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We present a group theoretic construction of the Virasoro algebra in the framework of wreath products. This can be regarded as a counterpart of a geometric construction of Lehn in the theory of Hubert schemes of points on a surface.

INTRODUCTION

It is by now well known that a direct sum $\oplus_{n \geq 0} R(S_n)$ of the Grothendieck rings of symmetric groups S_n can be identified with the Fock space of the Heisenberg algebra of rank one. One can construct vertex operators whose components generate an infinite-dimensional Clifford algebra, the relation known as boson–fermion correspondence [F] (also see [J]). A natural open problem which arises here is understanding the group theoretic meaning of more general vertex operators in a vertex algebra [B, FLM].

A connection between a direct sum $R_\Gamma = \oplus_{n \geq 0} R(\Gamma_n)$ of the Grothendieck rings of wreath products $\Gamma_n = \Gamma \sim S_n$ associated to a finite group Γ and vertex operators has been realized recently in [W, FJW] (also see [Z, M] for closely related algebraic structures on R_Γ). When Γ is trivial one recovers the above symmetric group picture. On the other hand, this

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wreath product approach turns out to be very much parallel to the development in the theory of Hilbert schemes of points on a surface (cf. [W, N2] and references therein). As one expects that new insight in one theory will shed new light on the other, this refreshes our hope of understanding the group theoretic meaning of general vertex operators.

The goal of this paper is to take the next step in this direction to produce the Virasoro algebra within the framework of wreath products. Denote by Γ_* the set of conjugacy classes of Γ, and by c^0 the identity conjugacy class. Recall [M, Z] that the conjugacy classes of the wreath product Γ_n are parameterized by the partition-valued functions on Γ_* of length n (also see Section 2). Given $c \in \Gamma_*$, we denote by λ_c the function which maps c to the one-part partition (2), c^0 to the partition (1^{n-2}), and other conjugacy classes to 0. We will define an operator Δ_c in terms of the convolution with the characteristic class function on Γ_n (for all n) associated to the conjugacy class parameterized by λ_c. We show that this operator can be identified with a differential operator which is the zero mode of a certain vertex operator when we identify R_Γ as in [M] with a symmetric algebra. A group theoretic construction of Heisenberg algebra has been given in [W] (also see [FJW]) which acts on R_Γ irreducibly. The commutator between Δ_c and the Heisenberg algebra generators on R_Γ provides us the Virasoro algebra generators.

Our construction is motivated in part by the work of Lehn [L] in the theory of Hilbert schemes. Among other results, he showed that an operator defined in terms of intersection with the boundary of Hilbert schemes may be used to produce the Virasoro algebra when combined with earlier construction of Heisenberg algebra due to Nakajima and independently to Grojnowski [N1, Gr]. It remains an important open problem to establish a precise relationship between Lehn's construction and ours.

We remark that the convolution operator in the symmetric group case (i.e., Γ trivial), when interpreted as an operator on the space of symmetric functions, is intimately related to the Hamiltonian in a Calegero–Sutherland integrable system and to the Macdonald operator which is used to define Macdonald polynomials [AMOS, M].

After we discovered our group theoretic approach toward the Virasoro algebra, we notice that our convolution operator in the symmetric group case has been considered by Goulden [Go] when studying the number of ways of writing permutations in a given conjugacy class as products of transpositions. We regard this as a confirmation of our belief that the connections between R_Γ and (general) vertex operators are profound. It is likely that oftentimes when we understand something deeper in this direction, we may realize that it is already hidden in the vast literature of combinatorics, particularly on symmetric groups and symmetric functions for totally different needs. Then the virtue of our point of view will be to serve as a unifying principle which patches together many pieces of mathematics which were not suspected to be related at all.

The plan of this paper is as follows. In Section 1, we recall some basics of Heisenberg and Virasoro algebras from the viewpoint of vertex algebras. In Section 2 we set up the background in the theory of wreath products which our main constructions are based on. In Section 3 we present our main results. Some materials in the paper are fairly standard to experts, but we have decided to include them in the hope that it might be helpful to readers with different backgrounds.

1. BASICS OF HEISENBERG AND VIRASORO ALGEBRAS

In the following we will present some basic constructions in vertex algebras which give us the Virasoro algebra from a Heisenberg algebra.

Let L be a rank N lattice endowed with an integral non-degenerate symmetric bilinear form. Indeed we will only need the case $L = \mathbb{Z}^N$ with the standard bilinear form.

Denote by $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} L$ the vector space generated by L with the bilinear form $\langle -, - \rangle$ induced from L. We define the Heisenberg algebra

$$\hat{\mathfrak{h}} = \mathbb{C}[t, t^{-1}] \otimes \mathfrak{h} \oplus \mathbb{C}C$$

with the commutation relations

$$[C, a_n] = 0,$$
$$[a_n, b_m] = n \delta_{n,-m} \langle a, b \rangle C,$$

where a_n denotes $t^n \otimes a$, $a \in \mathfrak{h}$.

