f and g as follows

$$\begin{aligned} &\lim_{\lambda_i\to\infty} f(\lambda_1,\lambda_2,\lambda_3) = \infty, \quad \lim_{\lambda_i\to\infty} f'(\lambda_1,\lambda_2,\lambda_3) = \infty, \\ &\lim_{\lambda_i\to 0} g(\lambda_1,\lambda_2,\lambda_3) = \infty, \quad \lim_{\lambda_i\to 0} g'(\lambda_1,\lambda_2,\lambda_3) = -\infty \end{aligned} \tag{8}$$

It is possible to find many functions for f and g in order to meet the constraints given in Eq. (8). For example, the functions subject to Eq. (8) can be presented in the forms of

$$\begin{aligned} &f = m_k(\lambda_1^{\alpha_k} + \lambda_2^{\alpha_k} + \lambda_3^{\alpha_k} - 3), \quad g = n_k\left(\lambda_1^{-\beta_k} + \lambda_2^{-\beta_k} + \lambda_3^{-\beta_k} - 3\right) \\ &f = m_k(\lambda_1^{\alpha_k} + \lambda_2^{\alpha_k} + \lambda_3^{\alpha_k} - 3), \quad g = \ln\left(\lambda_1^{-\beta_k} + \lambda_2^{-\beta_k} + \lambda_3^{-\beta_k} - 2\right) \\ &f = \ln(\lambda_1^{\alpha_k} + \lambda_2^{\alpha_k} + \lambda_3^{\alpha_k} - 2), \quad g = n_k\left(\lambda_1^{-\beta_k} + \lambda_2^{-\beta_k} + \lambda_3^{-\beta_k} - 3\right) \\ &f = m_k\ln\left(\frac{\lambda_1^{\alpha_k} + \lambda_2^{\alpha_k} + \lambda_3^{\alpha_k}}{3}\right), \quad g = n_k\ln\left(\frac{\lambda_1^{-\beta_k} + \lambda_2^{-\beta_k} + \lambda_3^{-\beta_k}}{3}\right) \end{aligned} \tag{9}$$

where α_k, β_k take real positive values and m_k, n_k are non-dimensional values.

Considering the case $g(\lambda_1,\lambda_2,\lambda_3) = 0$ in Eq. (8), makes the function f to be putted responsible for constraints on strain energy function as,

$$\begin{aligned} &\lim_{\lambda_i\to\infty} f(\lambda_1,\lambda_2,\lambda_3) = \infty, \quad \lim_{\lambda_i\to 0} f(\lambda_1,\lambda_2,\lambda_3) = \infty, \\ &\lim_{\lambda_i\to\infty} f'(\lambda_1,\lambda_2,\lambda_3) = \infty, \quad \lim_{\lambda_i\to 0} f'(\lambda_1,\lambda_2,\lambda_3) = -\infty \end{aligned} \tag{10}$$

Among the possible functions for this case, we can refer to the following forms

$$\begin{aligned} &f = m_k(\lambda_1^{\alpha_k} + \lambda_2^{\alpha_k} + \lambda_3^{\alpha_k} - 3) + n_k\left(\lambda_1^{-\beta_k} + \lambda_2^{-\beta_k} + \lambda_3^{-\beta_k} - 3\right), \quad g = 0 \\ &f = m_k\ln\left(\frac{\lambda_1^{\alpha_k} + \lambda_2^{\alpha_k} + \lambda_3^{\alpha_k}}{3}\right) + n_k\left(\lambda_1^{-\beta_k} + \lambda_2^{-\beta_k} + \lambda_3^{-\beta_k} - 3\right), \quad g = 0 \\ &f = m_k(\lambda_1^{\alpha_k} + \lambda_2^{\alpha_k} + \lambda_3^{\alpha_k} - 3) + n_k\ln\left(\frac{\lambda_1^{-\beta_k} + \lambda_2^{-\beta_k} + \lambda_3^{-\beta_k}}{3}\right), \quad g = 0 \\ &f = \ln(\lambda_1^{\alpha_k} + \lambda_2^{\alpha_k} + \lambda_3^{\alpha_k} - 2), \quad g = 0 \end{aligned} \tag{11}$$

Substituting Eqs. (9) and (11), as desired functions for f and g, into Eq. (7) gives the possible forms for strain energy density as

