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Maximum Internally Stable Sets of a Graph†

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I. INTRODUCTION

An undirected *graph* G is represented by an ordered pair (Γ, X) consisting of a finite set X and an *adjacency* relation Γ on X with the property that $y \in \Gamma x$ implies that $x \in \Gamma y$ for all $x \in X$ [1, 2].[1] The elements of X are called the *vertices* of G and are represented by points in a plane. If for some pair of vertices x and $y \in X$, $y \in \Gamma x$, then vertices x and y are called *adjacent* and their adjacency is represented by a line segment connecting the corresponding points in the plane. Such a line segment is denoted by an unordered pair (x, y) and is called an *arc* or a *branch* of G. If B is a subset of a set A, denoted by $B \subset A$, then, by $\bar{B} = B - A$ and $|A|$, we mean the set of all elements of B which are not in A and the number of elements in A, respectively. Let $S \subset X$ and define

$$\Gamma S = \{x \in X \mid x \in \Gamma y \text{ for some } y \in S\}.$$

A set $S \subset X$ is called an *internally stable* (IS) set of G if no two vertices in S are adjacent, i.e., $|\Gamma S \cap S| = 0$. An IS set S of G is a *maximum internally stable* (MIS) set, if there exists no IS set S_1 of G such that $|S_1| > |S|$.

It is easy to find an IS set of a graph $G = (\Gamma, X)$. However, the problem finding a MIS set of a graph remains substantially unsolved. To find an MIS set, one could consider all possible subsets of vertices of the graph. Naturally, such a procedure is highly impractical for a large graph. Maghout [3] proposed an algorithm, based on Boolean functions, to generate all possible internally

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[1] We have adopted Berge's notation {1} which at first may seem cumbersome, but is convenient for the presentation of results in this paper.

stable sets. The main difficulty with this algorithm is that is it extremely inefficient and computations become time consuming for even relatively small graphs. A more promising approach could be based on linear integer programming [4] or nonlinear programming. Again, the computational requirements for such a procedure rapidly become impractical. Furthermore, since both of the above methods are analytical in nature, they yield no direct relationships between topological structure and the maximum internally stable sets.

Berge [1] has shown that in some cases finding an MIS set of a graph is equivalent to finding a maximum matching of the graph. However, in most cases, there is no clear procedure to obtain an MIS set from a maximum matching. Other partial results have been obtained by Matthys [5, 6]. Also, interesting results on discrete optimization theory have been developed by Reiter and Sherman [7], and may be applied to obtain a heuristic procedure for finding an MIS set.

In this paper, we give a graph theoretic characterization of the MIS sets of a graph. This characterization is similar to one given by Edmonds [8] but leads to a sharper picture of the conditions under which an IS Set is maximum. Although this result does not immediately result in a satisfactory algorithm for finding an MIS set of an arbitrary graph, it does lead to a satisfactory algorithm for finding an MIS set of a bipartite [2] or simple [1] graph. The major portion of the paper is aimed at developing this algorithm and proving that at its termination an MIS set has been found.

A *bipartite* graph G is represented by an ordered triplet (Γ, X, Y) consisting of two finite sets X and Y and adjacency relation Γ mapping X onto Y and Y onto X with the property that $x \in X$ implies $x \notin \Gamma X$ and $y \in Y$ implies $y \notin \Gamma y$. It should be noted that a bipartite graph is a graph whose vertices can be divided into two disjoint subsets X and Y such that a vertex $x \in X$ can can only be adjacent to vertices $y \in Y$ and vice versa.

II. Definitions and the Statement of the Main Theorem

Let $G = (\Gamma, X)$ be a graph. A *subgraph* of $g \subset G$ is an ordered pair (Γ_1, X_1) with $X_1 \subset X$ and $\Gamma_1 x \subset \Gamma x \cap X_1$ for all $x \in X_1$. If $\Gamma_1 x = \Gamma x \cap X_1$ for all $x \in X_1$ we adopt the notation (Γ, X_1) for the subgraph g. If $g_1 = (\Gamma_1, X_1)$ and $g_2 = (\Gamma_2, X_2)$ are two subgraphs of $G = (\Gamma, X)$ then the subgraph represented by $g_1 \cup g_2$ is subgraph (Γ_3, X_3) where $X_3 = X_1 \cup X_2$ and $\Gamma_3 x = \Gamma_1 x \cup \Gamma_2 x$ for all $x \in X_3$. In the above definition, if g_2 consists of a single arc, say, (x_1, x_2), then we assume $X_2 = x_1 \cup x_2$ and $\Gamma_2 x_1 = x_2$ (and $\Gamma_2 x_2 = x_1$). If (x_1, x_2) is an arc in the subgraph $g_1 = (\Gamma_1, X_1)$, then by $g_1 - (x_1, x_2)$ we mean a subgraph $g_2 = (\Gamma_2, X_1)$ where $\Gamma_2 x = \Gamma_1 x$ for all