We denote by S_L the symmetric algebra generated by $\hat{\mathfrak{h}}^- = t^{-1}\mathbb{C}[t^{-1}] \otimes \mathfrak{h}$. It is well known that S_L can be given the structure of an irreducible module over Heisenberg algebra $\hat{\mathfrak{h}}$ by letting a_{-n}, $n > 0$ act multiplication and letting

$$a_n . a^1_{-n_1} a^2_{-n_2} \cdots a^k_{-n_k} = \sum_{i=1}^{k} \delta_{n, n_i} \langle a, a^i \rangle a^1_{-n_1} a^2_{-n_2} \cdots \check{a}^i_{-n_i} \cdots a^k_{-n_k},$$

where $n \geq 0$, $n_i > 0$, $a, a^i \in \mathfrak{h}$ for $i = 1, \ldots, k$, and $\check{a}^i_{-n_i}$ means the very term is deleted. A natural gradation on S_L is defined by letting

$$\deg\left(a^1_{-n_1} a^2_{-n_2} \cdots a^k_{-n_k}\right) = n_1 + \cdots + n_k .$$

We say an operator on S_L is of degree p if it maps any nth graded subspace of S_L to $(n+p)$th graded subspace.

The space S_L carries a natural structure of a vertex algebra [B, FLM]. It is convenient to use the generating function in a variable z:

$$a(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1}, \qquad a \in \mathfrak{h}.$$

In the language of vertex algebras, this is the vertex operator associated to the vector $a_{-1} \in S_L$. The normally ordered product between two vertex operators $a(z)$ and $b(z)$ is defined as

$$:a(z)b(z): = \sum_{n<0} a_n z^{-n-1} b(z) + b(z) \sum_{n\geq 0} a_n z^{-n-1}.$$

We remark that the commutation relations between a_n, b_m, $n, m \in \mathbb{Z}$ are encoded in the following operator product expansion (cf. [FLM]):

$$a(z)b(w) \sim \frac{\langle a, b\rangle}{(z-w)^2}.$$

We recall that the Virasoro algebra is spanned by L_n, $n \in \mathbb{Z}$ and a central element C, with the following commutation relations:

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{n^3-n}{12}\delta_{n,m}C.$$

It is convenient to denote

$$L(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}.$$

Given an orthonormal basis a^i, $i = 1, \dots, n$ of $\mathfrak{h}$, we define a series of operators $\tilde{L}^i_n$ $(i = 1, \dots, N,\ n \in \mathbb{Z})$ acting on S_L by

$$\tilde{L}^i(z) \equiv \sum_{n\in\mathbb{Z}} \tilde{L}^i_n z^{-n-2} = \tfrac{1}{2} :a^i(z)a^i(z):.$$

The following proposition is well known (cf., e.g., [FLM]).

Proposition 1. *The operators* $\tilde{L}^i_n$ $(i = 1, \dots, N,\ n \in \mathbb{Z})$ *generate* N *commutative copies of Virasoro algebra of central charge* 1. *The operators* $\tilde{L}_n$, $n \in \mathbb{Z}$ *generate the Virasoro algebra with central charge* N. *Namely we have*

$$\left[\tilde{L}^i_n, \tilde{L}^j_m\right] = \delta_{ij}(n-m)\tilde{L}^i_{n+m} + \frac{n^3-n}{12}\delta_{ij}\delta_{n,m}.$$

From now on we will simply write $\tilde{L}^i_n$ as L^i_n.

We introduce operators

$$\Delta^i = \tfrac{1}{6}\int : a^i(z)^3 : z^2\, dz.$$

These are well-defined operators of degree 0 acting on S_L. One can easily check that

$$\Delta^i = \tfrac{1}{2} \sum_{n, m > 0} \left(a^i_{-n-m} a^i_n a^i_m + a^i_{-n} a^i_{-m} a^i_{n+m} \right).$$

Here we omit on the right hand side the terms involving a_0 since a_0 acts as 0 on the Fock space S_L.

LEMMA 1. *We have*

$$\left[\Delta^i, a_n \right] = -n \langle a^i, a \rangle L^i_n.$$

Proof. It is clear that $[\Delta^i, a^j_n] = 0$ for $i \neq j$. So it suffices to prove that

$$\left[\Delta^i, a^i_n \right] = -n L^i_n.$$

We calculate that

$$\begin{aligned}
\left[\Delta^i, a^i(z) \right] &= \mathrm{Res}_{w=0} \tfrac{1}{6} \left[: a^i(w)^3 :, a^i(z) \right] w^2 \\
&= -\tfrac{1}{2} \mathrm{Res}_{w=0} : a^i(w)^2 : w^2 \sum_{m \in \mathbb{Z}} m w^{m-1} z^{-m-1} \\
&= -\mathrm{Res}_{w=0} \sum_{n \in \mathbb{Z}} L^i_n w^{-n-2} w^2 \sum_{m \in \mathbb{Z}} m w^{m-1} z^{-m-1} \\
&= -\sum_{n \in \mathbb{Z}} n L^i_n z^{-n-1}.
\end{aligned}$$

Therefore

$$\left[\Delta^i, a^i_n \right] = \mathrm{Res}_{z=0} \left[\Delta^i, a^i(z) \right] z^n = -n L^i_n.$$

∎

2. REPRESENTATION RINGS OF WREATH PRODUCTS

Given a finite group Γ, we denote by Γ^* the set of complex irreducible characters and by Γ_* the set of conjugacy classes. We denote by $R_{\mathbb{Z}}(\Gamma)$ the $\mathbb{Z}$-span of irreducible characters of Γ. Denote by c^0 the identity conjugacy class. We identify $R(\Gamma) = \mathbb{C} \otimes_{\mathbb{Z}} R_{\mathbb{Z}}(\Gamma)$ with the space of class functions on Γ.