$$
\begin{aligned}
&W=\sum_{k=1}^{\infty}A_k\left[\exp\left(m_k\left(\lambda_1^{\alpha_k}+\lambda_2^{\alpha_k}+\lambda_3^{\alpha_k}-3\right)\right)-1\right]\\
&\quad+\sum_{k=1}^{\infty}B_k\left[\exp\left(n_k\left(\lambda_1^{-\beta_k}+\lambda_2^{-\beta_k}+\lambda_3^{-\beta_k}-3\right)\right)-1\right]\\
&W=\sum_{k=1}^{\infty}A_k\left[\exp\left(m_k\left(\lambda_1^{\alpha_k}+\lambda_2^{\alpha_k}+\lambda_3^{\alpha_k}-3\right)\right)-1\right]+\sum_{k=1}^{\infty}B_k\left[\left(\lambda_1^{-\beta_k}+\lambda_2^{-\beta_k}+\lambda_3^{-\beta_k}-3\right)\right]\\
&W=\sum_{k=1}^{\infty}A_k\left(\lambda_1^{\alpha_k}+\lambda_2^{\alpha_k}+\lambda_3^{\alpha_k}-3\right)+\sum_{k=1}^{\infty}B_k\left[\exp\left(n_k\left(\lambda_1^{-\beta_k}+\lambda_2^{-\beta_k}+\lambda_3^{-\beta_k}-3\right)\right)-1\right]\\
&W=\sum_{k=1}^{\infty}A_k\left[\left(\frac{\lambda_1^{\alpha_k}+\lambda_2^{\alpha_k}+\lambda_3^{\alpha_k}}{3}\right)^{m_k}-1\right]+\sum_{k=1}^{\infty}B_k\left[\left(\frac{\lambda_1^{-\beta_k}+\lambda_2^{-\beta_k}+\lambda_3^{-\beta_k}}{3}\right)^{n_k}-1\right]\\
&W=\sum_{k=1}^{\infty}A_k\left[\exp\left(m_k\left(\lambda_1^{\alpha_k}+\lambda_2^{\alpha_k}+\lambda_3^{\alpha_k}-3\right)+n_k\left(\lambda_1^{-\beta_k}+\lambda_2^{-\beta_k}+\lambda_3^{-\beta_k}-3\right)\right)-1\right]\\
&W=\sum_{k=1}^{\infty}A_k\left[\left(\frac{\lambda_1^{\alpha_k}+\lambda_2^{\alpha_k}+\lambda_3^{\alpha_k}}{3}\right)^{m_k}\exp\left(n_k\left(\lambda_1^{-\beta_k}+\lambda_2^{-\beta_k}+\lambda_3^{-\beta_k}-3\right)\right)-1\right]\\
&W=\sum_{k=1}^{\infty}A_k\left[\left(\frac{\lambda_1^{-\beta_k}+\lambda_2^{-\beta_k}+\lambda_3^{-\beta_k}}{3}\right)^{n_k}\exp\left(m_k\left(\lambda_1^{\alpha_k}+\lambda_2^{\alpha_k}+\lambda_3^{\alpha_k}-3\right)\right)-1\right]\\
&W=\sum_{k=1}^{\infty}A_k\left(\lambda_1^{\alpha_k}+\lambda_2^{\alpha_k}+\lambda_3^{\alpha_k}-3\right)
\end{aligned}
\tag{12}
$$

We could present strain energy density function in terms of principal stretches. More, we make an effort to introduce strain energy in a simpler form in terms of strain invariants. Before this, we clarify the importance of second strain invariant in strain energy density for the behavior modeling of the hyperelastic materials.

Strain energy density function for incompressible elastomeric hyperelastic materials depends on the first two principal invariants of the strain tensor, I_1 and I_2 or depends on principal stretches as well as a hybrid form of them. There has been remarkable interest to know that what is the preference for modeling of elastomers in terms of first and second invariants in comparison to those are solely based on first strain invariant. These differences will be negligible when attention is just limited to simple extension or pure shear, where material response can be completely modeled by a strain energy density function in the form of $W(I_1)$ (Wineman, 2005), while in a real scenario, structures undergo 3D deformation. In fact, major difficulty in the contribution of the hyperelastic behavior of elastomers is to account for its dependence on the deformation (Arruda and Boyce, 1993). From the phenomenological point of view, this difficulty has been overcame by making W depends on two invariants of strain tensor for incompressible isotropic materials (Diani and Gilormini, 2005). Horgan and Smayda (2012) treated two models, Fung (1967) and Vito (1973), the first depends only on the first invariant, and second model include both of invariants. They realized that there is approximately 100% difference between the predictions of the models in some regions of deformation. Also they concluded significant role of second invariant in the deformation of simple shear. It is noted that, vanishing I_2 in the strain energy function can result in some limitations such as not reflecting poynting-type effect in simple shear and underestimating the axial force in pure torsion (Horgan and Saccomandi, 2006; Horgan and Smayda, 2012). Horgan and Saccomandi (1999) demonstrated that a model as $W(I_1)$ always will be failed to characterize simple torsion experiments. Therefore, inclusion of second invariant in hyperelastic models of rubber-like materials and soft biomaterials bring more accuracy in general.

Now, we face this issue that how to benefit from the existence of exponential terms in the strain energy density function such that it can be represented as simple as possible with mathematical function having few parameters and short terms. In order to provide a simple mathematical forms including second strain invariant as well as the first invariant for strain energy density, it is enough to set $k=1$ and $\alpha_1=\beta_1=2$ in the first six functions of Eq. (12). This setting provides the simple mathematical models in the following forms that some of them are new.

$$
\begin{aligned}
&W=A_1[\exp(m_1(I_1-3))-1]+B_1[\exp(n_1(I_2-3))-1]\\
&W=A_1[\exp(m_1(I_1-3))-1]+B_1(I_2-3)\\
&W=A_1(I_1-3)+B_1[\exp(n_1(I_2-3))-1]\\
&W=A_1\left[\left(\frac{I_1}{3}\right)^{m_1}-1\right]+B_1\left[\left(\frac{I_2}{3}\right)^{n_1}-1\right]\\
&W=A_1[\exp(m_1(I_1-3)+n_1(I_2-3))-1]\\
&W=A_1\left[\left(\frac{I_1}{3}\right)^{m_1}\exp(n_1(I_2-3))-1\right]\\
&W=A_1\left[\left(\frac{I_2}{3}\right)^{n_1}\exp(m_1(I_1-3))-1\right]
\end{aligned}
\tag{13}
$$