$x \in X_1$ except $x = x_1$ and $x = x_2$, and $\Gamma_2 x_1 = \Gamma_1 x_1 - x_2$ and $\Gamma_2 x_2 = \Gamma_1 x_2 - x_1$. By a *chain* of length n in $G = (\Gamma, X)$, we mean a sequence of distinct vertices $x_0 x_1 x_2 \cdots x_n$ such that $x_j \in \Gamma x_{j-1}$ for $j = 1, 2, \ldots, n$. Vertices x_0 and x_n in this chain are called the *initial* and *final* vertices of the chain. A graph (Γ, X) is *connected* if for any two distinct vertices x and $y \in X$, there exists a chain whose initial and final vertices are x and y. A *circuit* is a chain where all vertices except the initial and final vertices are distinct (and $x_0 = x_n$). A *subtree*[2] of a graph $G = (\Gamma, X)$ is a subgraph $t = (\Gamma_1, X_1)$ such that t contains no circuits and $|\Gamma_1 x| > 0$ for all $x \in X_1$. An IS set S of a graph $G = (\Gamma, X)$ is a *complete* IS set if there exists no vertex $x \in X \cap \bar{S}$ such that $|\Gamma x| \cap S = 0$. If an IS set S is not complete, then there exists a vertex $x_0 \in X \cap \bar{S}$ such that $S \cup x_0$ is an IS set. Thus, one can continue enlarging the resulting IS set until it is complete. If $g = (\Gamma_1, X_1)$ is a subgraph of G, then a *pendant vertex* of g is a vertex $x_1 \in X_1$ such that $|\Gamma_1 x_1| = 1$. In the sequel, we may call the vertices in a given IS set dark vertices and the remaining vertices light vertices. We are now ready to define an alternating tree.

DEFINITION 1. Let S be an IS set of a graph $G = (\Gamma, X)$. A subgraph $g = (\Gamma_1, X_1)$ is called an *alternating tree* of G with respect to S if g is a subtree of G which satisfies the following conditions:

(1) There exists no pair of distinct vertices x and $y \in X_1 \cap \bar{S}$ such that $y \in \Gamma x$. (No pair of light vertices in X_1 are adjacent in G.)

(2) The set of all pendant vertices of g, defined by

$$P = \{x \in X_1 \ni |\Gamma_1 x| = 1\}$$

is a subset of $\bar{S}$. (All pendant vertices of g are light.)

(3) $|\Gamma x \cap \{S \cap \bar{X}_1\}| = 0$ for all $x \in X_1 \cap \bar{S}$. (No light vertex in X_1 is adjacent to a dark vertex in $\bar{X}_1$.)

LEMMA 1. *If $g = (\Gamma_1, X_1)$ is an alternating tree of $G = (\Gamma, X)$ with respect to an* IS *set of G, then $|X_1 \cap \bar{S}| > |X_1 \cap S|$.*

The proof is elementary and can be carried out by induction of $|X_1|$ and the use of the fact that all pendant vertices of g are light. The following theorem gives a graph theoretic characterization of MIS sets.

THEOREM 1. *A complete* IS *set S_0 of a graph $G = (\Gamma, X)$ is an* MIS *set of G if and only if there exists no alternating tree in G with respect to S_0.*

REMARKS. This theorem resembles the results of Berge [1] and Norman and Rabin [1, 9] in connection with "maximum matchings" and "generalized

[2] A subtree need not be connected.

matchings" of a graph. Their results were based on the existence of an alternating chain between pairs of "exposed" vertices [10-12]. The concept of an alternating tree is a necessary extension of the notion of the alternating chain, and was first defined by Edmonds [8] in a slightly different manner. The definition given below eliminates some of the difficulties generated by Edmond's definition while at the same time yields an analogous characterization of the MIS sets.

PROOF. To prove the necessity, we assume there exists an alternating tree $g = (\Gamma_1, X_1)$ of G with respect to S_0. Consider the set of vertices $S = \{S_0 - S_0 \cap X_1\} \cup \{\bar{S}_0 \cap X_1\}$. Then, $S = \{S_0 \cap \bar{X}_1\} \cup \{\bar{S}_0 \cap X_1\}$ which is an IS set of G due to conditions (1) and (3) of Definition 1. From Lemma 1, $|S| > |S_0|$, hence S_0 is not an MIS set of G. The proof of sufficiency requires an additional result to be developed in the next section. We state this result here as Theorem 2 and then, based on the validity of this theorem, we prove the sufficiency of the condition.

THEOREM 2. *A bipartite graph* $G = (\Gamma, X, Y)$ *with* $|\Gamma z| \geqslant 1$ *for all* $z \in X \cup Y$ *contains an alternating tree with respect to* Y *if and only if there exist subsets* $X_1 \subset X$ *and* $Y_1 \subset Y$ *such that* $\Gamma X_1 = Y_1$ *and* $|X_1| > |Y_1|$.