For $c \in \Gamma_*$ let ζ_c be the order of the centralizer of an element in the class c. Denote by $|\Gamma|$ the order of Γ. The usual bilinear form $\langle -, - \rangle_\Gamma$ on $R(\Gamma)$ is defined as (often we will omit the subscript Γ)

$$\langle f, g \rangle = \langle f, g \rangle_\Gamma = \frac{1}{|\Gamma|} \sum_{x \in \Gamma} f(x) g(x^{-1}) = \sum_{c \in \Gamma_*} \zeta_c^{-1} f(c) g(c^{-1}), \quad (1)$$

where c^{-1} denotes the conjugacy class $\{x^{-1}, x \in c\}$. Clearly $\zeta_c = \zeta_{c^{-1}}$. It is well known that

$$\begin{aligned} \langle \gamma, \gamma' \rangle &= \delta_{\gamma, \gamma'}, \qquad \gamma, \gamma' \in \Gamma^* \\ \sum_{\gamma \in \Gamma^*} \gamma(c') \gamma(c^{-1}) &= \delta_{c, c'} \zeta_c, \qquad c, c' \in \Gamma_*. \end{aligned} \quad (2)$$

One may regard $\mathbb{C}[\Gamma]$ as the space of functions on Γ, and thus $R(\Gamma)$ as a subspace of $\mathbb{C}[\Gamma]$. Given $f, g \in \mathbb{C}[\Gamma]$, the convolution $f * g \in \mathbb{C}[\Gamma]$ is defined by

$$f * g(x) = \sum_{y \in \Gamma} f(xy^{-1}) g(y), \qquad f, g \in \mathbb{C}[\Gamma], \qquad x \in \Gamma.$$

In particular if $f, g \in R(\Gamma)$, then so is $f * g$. It is well known that

$$\gamma' * \gamma = \frac{|\Gamma|}{d_\gamma} \delta_{\gamma, \gamma'} \gamma, \qquad \gamma', \gamma \in \Gamma^*, \quad (3)$$

where d_γ is the *degree* of the irreducible character γ.

Denote by K_c the sum of all elements in the conjugacy class c. By abuse of notation, we also regard K_c the class function on Γ which takes value 1 on elements in the conjugacy class c and 0 elsewhere. It is clear that K_c, $c \in \Gamma_*$, forms a basis of $R(\Gamma)$. The elements K_c, $c \in \Gamma_*$ actually form a linear basis of the center in the group algebra $\mathbb{C}[\Gamma]$. But we will not need this fact.

For wreath products we basically follow the excellent presentation of Macdonald [M, Appendix B, Chap. 1], with the exception of Theorem 1 which is quoted from [W] (also see [FJW]). Given a positive integer n, let $\Gamma^n = \Gamma \times \cdots \times \Gamma$ be the nth direct product of Γ. The symmetric group S_n acts on Γ^n by permutations: $\sigma(g_1, \ldots, g_n) = (g_{\sigma^{-1}(1)}, \ldots, g_{\sigma^{-1}(n)})$. The wreath product of Γ with S_n is defined to be the semi-direct product

$$\Gamma_n = \{(g, \sigma) \,|\, g = (g_1, \ldots, g_n) \in \Gamma^n, \, \sigma \in S_n\}$$

with the multiplication

$$(g, \sigma) \cdot (h, \tau) = (g\sigma(h), \sigma\tau).$$

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ be a partition of integer $|\lambda| = \lambda_1 + \cdots + \lambda_l$, where $\lambda_1 \geq \cdots \geq \lambda_l \geq 1$. The integer l is called the *length* of the partition λ and is denoted by $l(\lambda)$. We will identify the partition $(\lambda_1, \lambda_2, \ldots, \lambda_l)$ with $(\lambda_1, \lambda_2, \ldots, \lambda_l, 0, \ldots, 0)$. We will also make use of another notation for partitions,

$$\lambda = (1^{m_1} 2^{m_2} \cdots),$$

where m_i is the number of parts in λ equal to i.

For a finite set X and $\rho = (\rho(x))_{x \in X}$ a family of partitions indexed by X, we write

$$\|\rho\| = \sum_{x \in X} |\rho(x)|.$$

Sometimes it is convenient to regard $\rho = (\rho(x))_{x \in X}$ as a partition-valued function on X. We denote by $\mathscr{P}(X)$ the set of all partitions indexed by X and by $\mathscr{P}_n(X)$ the set of all partitions in $\mathscr{P}(X)$ such that $\|\rho\| = n$.