Consequently, we could get simple mathematical forms for strain energy based on strain invariants. In fact, a decent strain energy density is one that can be illustrated as simple as a possible mathematical function having few parameters. The functions of Eq. (13) indicate the generality of proposed framework. For example, the last function of Eq. (12) signifies the Ogden model (1972) and in Eq. (13), the second one shows Veronda–Westmann model (1970) and the fifth one shows Vito model (1973). The others are new models to investigate. It was mentioned that having a complete form of strain energy in terms of exp-exp functions rather than exp-log or exp-polynomial and other forms, could bring acceptable results. Therefore, in this work among the functions of Eq. (13), it is preferred to investigate the performance of the following proposed model.

$$W=A_1[\exp(m_1(I_1-3))-1]+B_1[\exp(n_1(I_2-3))-1] \tag{14}$$

The model introduced in this work (Eq. (14)) is phenomenological based, mathematically justified and benefits from a mathematical and physical background, while the most of other models are presented based on a guesswork or a pure phenomenological combination method and only are verified via correlation with some experimental data (see e.g., Nunes, 2011; Lion, 1997; Yeoh and fleming, 1997; Amin et al., 2006; Pucci and Saccomandi, 2002). In addition, Due to adopting a different and self-containing approach, the proposed model benefits from these pivotal characteristics: the existence of second strain invariant, simplicity, stability of parameters through implementation of restrictions on strain energy density, and the state of being accurate based on the best phenomenological combination of functions (the latter will be examined in Section 4).

To show performance of the proposed model, by considering Eqs. (14) and (4), the Cauchy stress in principal directions is

$$\sigma_{ii}=-p\delta_{ii}+2\lambda_i^2A_1m_1\exp[m_1(I_1-3)]-2\lambda_i^{-2}B_1n_1\exp[n_1(I_2-3)],\quad i=\{1,2,3\},\ \text{(no sum on } i\text{)} \tag{15}$$

The non-dimensional values for Cauchy stresses have been plotted in Fig. 2 by adjusting *a single set* of parameter values. This figure shows the ability of Eq. (15) to differentiate mathematically between different applied deformation states for the following reasons. It is obvious that how this model can switch between different modes of deformation, for example, between strain hardening for uniaxial and biaxial. This is because the Cauchy stress in Eq. (15) includes the factors I_1 and I_2, the magnitude of which controls strain hardening. Additionally, there are two stretch-dependent

terms in Eq. (15), which potentiate the effect of these two factors: the first is proportional to λ_i^2; multiplying by I_1 makes larger amounts in extension state for $\lambda_i \gg 1$. The second is proportional to λ_i^{-2}; locating it before I_2 makes large values in compression for $\lambda_i \ll 1$. This is why this model can capture both tension and compression for materials, which exhibit strain softening. The multiplying λ_i^2 and λ_i^{-2} by I_1 and I_2 respectively, might enable the proposed model to deals precisely with stress–strain behavior of soft (or very soft) and hard (or very hard) materials.

3. Fitting procedure

In this work, the material parameters are determined based on the correlation between values of the strain energy density rebuilt from test data and the theory. This approach was introduced by Darijani and Naghdabadi (2010) and is preferred over the conventional method of stress correlation discussed by Drozdov (2007), Ogden et al. (2004), Hartmann (2001) and Gendy and Saleeb (2000). Although the material has different stress–stretch curves for different loading tests, there is a unique strain energy function for the material. The dependency of the strain energy density on the stretches for different modes of deformation including uniaxial extension, pure shear, and equi-biaxial tension has been presented in Darijani and Naghdabadi (2010). This means that the least squares regression analysis can be applied on the function of the strain energy instead of several different functions of the stresses. This mathematical operation has been done using the DataFit 9.0 software*. DataFit is a general tool for performing curve fitting, statistical analysis and data plotting. This software has the ability to fit equations with multiple parameters to the data and utilizes the Levenberg–Marquardt method with double precision to perform nonlinear regression.

In this paper, using this software the strain energy density function including the unknown material parameters has been fitted to the values of the strain energy density cast from the test data. The software extracts the material parameters for the best fit between the test data and the theory.

Based on this procedure, the hyper surface of the strain energy density of the theory tends to the real hyper surface of the strain energy density function of the material cast from the test data. Mathematically, the real hyper surface of the strain energy density of the material is positive definite, thus it necessitates all of the postulates governing the strain energy density to be satisfied. Therefore, there is no need to impose any constraint to solve the problem or optimize the fitting procedure. In this fitting procedure, the discrepancy between the model results and entered experimental data is defined by

$$\text{Residual sum of squares}: \ \text{RSS} = \sum_{i=1}^{n} \|W_{\text{data}} - W_{\text{model}}\|^2 \qquad (16)$$

A perfect fit would yield a residual sum of squares of 0.0.

4. Experimental data analysis

In this section, response of proposed model (Eq. (14)) to different deformation states is investigated for elastomers and soft biological tissues as hyperelastic isotropic materials.