PROOF OF SUFFICIENCY OF THEOREM 1. Let S_0 be a complete IS set and let S^* be an MIS set of $G = (\Gamma, X)$. We assume $|S^*| > |S_0|$ and we need to show the existence of an alternating tree with respect to S_0 in G. Consider the following disjoint subsets of X:

$$X_\alpha = S^* - S_0, \qquad X_\beta = S_0 - S^*, \qquad X_\gamma = S_0 \cap S^*.$$

From the hypothesis, we know that $|X_\alpha| > |X_\beta|$ and since S_0 is a complete IS set $|X_\beta| \neq 0$. The following properties of the above subsets of X are significant:

(1) Since X_α and X_β are both IS sets, no pair of vertices in X_α or in X_β are adjacent.

(2) Similarly, since $X_\alpha \cup X_\gamma$ and $X_\beta \cup X_\gamma$ are IS sets, no vertex in X_α or in X_β is adjacent to a vertex in X_γ.

(3) Since S^* and S_0 are both complete IS sets, each vertex in X_β (or X_α) is adjacent to a vertex in X_α (or X_β), respectively.

Let us now consider the bipartite subgraph $G_1 = (\Gamma_1, X_\alpha, X_\beta)$ of G where $\Gamma_1 x = \Gamma x \cap X_\beta$ for all $x \in X_\alpha$. Since $|X_\alpha| > |X_\beta|$, and $|\Gamma_1 z| \geqslant 1$ for all $z \in X_\alpha \cup X_\beta$, from Theorem 2, there exists an alternating tree $t_1 = (\Gamma_1{}^t, X_\alpha{}^t, X_\beta{}^t)$ of G_1 with respect to X_β. We claim that t_1 is also an alternating tree of G with respect to S_0. This follows directly from the definition of the alternating tree and statement (1), (2), and (3) of this proof, which completes the proof of the theorem.

III. Bipartite Graphs and Alternating Trees

Let $G = (\Gamma, X, Y)$ be a bipartite graph and assume $|\Gamma z| > 0$ for all $z \in X \cup Y$. Then, it is easy to see that Y (or X) is a complete IS set of G. In this section, we present an algorithm to find an alternating tree (if there exists any) of G with respect to Y. It should be noted that since G is bipartite all we must do is to construct a tree $t = (\Gamma_1, X_1, Y_1)$ such that all pendant vertices of t are in X_1 and $\Gamma X_1 \subset Y_1$ and, by Lemma 1, $|X_1| > |Y_1|$. The algorithm will be a bit stronger that it is necessary to prove Theorem 2. However, this will be needed to establish the results of the next section. The algorithm is designed to construct an alternating tree t with the most arcs. To do this, we construct each component[3] (connected part) t_j of t separately. Each component t_j of t is constructed by a number of chains each of which has exactly one vertex in common with one of the previous chains and each chain has its initial and final vertices in X. Let $G_j^i = (\Gamma_j^i, X_j^i, Y_j^i)$ and $t_j^i = (\Gamma_{tj}^i, X_{tj}^i, Y_{tj}^i)$ be the graph and the tree obtained after i "cycles" of the algorithm in the construction of jth component, t_j of t. Assume $G_1^0 = G$ and t_j^0 to be an empty subgraph of G_j^0, (an empty subgraph is a subgraph without vertices and as a result without arcs). We assume we have constructed components $t_1, t_2, \ldots, t_{j-1}$ of the desired tree t and we have also completed i "cycles" of the algorithm in the construction of t_j. After completion of each cycle the indices (i.e., superscript i) are advanced by one (replaced by $i + 1$). The algorithm consists of five steps of which the third step is the main part of the algorithm. Consider the graph G_j^0. Before the construction of t_j has begun, the vertices of G_j^0 are called *unlabeled vertices* and are not assigned symbolic representations. In the process of construction of t_j each vertex that has been encountered in a chain is called a *labeled vertex*, and is assigned the appropriate symbol x_k. By *deleting a subset* $A \subset X$ of vertices of a graph $G = (\Gamma, X)$, we mean constructing a graph $G_1 = (\Gamma, X_1)$ such that $X_1 = X - A$. The five steps of the algorithm are as follows:

1. Each vertex of X_j^i adjacent to a pendant vertex of G_j^i is deleted along with the pendant vertex. This step is repeated as long as there are such vertices in X_j^i. If pendant vertices are created in Y_j^i as a result of these deletions, these vertices as well as their adjacent vertices in X_j^i are deleted. If $i = 0$, go to step 2(a). Otherwise, to to step 2(b).

2. (a) Let x_1 be any vertex in X_j^i adjacent to an unlabeled vertex in Y_j^i. If X_j^i is empty go to step 4. Otherwise go to step 2(c).

(b) Let x_1 be any vertex in X_{tj}^i adjacent to an unlabeled vertex in Y_j^i. If no such vertex exists go to step 4. Otherwise go to step 2(c).

[3] Subgraph $g = (\Gamma, X_1)$ of $G = (\Gamma, X)$, $X_1 \subset X$, is a component of G if g is connected and $|\Gamma X_1 \cap \Gamma(X - X_1)| = 0$.