The conjugacy classes of Γ_n can be described in the following way. Let $x = (g, \sigma) \in \Gamma_n$, where $g = (g_1, \ldots, g_n) \in \Gamma^n$, $\sigma \in S_n$. The permutation σ is written as a product of disjoint cycles. For each such cycle $y = (i_1 i_2 \cdots i_k)$ the element $g_{i_k} g_{i_{k-1}} \cdots g_{i_1} \in \Gamma$ is determined up to conjugacy in Γ by g and y and will be called the *cycle-product* of x corresponding to the cycle y. For any conjugacy class c and each integer $i \geq 1$, the number of i-cycles in σ whose cycle-product lies in c will be denoted by $m_i(c)$. Denote by $\rho(c)$ the partition $(1^{m_1(c)} 2^{m_2(c)} \cdots)$, $c \in \Gamma_*$. Then each element $x = (g, \sigma) \in \Gamma_n$ gives rise to a partition-valued function $(\rho(c))_{c \in \Gamma_*} \in \mathscr{P}(\Gamma_*)$ such that $\sum_{i,c} i m_i(c) = n$. The partition-valued function $\rho = (\rho(c))_{c \in G_*}$ is called the *type* of x. It is known that any two elements of Γ_n are conjugate in Γ_n if and only if they have the same type.

Given a partition $\lambda = (1^{m_1} 2^{m_2} \cdots)$, we define $z_\lambda = \prod_{i \geq 1} i^{m_i} m_i!$. We note that z_λ is the order of the centralizer of an element of cycle-type λ in $S_{|\lambda|}$. The order of the centralizer of an element $x = (g, \sigma) \in \Gamma_n$ of the type $\rho = (\rho(c))_{c \in \Gamma_*}$ is

$$Z\rho = \prod_{c \in \Gamma_*} z_{\rho(c)} \zeta_c^{l(\rho(c))}. \tag{4}$$

Recall that $R(\Gamma_n) = R_{\mathbb{Z}}(\Gamma_n) \otimes_{\mathbb{Z}} \mathbb{C}$. We set

$$R_\Gamma = \bigoplus_{n \geq 0} R(\Gamma_n).$$

A symmetric bilinear form on R_Γ is given by

$$\langle u, v \rangle = \sum_{n \geq 0} \langle u_n, v_n \rangle_{\Gamma_n}, \tag{5}$$

where $u = \sum_n u_n$ and $v = \sum_n v_n$ with $u_n, v_n \in \Gamma_n$.

Since $R_{\mathbb{Z}}(\Gamma)$ may be regarded as an integral lattice with non-degenerate symmetric bilinear form given by (1) and with an orthonormal basis Γ^*, we can apply the constructions in Section 1 to the lattice $R_{\mathbb{Z}}(\Gamma)$. We denote the corresponding Heisenberg algebra by $\hat{\mathfrak{h}}_\Gamma$ with generators $a_n(\gamma)$, $n \in \mathbb{Z}$, $\gamma \in \Gamma^*$, and its irreducible representation by S_Γ.

By identifying $a_{-n}(\gamma)$ $(n > 0,\ \gamma \in \Gamma^*)$ with the nth power sum in a sequence of variables parameterized by $\gamma \in \Gamma^*$, we may regard S_Γ as the algebra of symmetric functions parameterized by $\gamma \in \Gamma^*$. In particular the operator $a_n(\gamma)$ $(n > 0,\ \gamma \in \Gamma^*)$ acts as the differential operator

$$n \frac{\partial}{\partial a_{-n}(\gamma)}.$$

For $m \in \mathbb{Z}$, $c \in \Gamma_*$ we define

$$a_m(c) = \sum_{\gamma \in \Gamma^*} \gamma(c^{-1}) a_m(\gamma). \tag{6}$$

From the orthogonality of the irreducible characters (2) it follows that

$$a_m(\gamma) = \sum_{c \in \Gamma_*} \zeta_c^{-1} \gamma(c) a_m(c).$$

Thus $a_n(c)$ $(n \in \mathbb{Z},\ c \in \Gamma_*)$ and C form a new basis for the Heisenberg algebra $\hat{\mathfrak{h}}_\Gamma$.

Given a partition $\lambda = (\lambda_1, \lambda_2, \dots)$ and $c \in \Gamma_*$, we define

$$p_\lambda(c) = p_{\lambda_1}(c) p_{\lambda_2}(c) \cdots.$$

For any $\rho = (\rho(c))_{c \in \Gamma_*} \in \mathscr{P}(\Gamma_*)$, we further define

$$P_\rho = \prod_{c \in \Gamma_*} p_{\rho(c)}(c).$$

The elements P_ρ, $\rho \in \mathscr{P}(\Gamma_*)$ form a $\mathbb{C}$-basis for S_Γ.

We define a bilinear form $\langle\ ,\ \rangle$ on the space S_Γ by letting

$$\langle P_\rho, P_\sigma \rangle = \delta_{\rho, \sigma} Z_\rho. \tag{7}$$

Let $\Psi\colon \Gamma_n \to S_\Gamma$ be the map defined by $\Psi(x) = P_\rho$ if $x \in \Gamma_n$ is of type ρ. We define a $\mathbb{C}$-linear map ch: $\oplus_{n \geq 0} \mathbb{C}[\Gamma_n] \to S_\Gamma$ by letting

$$\begin{aligned} \operatorname{ch}(f) &= \frac{1}{|\Gamma_n|} \sum_{x \in \Gamma_n} f(x)\Psi(x) \\ &= \sum_{\rho \in \mathscr{P}(\Gamma_*)} Z_\rho^{-1} f_\rho P_\rho \qquad \text{if } f \in R_\Gamma, \end{aligned} \tag{8}$$

where f_ρ is the value of f at elements of type ρ. Most often we will think of ch as a map from R_Γ to S_Γ as in [M], usually referred to as the *characteristic map*. It is well known that ch: $R_\Gamma \to S_\Gamma$ is an isometry for the bilinear forms on R_Γ and S_Γ defined in (5) and (7).