4.1. Rubber-like elastomers

In this subsection, we illustrate that Eq. (14) can capture the different modes of deformation for a wide range of stress-stretch data. Fitting this model to experimental data shows an acceptable

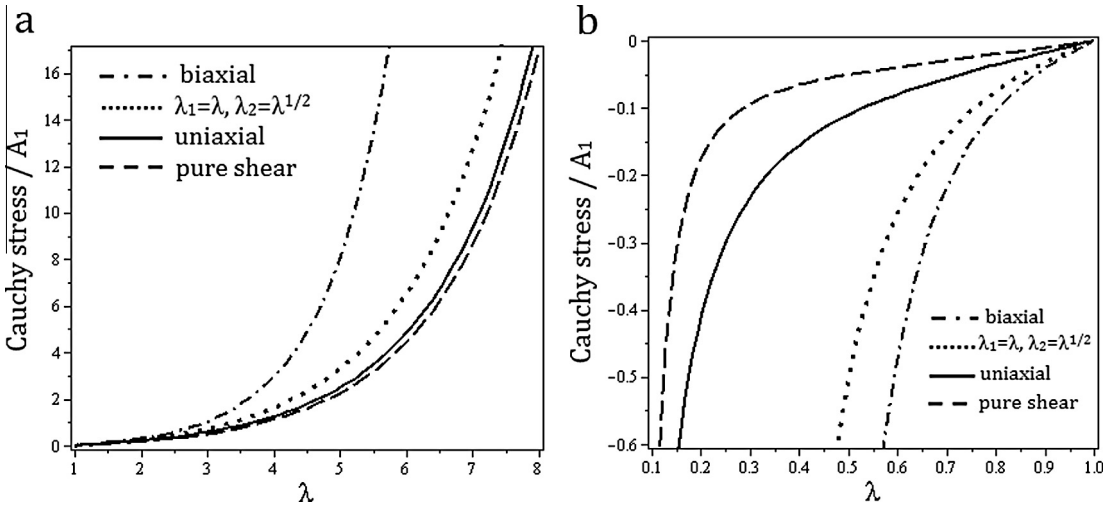

Fig. 2. The ability of model to differentiate between various loading states in (a) tension, (b) compression with values: $m_1 = 0.027$, $n_1 = 0.0005$, $B_1/A_1 = 2.94$.

Table 1
Unknown constants of the proposed model (Eq. (14)) for correlation with test data.

Material	A_1 (MPa)	B_1 (MPa)	m_1	n_1
VHB 4905	0.240	0.779	0.024	0.049
Silicone rubber (Arruda and Boyce)	59.84	1.485	0.0065	−0.058
Synthetic rubber neoprene	10.22	30.13	0.0271	0.0005
Yeoh vulcanizate	16.34	186.5	0.013	0.00006
Natural rubber	6.296	−1.67	0.018	−0.0091
Unfilled silicone rubber (Meunier)	1.54	5.57	0.089	0.004
Natural-rubber gum	64.5	304	0.0031	0.000035
b186	1.405	14.23	0.242	0.032

prediction for various regions of deformation, particularly at large deformations. The model is fitted to a large number of available experimental data such as: uniaxial and biaxial tension of a natural-rubber gum as reported by James et al. (1975), silicone rubber from Arruda and Boyce (1993), synthetic rubber neoprene from Alexander (1968), b186 rubber (a carbon black-filled rubber) from Miehe and Lulei (2001), an unfilled silicone rubber with a perfect non-linear and reversible behavior from Meunier et al. (2008),