(c) Form a chain $x_1 y_1 x_2 y_2 \cdots$ continuing as long as unlabeled vertices in G_j^i are encountered. If the chain terminates in a vertex $x_k \in X_j^i$, set $t_j^{i+1} = t_j^i \cup (x_1, y_1) \cup (x_2, y_2) \cup \cdots \cup (y_{k-1}, x_k)$ and return to step 2(b). If the chain terminates in $y_k \in Y_j^i$, let the last vertex in the chain in X_j^i be x_k and go to step 3(a).

3. (a) From step (1), y_k is not a pendant vertex. Therefore there exists a *labeled* vertex $x^a \neq x_k$ in X_j^i adjacent to y_k. The subgraph

$$T_j^i = (\Gamma_{Tj}^i, X_{Tj}^i, Y_{Tj}^i),$$

defined by

$$T_j^i = t_j^i \cup (x_1, y_1) \cup (y_1, x_2) \cup \cdots \cup (x_k, y_k) \cup (y_k, x^a),$$

contains a circuit. Let this circuit be $a_1 b_1 a_2 b_2 \cdots b_{r_i} a_1$ with $a_s \in X_j^i$ and $b_s \in Y_j^i$ for $s = 1, 2, \ldots, r_i$.

Define

$$\beta_1 = \{b_1, b_2, \ldots, b_{r_i}\},$$

and

$$\alpha_1 = \{a_1, a_2, \ldots, a_{r_i}\}.$$

Let $\ell = 1$ and go to step 3(b).

(b) If $|\alpha_\ell| = 0$ go to step 3(c). If $|\alpha_\ell| \neq 0$ and α_ℓ contains no unlabeled vertices go to step 3(d). Otherwise go to step (3e).

(c) Set t_j equal to the empty set and construct G_{j+1}^0 from G_j^i by deleting the vertices in $\{\bigcup_{p=1}^{\ell} \alpha_p\}$ from G_j^i; return to step 1 and start the first cycle in the construction of t_{j+1}.

(d) Let $\ell = \ell + 1$ and redefine β_ℓ and α_ℓ by

$$\beta_\ell = \Big\{ y \in Y_{Tj}^i - \bigcup_{p=1}^{\ell-1} \beta_p \mid y \text{ is in a circuit in } T_j^i \cup (\hat{x}, \hat{y}) \text{ for some } \hat{x} \in \Gamma_j^i y \text{ and } \hat{y} \in \beta_{\ell-1} \Big\}$$

and

$$\alpha_\ell = \Gamma_j^i \beta_{\ell-1} \cup \{x \in X_{Tj}^i \mid x \text{ is in a circuit in } T_j^i \cup (\hat{x}, \hat{y}) \text{ for some } \hat{x} \in \Gamma_j^i y \text{ and } \hat{y} \in \beta_{\ell-1}\} - \bigcup_{p=1}^{\ell-1} \alpha_p .$$

Go to step 3(b).

(e) α_ℓ contains at least one unlabeled vertex; let one such vertex be labeled x_{k+1}. Let the (or a) vertex in $\beta_{\ell-1}$ which is adjacent to x_{k+1} be $b'_{\ell-1}$. We now define a set of vertices $\{a'_p, b'_p \mid p = 0, 1, \ldots, \ell - 2\}$. In this set $a'_p \in \{\Gamma_j{}^i b'_p\} \cap \alpha_p$ and $b'_p \in \beta_p$ and $b'_{\ell-r}$ is any vertex such that $b'_{\ell-r}$ is in the circuit in $T_j{}^i \cup (b'_{\ell-r}, a'_{\ell-r+1})$ for $r = 2, 3, \ldots, \ell - 1$. Let a''_p be an element of α_p such that the arc $(b'_p, a''_p) \in T_j{}^i$ for $p = 1, 2, 3, \ldots, \ell - 1$. Then construct t_j^{i+1} from $t_j{}^i$ as follows and return to step 2(b), setting $G_j^{i+1} = G_j{}^i$.

$$t_j^{i+1} = T_j{}^i \cup (x_{k+1}, b'_{\ell-1}) \cup (b'_{\ell-2}, a'_{\ell-1}) \cup \cdots \cup (b'_2, a'_3) \cup (b'_1, a'_2)$$

$$- (b'_{\ell-1}, a''_{\ell-1}) \cup (b'_{\ell-2}, a''_{\ell-2}) \cup \cdots \cup (b'_2, a''_2) \cup (b'_1, a''_1).$$

4. Construct G_{j+1}^0 from $G_j{}^i$ be deleting all vertices in Y_{tj}^i from $G_j{}^i$ and return to step 2(a) if $Y_j{}^i$ is not empty; if $\mid Y_j{}^i - Y_{tj}^i \mid = 0$, go to step 5.

5. Let $t' = \bigcup_{p=1}^{j} t_p = (\Gamma'_t, X'_t, Y'_t)$. Construct the final tree t from t' by adding any arc (x, y) of $G = (\Gamma, X, Y)$ for each vertex $x \in X$ such that $\Gamma x \subset Y'_t$ with $y \in \Gamma x$, and no circuit is formed. This represents the final step of the algorithm.