As we identify $a_{-n}(\gamma)$ $(n > 0,\ \gamma \in \Gamma^*)$ with the nth power sum $p_n(\gamma)$ and the space S_Γ with the space of symmetric functions indexed by Γ^*, we may regard the Schur function $s_\lambda(\gamma)$ associated to γ and a partition λ as a corresponding element in S_Γ. For $\lambda \in \mathscr{P}(\Gamma^*)$, we denote

$$s_\lambda = \prod_{\gamma \in \Gamma^*} s_{\lambda(\gamma)}(\gamma) \in S_\Gamma.$$

Then s_λ is the image under the characteristic map ch of the character of an irreducible representation χ^λ of Γ_n (cf. [M]).

Denote by c_n $(c \in \Gamma_*)$ the conjugacy class in Γ_n of elements $(x, s) \in \Gamma_n$ such that s is an n-cycle and the cycle product of x lies in the conjugacy class c. Denote by $\sigma_n(c)$ the class function on Γ_n which takes value $n\zeta_c$ (i.e., the order of the centralizer of an element in the class c_n) on elements in the class c_n and 0 elsewhere. For $\rho = \{m_r(c)\}_{r \geq 1,\ c \in T_*} \in \mathscr{P}_n(\Gamma_*)$, $\sigma_\rho = \prod_{r \geq 1,\, c \in \Gamma_*} \sigma_r(c)^{m_r(c)}$ is the class function on Γ_n which takes value Z_ρ on the conjugacy class of type ρ and 0 elsewhere. Given $\gamma \in R(\Gamma)$, we denote by $\sigma_n(\gamma)$ the class function on Γ_n which takes value $n\gamma(c)$ on elements in the class c_n, $c \in \Gamma_*$, and 0 elsewhere. We define an operator $\tilde{a}_{-n}(\gamma)$, $n > 0$ to be a map from R_Γ to itself by the following composition:

$$R(\Gamma_m) \xrightarrow{\sigma_n(\gamma)\otimes} R(\Gamma_n) \otimes R(\Gamma_m) \xrightarrow{\text{Ind}} R(\Gamma_{n+m}).$$

We also define another operator $\tilde{a}_n(\gamma)$, $n > 0$ to be a map from R_Γ to itself (which is the adjoint of $\tilde{a}_{-n}(\gamma)$ with respect to the bilinear form (5)) as the composition

$$R(\Gamma_m) \xrightarrow{\text{Res}} R(\Gamma_n) \otimes R(\Gamma_{m-n}) \xrightarrow{\langle \sigma_n(\gamma), \cdot \rangle} R(\Gamma_{m-n}).$$

The following theorem was established in [W] (also see [FJW]).

THEOREM 1. *The space R_Γ affords a representation of the Heisenberg algebra $\hat{\mathfrak{h}}_\Gamma$ by letting $a_n(\gamma)$ $(n \in \mathbb{Z}\backslash 0)$ act as $\tilde{a}_n(\gamma)$, and C as 1. The characteristic map* ch *is an isomorphism of R_Γ and S_Γ as representations over the Heisenberg algebra.*

3. VIRASORO ALGEBRA AND GROUP CONVOLUTION

We first look at the case when Γ is trivial and so Γ_n becomes the symmetric group S_n. We simply write the ith power sum $p_i(\gamma)$ as p_i.

We consider the convolution product on the space of class functions on S_n with the class function $K_{(1^{n-2}2)}$, which takes value 1 at elements of cycle type $(1^{n-2}2)$ and 0 otherwise. It follows from (8) that

$$\operatorname{ch}\left(K_{(1^{n-2}2)}\right) = \frac{1}{2(n-2)!} p_1^{n-2} p_2 .$$

We denote

$$\Delta = \frac{1}{2} \sum_{i,j \geq 1} \left(ijp_{i+j} \frac{\partial}{\partial p_i} \frac{\partial}{\partial p_j} + (i+j) p_i p_j \frac{\partial}{\partial p_{i+j}} \right). \tag{9}$$

Given $f, g \in R(S_n)$, it is natural to define the convolution of two symmetric functions $\operatorname{ch}(f)$ and $\operatorname{ch}(g)$ as

$$\operatorname{ch}(f) * \operatorname{ch}(g) := \operatorname{ch}(f * g).$$

The next theorem describes the effect of the convolution with $K_{(1^{n-2}2)}$ on the space of symmetric polynomials by means of the characteristic map.

THEOREM 2. *For any $f \in \mathbb{C}[S_n]$, we have*

$$\operatorname{ch}\left(K_{(1^{n-2}2)} * f\right) = \Delta \operatorname{ch}(f).$$

Equivalently we have

$$\frac{1}{2(n-2)!} p_1^{n-2} p_2 * \operatorname{ch}(f) = \Delta \operatorname{ch}(f).$$

Proof. Take a transposition (a, b) and a permutation τ.