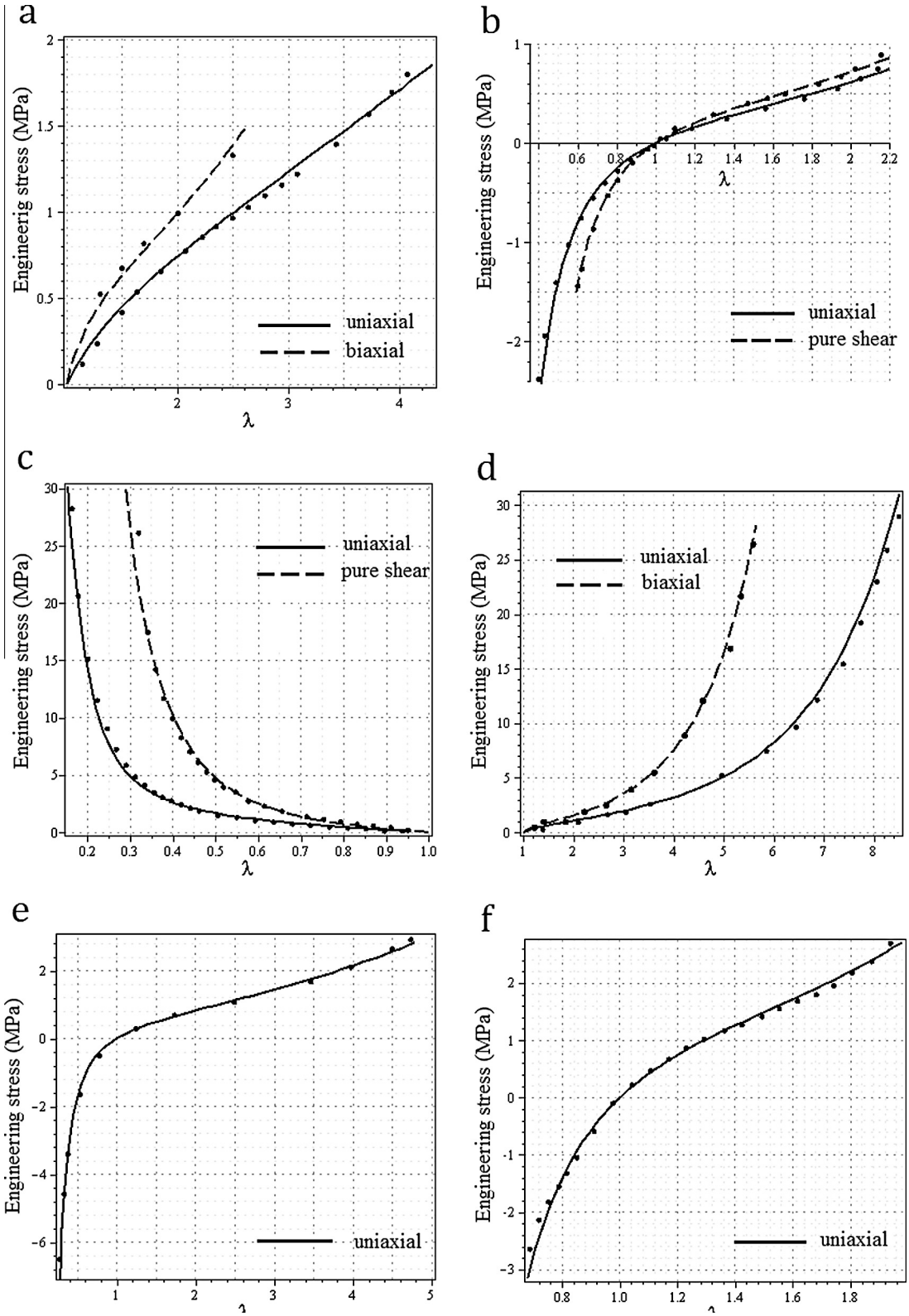

Fig. 3. Comparison of the theory with the experimental results on (a) natural-rubber gum (James et al., 1975), (b) unfilled silicone rubber (Meunier et al., 2008), (c) silicone rubber (Arruda and Boyce, 1993), (d) synthetic rubber neoprene (Alexander, 1968), (e) Yeoh vulcanizate data (1997), (f) b186 rubber (Miehe and Lulei, 2001), (g) natural rubber (Treloar, 1944), (h) VHB 4905 (Fox and Goulbourne, 2008).

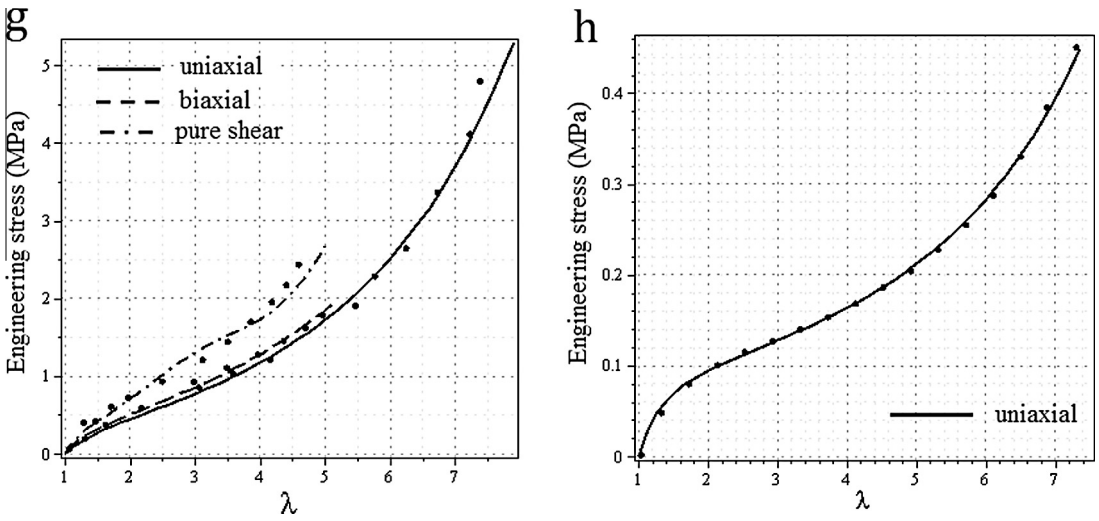

Fig. 3 (*continued*)

natural rubber from Treloar (1944), VHB 4905 (polyacrylate rubber) from Fox and Goulbourne (2008) and Yeoh vulcanizate data (1997). In the next subsection, the proposed model is applied to the modeling of mechanical behavior of a very soft biological tissue data named porcine liver tissue from Gao et al. (2010).

The James data includes both uniaxial and biaxial tension and Arruda–Boyce data is for the compressive mode of uniaxial and pure shear deformations. Alexander data contains three regions of loading for uniaxial and biaxial modes. The b186 carbon black-filled rubber possesses a large softening region and has an abrupt upturn in strain hardening region. Here, we bring unfilled silicone rubber because it experiences two modes of deformation in tension as well as compression. Silicone rubbers being increasingly used in bio-medical applications. VHB 4905 is much softer than the other rubber materials and its strain softening behavior in uniaxial tension is remarkable. VHB is used widely in dielectric elastomers. The Treloar data includes uniaxial, pure shear, and biaxial loading. Yeoh vulcanizate data are especially remarkable for covering a large range of extension and compression, which let us to have an overall investigation for uniaxial loading.