Our immediate goal is to show that the algorithm generates an alternating tree, and eventually to show that it indeed produces an alternating tree with the most arcs. To reach this goal we need the following results.

LEMMA 2. *In the algorithm, for each j, if $t_j{}^i = (\Gamma_{tj}^i, X_{tj}^i, Y_{tj}^i)$ is not empty, then* (a) *subgraph $t_j{}^i$ is a connected tree,* (b) $\mid \Gamma_{tj}^i y \mid = 2$ *for all* $y \in Y_{tj}^i$, *and* (c) $\mid X_{tj}^i \mid = \mid Y_{tj}^i \mid + 1$.

PROOF. We prove the lemma by induction on i. Thus, we assume the lemma is correct for $t_j{}^i$ and we prove it is also correct for t_j^{i+1}. If t_j^{i+1} is obtained from $t_j{}^i$ by step 2 of the algorithm, then clearly statements (a), (b), and (c) are correct. Suppose t_j^{i+1} is obtained from $t_j{}^i$ by step 3(c) of the algorithm. From the algorithm it easily follows that $T_j{}^i = (\Gamma_{Tj}^i, X_{Tj}^i, Y_{Tj}^i)$ is connected and $\mid \Gamma_{Tj}^i y \mid = 2$ for all $y \in Y_{Tj}^i$ and $\mid X_{Tj}^i \mid = \mid Y_{Tj}^i \mid$. We now prove that t_j^{i+1} obtained from $T_j{}^i$ satisfies statements (a), (b), and (c) of the lemma by induction on ℓ. We note that if $\ell = 2$, then $t_j^{i+1} = T_j{}^i \cup (b'_1, a'_2) - (b'_1, a''_1)$, where $a'_2 = x_{k+1}$. Since the arc (b'_1, a''_1) is an arc in the circuit in $T_j{}^i$, its deletion keeps the resulting graph connected and circuitless. To prove statements (b) and (c), we note that $\mid \Gamma_{tj}^{i+1} y \mid = \mid \Gamma_{Tj}^i y \mid$ for all $y \in Y_{tj}^{i+1} = Y_{Tj}^i$ and finally that $X_{tj}^{i+1} = X_{tj}^i \cup x_{k+1}$, where $x_{k+1} \in X_j{}^i - X_{tj}^i$. Thus the lemma is correct for $\ell = 2$. Suppose it is also true for $\ell = r$, we will prove it for $\ell = r + 1$. In this case, let $T'_j{}^i = T_j{}^i \cup (b'_1, a'_2) - (b'_1, a''_1)$, where $a'_2 \in \alpha_2$ is now a labeled vertex. It can be seen that $T'_j{}^i$ satisfies the same conditions that were satisfied by T_j^i. The process of obtaining t_j^{i+1} from $T_j'^i$ is identical with the process outlined in step 3(e) of the algorithm but in this case $\ell = r$. This completes the induction, hence the lemma.

LEMMA 3. *In the algorithm, for each i and j if α_p does not contain any unlabeled vertices for $p = 1, 2, \ldots, r$, then* (a) $| \bigcup_{i=1}^{r} \beta_p | = | \bigcup_{p=1}^{r} \alpha_p |$ *and* (b) *if, in addition* $| \alpha_{r+1} | = 0$, *then*

$$\Gamma_j^i \left\{ \bigcup_{p=1}^{r} \beta_p \right\} \subset \left\{ \bigcup_{p=1}^{r} \alpha_p \right\}.$$

PROOF. (a) Consider the graph T_j^i and construct from it a graph T_{js}^i by deleting the vertices in $\bigcup_{p=1}^{r} \beta_p$ from T_j^i and replacing the vertices in $\bigcup_{p=1}^{r} \alpha_p$ by a single vertex, say x_α. It can be easily shown that

$$T_{js}^i = (\Gamma_{js}^i, X_{js}^i, Y_{js}^i)$$

is a connected tree and $| \Gamma_{js}^i y | = 2$ for all $y \in Y_{js}^i$. Thus, from Lemma 2, $| X_{js}^i | = | Y_{js}^i | + 1$, and we already know $| X_{Tj}^i | = | Y_{Tj}^i |$. However,

$$| X_{Tj}^i | = | X_{js}^i | + \left| \bigcup_{p=1}^{r} \alpha_p \right| - 1 \qquad \text{and} \qquad | Y_{Tj}^i | = | Y_{js}^i | + \left| \bigcup_{p=1}^{r} \beta_p \right|.$$

The above three equalities imply that $| \bigcup_{p=1}^{r} \beta_p | = | \bigcup_{p=1}^{r} \alpha_p |$, and hence part (a) of the lemma.

(b) If $| \alpha_{r+1} | = 0$, then from step 3(c) of the algorithm, $\Gamma_j^i \beta_r \subset \bigcup_{p=1}^{r} \alpha_p$. However, in any case, we have $\Gamma_j^i \{ \bigcup_{p=1}^{r-1} \beta_p \} \subset \bigcup_{p=1}^{r} \alpha_p$ from which part (b) of lemma is directly deduced. We now restate and prove Theorem 2 which was quoted in the previous section[4].