If a and b lie in different cycles of τ, the product $(a, b)\tau$ will have the effect of combining the two cycles, say of cycle length i and j, respectively, in τ containing respectively a and b into a single cycle. For example, let $n = 7$, $a = 3$, $b = 5$, and $\tau = (1,3)(2,6,5)(4,7)$; then $(3,5)\tau = (1,5,2,6,3)(4,7)$. Thus one $p_i p_j$ in $\operatorname{ch}(\tau)$ is replaced by one p_{i+j} in $\operatorname{ch}((a,b)\tau)$. Among all transpositions $(a,b) \in S_n$, there are exactly ij of which have the effect of replacing one $p_i p_j$ by one p_{i+j}.

On the other hand, if a and b lie in a same cycle, say of cycle length k, of τ, then the product $(a, b)\tau$ will have the effect of splitting this cycle in τ containing a and b into two disjoint cycles. For example, let $n = 8$, $a = 3$,

$b = 5$, and $\tau = (8,7,3,1,4,5)(2,6)$; then $(3,5)\tau = (8,7,5)(1,4,3)(2,6)$. More precisely the cycle of τ splits into two disjoint cycles of length i and $k-i$ if a and b in the cycle of τ are separated by $i-1$ and $k-i-1$ elements. Thus one p_k in $\mathrm{ch}(\tau)$ is replaced by one $p_i p_{k-i}$ in $\mathrm{ch}((a,b)\tau)$. We can see easily among all possible transpositions there are k (resp. $k/2$) of them which have the effect of replacing one p_k by one $p_i p_{k-i}$ when $k/2 \neq i$ (resp. $k/2 = i$).

Combining the above considerations together, we have proved the theorem. ∎

Remark 1. Our purpose of studying this convolution is to find a group theoretic construction of the Virasoro algebra. It turns out that such a convolution appeared earlier in a paper of Goulden [Go, Proposition 3.1] in his study of the number of ways of writing permutations in a given conjugacy class as products of transpositions.

Recall that s_λ denotes the Schur functions associated to the partition λ. We denote by f^λ the degree of s_λ, i.e., the dimension of the irreducible representation of S_n corresponding to s_λ.

PROPOSITION 2. *We have*

$$\Delta(s_\lambda) = \frac{n(n-1)}{2f^\lambda}\langle p_1^{n-2}p_2, s_\lambda\rangle s_\lambda.$$

Proof. By the orthogonality relation of characters (2) we have $\langle s_\mu, s_\lambda\rangle = \delta_{\mu,\lambda}$, and thus $p_1^{n-2}p_2 = \sum_\mu \langle p_1^{n-2}p_2, s_\mu\rangle s_\mu$. In addition Eq. (3) implies that $s_\mu * s_\lambda = (n!/f^\lambda)\,\delta_{\mu,\lambda}s_\lambda$. Thus by Theorem 2, we have

$$\begin{aligned}\Delta(s_\lambda) &= \frac{1}{2(n-2)!}p_1^{n-2}p_2 * s_\lambda \\ &= \frac{1}{2(n-2)!}\sum_\mu \langle p_1^{n-2}p_2, s_\mu\rangle s_\mu * s_\lambda \\ &= \frac{n(n-1)}{2f^\lambda}\langle p_1^{n-2}p_2, s_\lambda\rangle s_\lambda.\end{aligned}$$

∎

Now we return to the general case of wreath product Γ_n. Given $c \in \Gamma_*$, we denote by $\lambda_c \in \mathscr{P}_n(\Gamma_*)$ the function which maps c to the one-part partition (2), c^0 to the partition (1^{n-2}), and other conjugacy classes to $\emptyset$. We denote by K_{λ_c} the sum of all the elements in the conjugacy class corresponding to λ_c.

We introduce the operator acting on R_Γ

$$\Delta^\gamma = \frac{1}{6}\int :\tilde{a}^\gamma(z)^3: z^2\,dz,$$

where $\tilde{a}^\gamma(z) = \sum_{n\in\mathbb{Z}}\tilde{a}_n(\gamma)z^{-n-1}$.

We define

$$\Delta_c = \sum_{\beta\in\Gamma^*}\frac{|\Gamma|^2\beta(c^{-1})}{d_\beta^2\zeta_c}\Delta^\beta. \tag{10}$$

The operator Δ_c reduces to (9) for Γ trivial. By the characteristic map, the operator Δ^γ can be identified with the differential operator

$$\frac{1}{2}\sum_{i,j\geq 1}\left(ijp_{i+j}(\gamma)\frac{\partial}{\partial p_i(\gamma)}\frac{\partial}{\partial p_j(\gamma)} + (i+j)p_i(\gamma)p_j(\gamma)\frac{\partial}{\partial p_{i+j}(\gamma)}\right).$$

Note that

$$\mathrm{ch}(K_{\lambda_c}) = \frac{1}{Z_{\lambda_c}}p_1(c^0)^{n-2}p_2(c),$$

where

$$Z_{\lambda_c} = 2(n-2)!|\Gamma|^{n-2}\zeta_c \tag{11}$$

is the order of the centralizer of an element in the conjugacy class associated to λ_c, cf. (4). The following theorem generalizes Theorem 2.

THEOREM 3. *Given $f\in R_\Gamma$, we have*

$$\mathrm{ch}(K_{\lambda_c} * f) = \Delta_c\,\mathrm{ch}(f).$$

Equivalently we have

$$\frac{1}{Z_{\lambda_c}}p_2(c)p_1(c^0)^{n-2} * \mathrm{ch}(f) = \Delta_c\,\mathrm{ch}(f).$$

We need some preparation for the proof of the theorem. Recall that the image of the irreducible character associated to $\lambda\in\mathscr{P}_n(\Gamma^*)$ under the characteristic map is $s_\lambda = \prod_\gamma s_{\lambda(\gamma)}(\gamma)$. We set $n_\gamma = |\lambda(\gamma)|$. The first lemma below is straightforward.