The calculation method of the material parameters was discussed in Section 3 and is listed in Table 1. As depicted in Fig. 3, the developed model is able to exhibit the strain softening and strain hardening regions accurately and takes advantage of modeling of the different loadings, as well as distinguishing between different materials.

Table 2
unknown constants given in Eq. (14) for the behavior modeling of the porcine liver tissue.

Material	A_1 (kPa)	B_1 (kPa)	m_1	n_1
Porcine liver tissue	0.0000049	0.018	6.647	2.367

In Fig. 4, the performance of the proposed model has been compared with 4-parameters Ogden model (1972) and Arruda and Boyce (1993) 8-chain model for VHB 4905. The motivation of selecting these two models for comparison is that they are safe for uniaxial deformation, while Fig. 4 shows that Ogden model does not handle strain hardening for large stretches and 8-chain model cannot follow data for initial deformation.

In the case of applications where constructions undergo simple state of deformation, it may be feasible to obtain suitable parameters for materials in fitting process using just tensile or compressive data. However, accurate modeling of a real scenario demands to consider at least two experiments (e.g., simple extension and pure shear) to determine the parameters of every model (Horgan and Saccomandi, 2002). Urayama (2006) has attempted to show that a single uniaxial test could not be reliable for extracting the strain energy parameters for all materials.

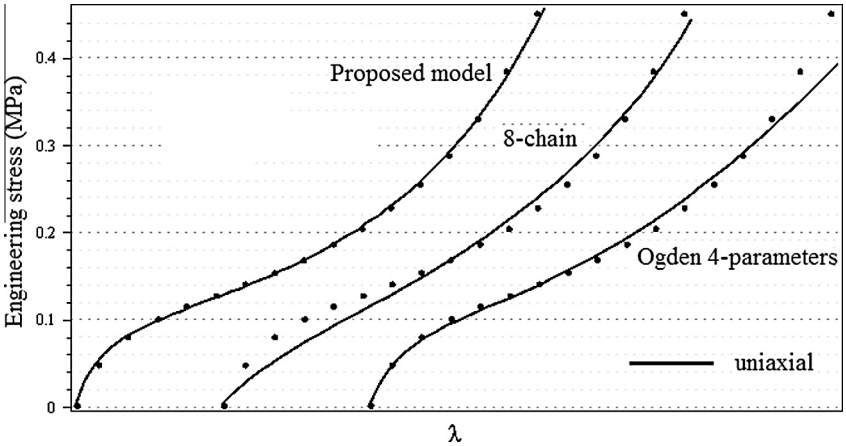

Fig. 4. The proposed model, Arruda–Boyce model (1993), and 4-parameters Ogden model (1972) applied to VHB 4905. Each curve is shifted to the right by a stretch of 2.

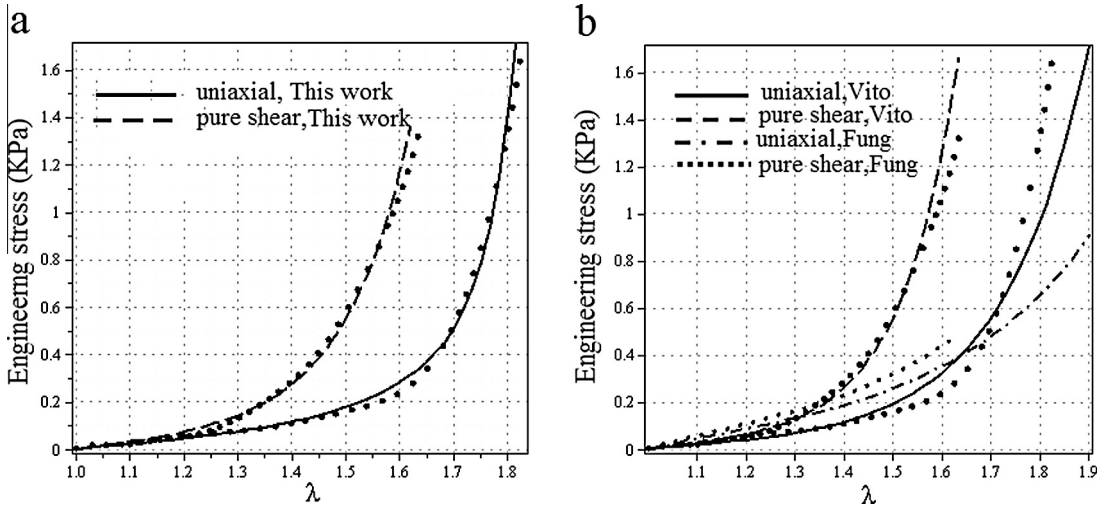

Fig. 5. Comparison of the theoretical and experimental results on the porcine liver tissue (Gao et al., 2010).

Table 3
Values of RSS for showing the significance of second term in the proposed model.

Material	Proposed model RSS	Fung model ($n_1 = 0$) RSS
VHB 4905	0.0001	0.007
Silicone rubber (Arruda and Boyce)	0.06	0.88
Synthetic rubber neoprene	1.13	749
Yeoh vulcanizate	0.01	0.02
Natural rubber	0.23	4.49
Unfilled silicone rubber (Meunier)	0.002	0.006
Natural-rubber gum	0.06	0.6
Porcine liver tissue	0.0001	0.02

4.2. Soft biomaterials

In general, the mechanical properties of tissues are described base on their characteristics such as time dependency, anisotropic response and viscoelastic behavior. Considering very soft biological tissue such as liver, brain, kidney, and prostate as *isotropic* materials may constitute a simplification of the problem for understanding the mechanical behavior of very soft tissues. Another example of this assumption is solid intra-abdominal organs (Carter et al., 2001). Moreover, under low strain rate, which is typical in surgery, much of soft tissues have been reported to be relatively rate insensitive (i.e. see, Miller, 2000; Chui et al., 2004).