THEOREM 2. *A bipartite graph $G = (\Gamma, X, Y)$ with $| \Gamma z | \geqslant 1$ for all $z \in X \cup Y$ contains an alternating tree with respect to Y if and only if there exist subsets $X_1 \subset X$ and $Y_1 \subset Y$ such that $\Gamma X_1 = Y_1$ and $| X_1 | > | Y_1 |$.*

PROOF. If $G = (\Gamma, X, Y)$ has an alternating tree $t = (\Gamma^*, X^*, Y^*)$ with respect to Y, then from Definition 1, $\Gamma X^* \subset Y^*$; and by Lemma 1, $| X^* | > | Y^* |$. This proves the necessity. To prove the converse, we assume

[4] J. E. Desler of Northwestern University has privately communicated to the authors a direct proof of Theorem 2, which is independent of the algorithm. However, discussion of Desler's method of proof would be a diversion from our main goal which is to eventually prove that the algorithm finds an MIS set of G.

there exist subsets $X_1 \subset X$ and $Y_1 \subset Y$ such that $\Gamma X_1 = Y_1$ and $|X_1| > |Y_1|$ and construct the graph $G_1 = (\Gamma_1, X_1, Y_1)$ where $\Gamma_1 x = \Gamma x$ for all $x \in X_1$ and $\Gamma X_1 = Y_1$. Now, it suffices to show that there exists an alternating tree in G, with respect to Y_1. If $|Y_1| = 1$, then the theorem is trivially true. Suppose it is true for $|Y_1| = k$, we prove it for $|Y_1| = k + 1$. Consider graph $G_1 = (\Gamma_1, X_1, Y_1)$. We follow the algorithm. In step (1), if there exists a vertex $x \in X_1$ adjacent to a pendant vertex we delete x. Then, in the resulting graph, $G_1' = (\Gamma_1, X_1 - x, Y_1')$, $|Y_1'| \leqslant k$ and $|X_1 - x| > |Y_1'|$; hence, by the induction hypothesis, the lemma is correct. Let us assume there are no such vertices, then we go to step 2 of the algorithm. Either a desired tree is formed in step 2 or we are referred to step 3 where again either a desired tree is formed or we must delete a set of vertices $X^* \subset X_1$ (in the algorithm $X^* = \bigcup_{p=1}^{\ell} \alpha_p$) from G_1. Let the resulting graph (if it is not empty) be $G_1' = (\Gamma_1, X_1 - X^*, Y_1')$, where as a consequence of Lemma 3, $|Y_1'| < |Y_1| - |X^*|$. Now, in G_1', $|\Gamma_1 x| > 1$ for all $x \in X_1 - X^*$ and $|X_1 - X^*| = |X_1| - |X^*| > |Y_1| - |X^*| > |Y_1'|$. Thus, by the induction hypothesis there exists an alternating tree in G_1' with respect to Y_1' which in turn proves the existence of the tree in G_1 and therefore in G. We must show that G_1' is not empty. To do this, one only needs to show that $|X_1 - X^*| > 0$. Let us return to step 3(c) of the algorithm. We decided to remove a set of vertices $X^* \subset X_1$ from G_1 because there existed a set $Y^* \subset Y_1$, (in the algorithm $Y^* = \bigcup_{p=1}^{\ell} \beta_p$), such that $\Gamma_1 Y^* \subset X^*$, and due to Lemma 3, $|Y^*| = |X^*|$. But $|Y^*| \leqslant |Y_1| < |X_1|$, from which one concludes that $|X_1| - |Y^*| > 0$ which implies that $|X_1| - |X^*| > 0$. This is the desired result since $|X_1| - |X^*| = |X_1 - X^*|$.

IV. Maximum Internally Stable Sets of a Bipartite Graph

In this section, we show that the algorithm, as stated in Section III, when it is allowed to run to its completion, provides an efficient computational method for finding an MIS set of a bipartite graph. To the authors' knowledge, no "acceptable" computational procedure for this problem is available, as yet. We state the required result in the form of the following theorem.

Theorem 3. *Let $G = (\Gamma, X, Y)$ be a bipartite graph and assume, without the loss of generality, that $|\Gamma z| > 0$ for all $z \in X \cup Y$. (Then, Y is a complete* IS *set of G.) Let $t = (\Gamma_t, X_t, Y_t)$ be an alternating tree of G with respect to Y found by completing the algorithm, (t may be empty). Then $S = (Y \cup X_t) - Y_t$ is an* MIS *set of G.*

To prove Theorem 3, we need some additional results which will be developed in this section.