LEMMA 2. *Given $\beta\in\Gamma^*$ and a sequence of non-negative integers m_γ, $\gamma\in\Gamma^*$ such that $m_\gamma\neq n_\gamma$ for at least one γ, we have*

$$\left\langle p_1(\beta)^{m_\beta-2}p_2(\beta)\prod_{\gamma\neq\beta}p_1(\gamma)^{m_\gamma}, s_\lambda\right\rangle = 0.$$

LEMMA 3. *Given* $\beta \in \Gamma^*$, $\lambda \in \mathscr{P}_n(\Gamma^*)$, *and setting* $n_\gamma = |\lambda(\gamma)|$ (*for all* $\gamma \in \Gamma^*$), *we have*

$$\frac{n_\beta(n_\beta - 1)}{2} \left\langle p_1(\beta)^{n_\beta - 2}(\beta) \prod_{\gamma \neq \beta} p_1(\gamma)^{n_\gamma}, s_\lambda \right\rangle s_\lambda$$
$$= \left(\prod_\gamma f^{\lambda(\gamma)} \right) \Delta^\beta s_{\lambda(\beta)}(\beta) \cdot \prod_{\gamma \neq \beta} s_{\lambda(\gamma)}(\gamma).$$

Proof. It follows from Proposition 2 that

$$\Delta^\beta\left(s_{\lambda(\beta)}(\beta)\right) = \frac{n_\beta(n_\beta - 1)}{2 f^{\lambda(\beta)}} \left\langle p_1(\beta)^{n_\beta - 2} p_2(\beta), s_{\lambda(\beta)}(\beta) \right\rangle s_{\lambda(\beta)}(\beta). \tag{12}$$

Noting also that

$$\left\langle p_1(\gamma)^{n_\gamma}, s_{\lambda(\gamma)}(\gamma) \right\rangle = f^{\lambda(\gamma)}, \tag{13}$$

we calculate that

$$\frac{n_\beta(n_\beta - 1)}{2} \left\langle p_1(\beta)^{n_\beta - 2} p_2(\beta) \prod_{\gamma \neq \beta} p_1(\gamma)^{n_\gamma}, s_\lambda \right\rangle s_\lambda$$
$$= \frac{n_\beta(n_\beta - 1)}{2} \left\langle p_1(\beta)^{n_\beta - 2} p_2(\beta), s_{\lambda(\beta)}(\beta) \right\rangle s_{\lambda(\beta)}(\beta)$$
$$\cdot \prod_{\gamma \neq \beta} \left\langle p_1(\gamma)^{n_\gamma}, s_{\lambda(\gamma)}(\gamma) \right\rangle s_{\lambda(\gamma)}(\gamma)$$
$$= f^{\lambda(\beta)} \Delta^\beta s_{\lambda(\beta)}(\beta) \cdot \prod_{\gamma \neq \beta} f^{\lambda(\gamma)} s_{\lambda(\gamma)}(\gamma) \qquad \text{by Eqs. (12) and (13)},$$
$$= \left(\prod_{\gamma \in \Gamma^*} f^{\lambda(\gamma)} \right) \Delta^\beta s_{\lambda(\beta)}(\beta) \cdot \prod_{\gamma \neq \beta} s_{\lambda(\gamma)}(\gamma).$$

This proves the lemma. ∎

Proof of Theorem 3. Consider the irreducible character associated to $\lambda \in \mathscr{P}_n(\Gamma^*)$ which maps by the characteristic map ch to $s_\lambda = \Pi_\gamma s_{\lambda(\gamma)}(\gamma)$. We set $n_\gamma = |\lambda(\gamma)|$.

We calculate that

$$\begin{aligned}
&\frac{1}{2(n-2)!} p_1(c^0)^{n-2} p_2(c) \\
&\quad = \frac{1}{2(n-2)!} \left(\sum_{\gamma \in \Gamma^*} d_\gamma p_1(\gamma) \right)^{n-2} \left(\sum_{\beta \in \Gamma^*} \beta(c^{-1}) p_2(\beta) \right) \\
&\qquad \text{by Eq. (6),} \\
&\quad = \frac{1}{2} \left(\sum_{\{m_\gamma\}} \prod_{\gamma \in \Gamma^*} \frac{\left(d_\gamma p_1(\gamma)\right)^{m_\gamma}}{m_\gamma !} \right) \left(\sum_{\beta \in \Gamma^*} \beta(c^{-1}) p_2(\beta) \right) \\
&\quad = \frac{1}{2} \sum_{\{m_\gamma\}} \sum_{\beta \in \Gamma^*} \beta(c^{-1}) p_2(\beta) \frac{\left(d_\beta p_1(\beta)\right)^{m_\beta}}{(m_\beta)!} \prod_{\gamma \neq \beta} \frac{\left(d_\gamma p_1(\gamma)\right)^{m_\gamma}}{m_\gamma !},
\end{aligned} \tag{14}$$

where the sum $\sum_{\{m_\gamma\}}$ is taken over all sequences of non-negative integers m_γ, $\gamma \in \Gamma^*$ such that $\sum_\gamma m_\gamma = n - 2$.