Table 4
The models and relative errors (RSS).

Synthetic rubber neoprene		Natural rubber		Porcine liver tissue		Natural rubber gum	
Proposed model	1.1	James et al. (1975)	0.12	Proposed model	0.00014	Ogden-2term (1972)	0.007
James et al. (1975)	1.6	Biderman (1958)	0.4	James et al. (1975)	0.00015	James et al. (1975)	0.009
Biderman (1958)	3.3	Proposed model	0.46	Vito (1973)	0.0008	Proposed model	0.014
Hartmannand Neff (3-term, 2003)	8	Ogden-2term (1972)	1.01	Hartmann and Neff (3-term, 2003)	0.0018	Attard and Hunt (2004)	0.018
Attard and Hunt (2004)	9.6	Attard and Hunt (2004)	1.06	Attard and Hunt (2004)	0.0019	Biderman (1958)	0.02
Isihara et al. (1951)	10.9	Swanson (1985)	1.4	Biderman (1958)	0.0019	Nunes (2011)	0.046
Swanson (1985)	12	Hartmann and Neff (3-term, 2003)	1.6	Isihara et al. (1951)	0.002	Isihara et al. (1951)	0.047
Vito (1973)	14.8	Isihara et al. (1951)	1.6	Swanson (1985)	0.004	Hartmann and Neff (3-term, 2003)	0.06
Carroll (2011)	228	Vito (1973)	1.6	Carroll (2011)	0.009	Gent and Thomas (1958)	0.06
Yeoh (1990)	679	Carroll (2011)	4.2	Veronda and Westmann (1970)	0.01	Swanson (1985)	0.11
Nunes (2011)	711	Yeoh and Fleming (1997)	7.04	Yeoh and Fleming (1997)	0.02	Horgan and Saccomandi (2004)	0.11
Fung (1967)	749	Nunes (2011)	7.05	Arruda and Boyce (1993)	0.02	Yeoh-modified (1993)	0.12
Yeoh-modified (1993)	692	Lopez-Pamies (2010)	7	Yeoh-modified (1993)	0.021	Yeoh (1990)	0.3
Unfilled silicone rubber		Silicone (Arruda–Boyce)		VHB		Yeoh vulcanizate	
Proposed model	0.0009	Proposed model	0.04	James et al. (1975)	4E-006	Lopez-Pamies (2010)	0.002
Attard and Hunt (2004)	0.001	James et al. (1975)	0.06	Biderman (1958)	6E-006	Yeoh and Fleming (1997)	0.002
James et al. (1975)	0.001	Swanson (1985)	0.07	Yeoh and Fleming (1997)	8E-006	James et al. (1975)	0.002
Ogden-2term (1972)	0.001	Attard and Hunt (2004)	0.08	Proposed model	9E-006	Biderman (1958)	0.003
Isihara et al. (1951)	0.002	Biderman (1958)	0.13	Hartmann and Neff (3-term, 2003)	2E-005	Proposed model	0.003
Biderman (1958)	0.002	Gent and Thomas (1958)	0.17	Lopez-Pamies (2010)	4E-005	Ogden-2term (1972)	0.004
Hartmann and Neff (3-term, 2003)	0.002	Isihara et al. (1951)	0.19	Attard and Hunt (2004)	7E-005	Yeoh (1990)	0.004
Gent and Thomas (1958)	0.003	Nunes (2011)	0.23	Yeoh-modified (1993)	2E-004	Swanson (1985)	0.009
Nunes (2011)	0.004	Vito (1973)	0.25	Isihara et al. (1951)	3E-004	Attard and Hunt (2004)	0.009
Horgan and Saccomandi (2004)	0.005	Hartmann and Neff (3-term, 2003)	0.26	Yeoh (1990)	4E-004	Hartmann and Neff (3-term, 2003)	0.01

(*continued on next page*)

Table 4 (*continued*)

Unfilled silicone rubber		Silicone (Arruda–Boyce)		VHB		Yeoh vulcanizate	
Yeoh-3term (1990)	0.006	Ogden-2term (1972)	0.32	Ogden-2term (1972)	0.001	Isihara et al. (1951)	0.01
Arruda and Boyce (1993)	0.006	Yeoh-modified (1993)	0.83	Swanson (1985)	0.002	Arruda and Boyce (1993)	0.02
Lopez-Pamies (2010)	0.006	Yeoh and Fleming (1997)	0.83	Martins et al. (2006)	0.004	Amin et al. (2006)	0.05

Generally, soft tissues are composed of two main parts: elastin and collagen fibers. This is sufficiently general to divide the energy function of biological tissue into two parts: a part ψ_{iso} associated with isotropic deformation and a part ψ_{aniso} associated with anisotropic deformation (Holzapfel, 2001). Since the collagen fibers of tissue are not active at low stresses (they do not store strain energy), ψ_{iso} is associate with them.