LEMMA 4. *Let* $G = (\Gamma, X, Y)$ *be a connected bipartite graph with* $|X| \leqslant |Y|$. *If* G *contains a connected subgraph* $G_1 = (\Gamma_p, X, Y)$ *such that* $|\Gamma_p y| = 2$ *for all* $y \in Y$, *then* G *contains no alternating tree with respect to* Y*; or equivalently, there exists no subsets* $X_1 \subset X$ *and* $Y_1 \subset Y$ *such that* $\Gamma X_1 \subset Y_1$ *and* $|X_1| > |Y_1|$.

PROOF. Let us assume otherwise. Thus, there exist subsets $X_1 \subset X$ and $Y_1 \subset Y$ such that $\Gamma X_1 \subset Y_1$ and $|X_1| > |Y_1|$. Since $\Gamma_p X_1 \subset \Gamma X_1$, then $\Gamma_p X_1 \subset Y_1$. This implies (from Theorem 2) there exists an alternating tree $t_1 = (\Gamma_{pt}, X_{1t}, Y_{1t})$ in G_1 with respect to Y where $X_{1t} \subset X_1$ and $Y_{1t} \subset Y_1$. From the definition of an alternating tree, $|X_{1t}| > |Y_{1t}|$ and $|\Gamma_{pt} y| \geqslant 2$ for all $y \in Y_{1t}$ and $\Gamma_p X_{1t} \subset Y_{1t}$. Since $|\Gamma_{pt} y| \leqslant |\Gamma_p y| = 2$ for all $y \in Y_{1t}$, then $\Gamma_{pt} y = \Gamma_p y$ for all $y \in Y_{1t}$. This implies that $\Gamma_p Y_{1t} = \Gamma_{pt} Y_{1t} = X_{1t}$. We also have $\Gamma_p X_{1t} \subset Y_{1t}$, and the last two relations imply that there exists no chain beginning with an vertex in $X_{1t} \cup Y_{1t}$ and ending at a vertex in $(X_1 \cup Y_1) - (X_{1t} \cup Y_{1t})$ in $G_1 = (\Gamma_p, X_1, Y_1)$. Thus, if $(X_1 \cup Y_1) - (X_{1t} \cup Y_{1t})$ is not empty, G_1 is not connected, which is a contradiction to the hypothesis. However, if $X_1 \cup Y_1 - X_{1t} \cup Y_{1t}$ is empty, then $X_1 = X_{1t}$ and $Y_1 = Y_{1t}$ which implies that $|X_{1t}| \leqslant |Y_{1t}|$. This is again a contradiction, since we have $|X_{1t}| > |Y_{1t}|$; hence the lemma.

THEOREM 4. *Let* $G = (\Gamma, X, Y)$ *be a bipartite graph with* $|\Gamma z| > 0$ *for all* $z \in X \cup Y$. *If* $t = (\Gamma_t, X_t, Y_t)$ *is an alternating tree of* G *with respect to* Y *found by the completion of the algorithm, (if* t *is empty, assume* $|Y_t| = |X_t| = 0$*), then there exists no alternating tree* $t_1 = (\Gamma_{t1}, X_{t1}, X_{t1})$ *of* G *with respect to* Y *such that* $|Y_{t1} - Y_t| > 0$.

PROOF. We prove the theorem by induction on $|Y|$. If $|Y| = 1$, then the theorem can be easily shown to be correct. Assume the theorem is correct for $|Y| \leqslant k$. Suppose $|Y| = k + 1$. If $|Y_t| = |Y|$, then again the theorem is correct. Let us assume for the remainder of this proof that $|Y_t| < |Y|$. If there exists a pendant vertex $y \in Y$ of $G = (\Gamma, X, Y)$, then both trees t and t_1 will be alternating trees of $G_1 = (\Gamma, X - x, Y')$ where x is adjacent to y and $\Gamma\{X - x\} = Y'$. But $|Y'| \leqslant k$, and by induction hypothesis such a pair of alternating trees cannot exist. Now, let us assume there exist no pendant vertices in Y. Since $|Y_t| < |Y|$, a set of vertices X^* were deleted by the algorithm in step 3(c) on the construction of alternating tree t. This implies that there exist nonempty sets of vertices $X^* \subset X$ and $Y^* \subset Y$ such that $\Gamma Y^* \subset X^*$ and $|Y^*| \geqslant |X^*|$. (In the algorithm step 3(c) $X^* = \bigcup_{p=1}^{\ell} \alpha_p$ and $Y^* = \Gamma X^* \supset \bigcup_{p=1}^{\ell} \beta_p$.) Consider the bipartite graph (a subgraph of G) $G^* = (\Gamma, X^*, Y^*)$. From the algorithm, it can be seen that this graph contains a connected subgraph (Γ_p, X^*, Y^*) such that $|\Gamma_p y| = 2$ for all $y \in Y^*$. Thus, from Lemma 4, G^* does not contain an alternating tree with respect to Y^*. Since vertices X^* are deleted from the graph by the algorithm, alternating tree t is also an alternating tree of $G_1 = (\Gamma, X - X^*, Y - Y^*)$. If t_1 is also an alternating tree of G_1, then since $| Y - Y^* | \leqslant k$, this is a contradiction to the induction hypothesis. If t_1 is not an alternating tree of G_1, then a vertex in Y^* must be in the alternating tree t_1. This implies that there exists an alternating tree in $G^* = (\Gamma, X^*, Y^*)$ with respect to Y which violates Lemma 4. Hence the Theorem.