Recall [M, Z] that the degree of the irreducible character associated to $\lambda \in \mathscr{P}_n(\Gamma^*)$ is

$$f^\lambda = n! \prod_\gamma d_\gamma^{n_\gamma} f^{\lambda(\gamma)} / n_\gamma !. \tag{15}$$

We also obtain by using (3) and $|\Gamma_n| = n! |\Gamma|^n$ that

$$\begin{aligned}
&\frac{f^\lambda}{n! |\Gamma|^n} \left(p_2(\beta) p_1(\beta)^{n_\beta - 2} \prod_{\gamma \neq \beta} p_1(\gamma)^{n_\gamma} \right) * s_\lambda \\
&\quad = \left\langle p_2(\beta) p_1(\beta)^{n_\beta - 2} \prod_{\gamma \neq \beta} p_1(\gamma)^{n_\gamma}, s_\lambda \right\rangle s_\lambda.
\end{aligned} \tag{16}$$

We then have

$$\begin{aligned}
&\frac{1}{Z_{\lambda_c}} p_1(c^0)^{n-2} p_2(c) * s_\lambda \\
&\quad = \frac{1}{2|\Gamma|^{n-2} \zeta_c} \sum_{\beta \in \Gamma^*} \beta(c^{-1}) p_2(\beta) \frac{\left(d_\beta p_1(\beta)\right)^{n_\beta - 2}}{(n_\beta - 2)!} \\
&\qquad \times \prod_{\gamma \neq \beta} \frac{\left(d_\gamma p_1(\gamma)\right)^{n_\gamma}}{n_\gamma !} * s_\lambda \\
&\qquad \text{by Eqs. (11), (14), and Lemma 2,}
\end{aligned}$$

$$= \frac{1}{|\Gamma|^{n-2}\zeta_c} \prod_{\gamma} \frac{d_\gamma^{n_\gamma}}{n_\gamma!}$$

$$\times \sum_{\beta \in \Gamma^*} \frac{n_\beta(n_\beta - 1)\beta(c^{-1})}{2d_\beta^2} \left(p_2(\beta) p_1(\beta)^{n_\beta - 2} \prod_{\gamma \neq \beta} p_1(\gamma)^{n_\gamma} \right) * s_\lambda$$

$$= \frac{f^\lambda}{n!|\Gamma|^{n-2}\zeta_c}$$

$$\times \sum_{\beta \in \Gamma^*} \frac{n_\beta(n_\beta - 1)\beta(c^{-1})}{2d_\beta^2 \left(\prod_\gamma f^{\lambda(\gamma)}\right)} \left(p_2(\beta) p_1(\beta)^{n_\beta - 2} \prod_{\gamma \neq \beta} p_1(\gamma)^{n_\gamma} \right) * s_\lambda$$

by Eq. (15),

$$= \frac{|\Gamma|^2}{\zeta_c} \sum_{\beta \in \Gamma^*} \frac{n_\beta(n_\beta - 1)\beta(c^{-1})}{2d_\beta^2 \left(\prod_\gamma f^{\lambda(\gamma)}\right)}$$

$$\times \left\langle p_2(\beta) p_1(\beta)^{n_\beta - 2} \prod_{\gamma \neq \beta} p_1(\gamma)^{n_\gamma}, s_\lambda \right\rangle s_\lambda$$

by Eq. (16),

$$= \sum_{\beta \in \Gamma^*} \frac{|\Gamma|^2 \beta(c^{-1})}{d_\beta^2 \zeta_c} \left(\Delta^\beta s_{\lambda(\beta)}(\beta) \cdot \prod_{\gamma \neq \beta} s_{\lambda(\gamma)}(\gamma) \right)$$

by Lemma 3,

$$= \Delta_c s_\lambda.$$

The last equation holds since $\Delta^\beta s_{\lambda(\gamma)}(\gamma) = 0$ for $\gamma \neq \beta$. This finishes the proof, since s_λ, $\lambda \in \mathscr{P}(\Gamma^*)$ forms a basis of the space R_Γ. ∎

COROLLARY 1. *For the identity conjugacy class $c^0 \in \Gamma_*$, we have $\Delta_{c^0} = \sum_{\beta \in \Gamma^*} (|\Gamma|/d_\beta) \Delta^\beta$.*

Combining (10), Lemma 1, Theorem 1, and Theorem 3, we obtain the following.

THEOREM 4. *The operator Δ_c acting on R_Γ is realized by the convolution with K_{λ_c}. The commutation relation between Δ_c and the Heisenberg algebra generator $\tilde{a}_n(\gamma)$ (constructed in a group theoretic manner) is given by*

$$[\Delta_c, \tilde{a}_n(\gamma)] = -\frac{n|\Gamma|^2 \gamma(c^{-1})}{d_\gamma^2 \zeta_c} L_n^\gamma,$$

where the operators L_n^γ acting on the space R_Γ satisfy the Virasoro commutation relations:

$$\left[L_n^\beta, L_m^\gamma\right] = \delta_{\beta\gamma} L_{n+m}^\gamma + \frac{n^3 - n}{12} \delta_{\beta\gamma} \delta_{n,-m}.$$

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