To develop a realistic soft tissue model it is necessary to have various modes of deformation. Hence, in this paper, we use the test data of pure shear and uniaxial tension for porcine liver tissue from Gao et al. (2010). Gao et al. have extracted these experimental results on liver tissue for the pure shear test for the first time. This experiment conducted on 130 samples from 12 porcine livers, and among them, we choose the stress–strain curves for uniaxial and pure shear deformations, which achieve maximum stress. All modes of deformation at Gao's experiment were conducted at nominal strain rate of 0.05 s^{-1}, which is close to the strain rate of 0.01 s^{-1} reported as typical for neurosurgery (Miller, 2000). Because of high water content of liver tissue, it is considered to be incompressible. With the assumption of isotropic and homogeneous and because of low strain rate, it is feasible to describe the response of such materials in hyperelasticity framework generally.

Simultaneous fitting to the pure shear and uniaxial deformation states has conducted to obtain parameters in Table 2. Observations from Fig. 5 shows that our simple model captures all the relevant characteristics of liver tissue and can accurately follow *J*-shape form of the mechanical behavior of soft tissue.

5. Discussion

As we know, most of phenomenological models have been presented in the forms of polynomial, power law, logarithmic and exponential functions, or a linear combination of them. There are many strain energy functions around here, which are composed by combination of the previous proposed models. Although, these hyperelastic models based on combination viewpoint have been accepted. The methodology presented in this paper is different from a guesswork or combination viewpoint. A collected research has done by Darijani and Naghdabadi (2010) to examine all combinations of power law, polynomial, logarithmic, and exponential functions. As a result of this investigation, they concluded that a strain energy function in terms of exponential forms in comparison to other combinations brings a good agreement with experimental data. By inspiring from this significant result, we proposed a general exponential framework for strain energy density function. By considering and applying the governing postulates of strain energy on its mathematical form, we could extract a self-contained form for strain energy density. It can be stated that there is a challenge among researchers for introducing a strain energy density which can capture the material behavior and be as simple as a possible mathematical function having few parameters. Simplicity of proposed model due to the number of its terms with a good correlation with experimental data can be appeared as a simple and accurate model in computational solid mechanics field.

It is notable that the second term in Eq. (14) is new and this term is not seen in the popular hyperelastic models collected and reviewed in papers (see e.g., Marckmann and Verron, 2006; Hoss and Marczak, 2010; Steinmann et al., 2012; Beda, 2014). Hence, the second term in the proposed model and putting it together with first term (Fung model) is not a guesswork combination. In order to show the importance of existence of the second term in the proposed model, the values of RSS related to the proposed model and Fung model (the non-existence the second term in the proposed model) are compared in Table 3.

Moreover, Table 4 shows the range for superiority of proposed model, in comparison to almost 30 well-known models, for different experimental data. This table comprises 12 of these models according to the less RSS for each experimental data (the models that are not listed have more errors).

As it can be seen from this table, some models have less RSS rather than the proposed model especially for the materials with S-shaped mechanical behavior. In contrary, the proposed model has less RSS rather than others specially for the materials with J-shaped mechanical behaviors (these materials have high strain stiffening). Nonetheless, it is notable that the models proposed by James, Biderman, and Attard and Hunt (with 5, 4 and 4 parameters, respectively) are in the form of a general Rivlin's expansion (Rivlin and Saunders, 1951). As mentioned earlier, these models almost can be fitted to any hyperelastic experimental curve but they have numerical problems. Because of weighted model and instable parameters, there is not a tendency between researchers to use such models in numerical solutions (see e.g., Selvadurai, 2006; Goulbourne et al., 2007; Selvadurai and Shi, 2012; Albrecht and Ravi-Chandar, 2013; Kumar and DasGupta, 2013; Li et al., 2013; Tamadapu and Dasgupta, 2013; Wang et al., 2013; Li et al., 2014; Mao et al., 2014). Despite having a low RSS, because of instability, some of these models do not have a desirable trend for stress–strain curves and need to fulfill parameter identification. The proposed model is free from optimization of parameters. This is because of that we implement some restrictions in the form of strain energy as well as fitting conditions.

6. Conclusions

In this paper, we suggested a new framework for strain energy density of the isotropic hyperelastic materials. Based on this framework, using a different methodology from a guesswork or combination viewpoint, a set strain energy density functions is constructed. Among the constructed set for strain energy density functions, we examined a constitutive model in exponential form to deal with large deformation of hard and soft elastomers and biological tissues. In fact, we used a self-contained approach which led to propose a multiaxial form for strain energy density, having pivotal characteristics, e.g., the simplicity, stability of the parameters, having second strain invariant, and being accurate for different materials. This model is phenomenological based, mathematically justified and correlated with the experimental data and benefits from a mathematical and physical background. It was demonstrated that this model is able to provide an acceptable prediction for stress-stretch response of elastomers and soft tissues with a wide range in properties by applying it to VHB 4905 (polyacrylate rubber), two various silicone rubbers, synthetic rubber neoprene, two different natural rubbers, b186 rubber (a carbon black-filled rubber), Yeoh vulcanizate rubber, and porcine liver tissue (a very

soft tissue). Moreover, the performance of the proposed model and role of its second term (term including of second strain invariant) in comparison to other hyperelastic models (almost 30 models) was examined for modeling of materials with J-shaped and S-shaped mechanical behaviors. The results show that the model is a capable model in its ability to illustrate precisely stress–strain behavior of soft (or very soft), hard (or very hard) elastomers and soft tissues.

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