COROLLARY. *Let $G = (\Gamma, X, Y)$ be a bipartite graph with $| \Gamma z | > 0$ for all $z \in X \cup Y$. Let $t = (\Gamma_t, X_t, Y_t)$. Then there exists no alternating tree of G with more arcs than t.*

PROOF. From Theorem 4, if $t_1 = (\Gamma_1, X_{t1}, Y_{t1})$ is any alternating tree of G, then $Y_{t1} \subset Y_t$. Thus, any arc that can be included in t_1 is (or can be) also included in t, (see step 5 of the algorithm). This completes the proof.

LEMMA 5. *Let $G = (\Gamma, X, Y)$ be a bipartite graph with an alternating tree $t = (\Gamma_t, X, Y)$ with respect to Y such that $| \Gamma_t z | > 0$ for all $z \in X \cup Y$. Then, there exists no alternating tree of G with respect to X.*

PROOF. We prove this lemma also be induction on $| Y |$. If $| Y | = 1$, then the lemma is true. Assume that if $| Y | \leqslant k$, the lemma is correct. Now let $G = (\Gamma, X, Y)$ be a graph, with $| Y | = k + 1$, and $t = (\Gamma_t, X, Y)$ and an alternating tree of G with respect to Y. Let $t_1 = (\Gamma_{t1}, X_1, Y_1)$ be an alternating tree of G with respect to X. We first claim that $Y_1 \neq Y$, because if $Y_1 = Y$, then $\Gamma Y_1 = X$ and from the definition of an alternating tree $\Gamma Y_1 = X_1$; thus $X = X_1$. The fact that alternating trees t and t_1 exist with respect to Y and X, imply that $| X | > | Y |$ and $| Y_1 | > | X_1 |$ which implies $| Y_1 | > | Y |$; this is a contradiction, hence $Y_1 \subset Y$. Let $y \in Y - Y_1$ and consider the graph $G^* = (\Gamma, X', Y - y)$ where $X' = \Gamma\{Y - y\}$. Then $t^* = (\Gamma_t, X', Y - y)$ is also an alternating tree of G^* with respect to $Y - y$ and also $t_1 = (\Gamma_{t1}, X_1, Y_1)$ is an alternating tree of G^* with respect to X'. Since $| Y - y | = k$, the existence of these two alternating trees in G^* violates the induction hypothesis. Hence the lemma.

PROOF OF THEOREM 3. Clearly $(Y \cup X_t) - Y_t = S = (Y - Y_t) \cup X_t$ is an IS set; the question is, is it also an MIS set of G? If t is empty, then by Theorem 4, there exists no alternating tree of G with respect to Y; hence by Theorem 1, S is a MIS set of G and the Theorem is correct. Suppose t is not empty; then, we know that $| S | > | Y |$. It can be shown that S is a complete IS set (see step 5 of the algorithm). Let $X_1 = X_t$, $Y_1 = Y_t$, $X_2 = X - X_t$, and $Y_2 = Y - Y_t$, then $S = X_1 \cup Y_2$. If S is not an

MIS set of G then there exists, from Theorem 1, an alternating tree of G with respect to S. Since such an alternating tree, if it exists, cannot contain any arc (x, y) with either $x \in X_1$ and $y \in Y_2$ or $x \in X_2$ and $y \in Y_1$, existence of the alternating tree with respect to S implies one of the following possibilities:

(1) There exists an alternating tree with respect to Y_2 in $G_2 = (\Gamma, X_2, Y_2)$. This however contradicts Theorem 4.

(2) There exists an alternating tree with respect to X_1 in $G_1 = (\Gamma, X_1, Y_1)$. But the original alternating tree $t = (\Gamma_t, X_1, Y_1)$ is a tree with respect to Y_1; thus G_1, by Lemma 5, contains no alternating trees with respect to X_1. This ends the proof of the theorem.

V. Conclusions

The problem of finding a maximum internally stable (MIS) set of an arbitrary graph is reduced to that of finding an alternating tree with respect to a complete internally stable set. This is conceptually similar to Norman Rabin [9] and Berge [1] theory which reduced the problem of finding a "maximum matching" or, in general, finding "maximum degree constrained subgraphs" of a given graph to that of finding an alternating chain between "exposed" vertices [10, 11, 12]. Our theory does not solve the problem of finding an MIS set of a graph and, in a similar manner, Norman, Rabin, and Berge theory didn't solve the problem of finding a maximum matching. However Edmonds [10] proposed an algorithm for finding alternating chains between exposed vertices of a graph resulting in a practical computational method for finding a maximum matching of a graph. Similarly, an algorithm for finding an alternating tree of a graph, if it is found, will solve the problem of finding an MIS set of a graph. In the meantime, our algorithm is certainly an acceptable method for finding the maximum internally stable set of a bipartite graph.

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