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Integral equations for thermodynamics of the $osp(1|2s)$ integrable spin chain

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Abstract

We propose a system of nonlinear integral equations (NLIE), which gives the free energy of the $osp(1|2s)$ integrable spin chain at finite temperatures. In contrast with usual thermodynamic Bethe ansatz equations, our new NLIE contain only a *finite* number of unknown functions. On deriving NLIE, we use our $osp(1|2s)$ version of the T-system and the quantum transfer matrix method. Based on our NLIE, we also calculate the high temperature expansion of the free energy and the specific heat.

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1. Introduction

In recent years, thermodynamics of solvable lattice models related to superalgebras have been studied by thermodynamic Bethe ansatz (TBA) method, and TBA equations are examined from various point of view (see Section 1 in [1] for a comment on the present status of TBA equations for models related to superalgebras). In our previous papers, we have derived TBA equations for the $osp(1|2)$ model [2–4] and the $osp(1|2s)$ model [1] from the string hypothesis and the T-system [5] of the quantum transfer matrix (QTM). These TBA equations contain an infinite number of unknown functions, and thus are not always easily treated. It is important to reduce the TBA equations to tractable integral equations which contain only a finite number of unknown functions.

As for the *XXZ* spin chain, which is related to the algebra $U_q(A_1^{(1)})$ of rank one, Takahashi proposed [6] a nonlinear integral equation (NLIE) recently. This NLIE contains only one unknown function. Due to its simplicity, we can calculate [7] the high temperature expansion of physical quantities to very high order from this NLIE. The purpose of this Letter is to derive a system of NLIE with only a *finite* number (the number of rank s) of unknown

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functions from our $osp(1|2s)$ version of the T-system [5]. This Letter is the first attempt to derive this type of NLIE for a vertex model associated with an algebra of *arbitrary* rank.[1]

In Section 2, we briefly mention the T-system for the $osp(1|2s)$ model and the QTM method [13–17]. This section overlaps with Section 2 and Section 4 in [1], but normalization of the fused QTM is different from [1]. In Section 3, we derive the NLIE. The normalized QTM defined in Section 2 play the role of the unknown functions of these NLIE. These NLIE contain a parameter m, which corresponds to the fusion degree of the model. For $m = 1$, these NLIE form a closed set of equations, which contains only a finite number of unknown functions. On the other hand, for $m \geqslant 2$, they couple with the ones for $m = 1$, and contain an infinite number of unknown functions. This type of equations (for $m \geqslant 2$) has never been considered before even in the case of Takahashi's NLIE for the *XXZ* spin chain [6]. These NLIE relate to traditional TBA equations through the dependent variable transformation (5.1). Using our new NLIE, we calculate the high-temperature expansion of the free energy in Section 4. Section 5 is devoted to discussions.

2. *T*-system and QTM method

In this section, we consider [1] an integrable spin chain [11,12] associated with the fundamental representation of $osp(1|2s)$, and introduce the QTM [13–17] and the T-system [5] for this model. The $\check{R}$-matrix [12,18–21] of this model is given as

$$\check{R}(v) = I + vP - \frac{2v}{2v-g}E, \tag{2.1}$$

where $g = 2s + 1$; $v \in \mathbb{C}$; $P_{ab}^{cd} = (-1)^{p(a)p(b)}\delta_{ad}\delta_{bc}$; $E_{ab}^{cd} = \alpha_{ab}(\alpha^{-1})_{cd}$; $a, b, c, d \in \{1, 2, \dots, s, 0, \bar{s}, \dots, \bar{1}\}$; $1 \prec 2 \prec \cdots \prec s \prec 0 \prec \bar{s} \prec \cdots \prec \bar{2} \prec \bar{1}$; α is $(2s+1) \times (2s+1)$ anti-diagonal matrix whose non-zero elements are $\alpha_{a,\bar{a}} = 1$ for $a \in \{1, 2, \dots, s, 0\}$ and $\alpha_{a,\bar{a}} = -1$ for $a \in \{\bar{s}, \overline{s-1}, \dots, \bar{1}\}$; $\bar{\bar{a}} = a$; $p(a) = 0$ for $a = 0$; $p(a) = 1$ for $a \in \{1, 2, \dots, s\} \sqcup \{\bar{s}, \dots, \bar{2}, \bar{1}\}$. The Hamiltonian of the present model for the periodic boundary condition is given by

$$H = J\sum_{k=1}^{L}\left(P_{k,k+1} + \frac{2}{g}E_{k,k+1}\right), \tag{2.2}$$

where J is a coupling constant: $J > 0$ and $J < 0$ correspond to the ferromagnetic and antiferromagnetic regimes, respectively; L is the number of the lattice sites; $P_{k,k+1}$ and $E_{k,k+1}$ act nontrivially on the kth site and $(k+1)$th site.

The QTM is defined as

$$t_1^{(1)}(v) = \mathrm{Tr}_j \prod_{k=1}^{N/2} R_{a_{2k},j}(u+iv)\widetilde{R}_{a_{2k-1},j}(u-iv), \tag{2.3}$$

where $R_{ab}^{cd}(v) = \check{R}_{ba}^{cd}(v)$; $\widetilde{R}_{jk}(v) = {}^{t_k}R_{kj}(v)$ (t_k is the transposition in the kth space); N is the Trotter number and assumed to be even. By using the largest eigenvalue $T_1^{(1)}(0)$ of the QTM (2.3) at $v = 0$, the free energy per site is expressed as

$$f = -T\lim_{N\to\infty}\log T_1^{(1)}(0), \tag{2.4}$$

[1] In the case of the deformation parameter q of an underlining quantum affine algebra is root of unity ($|q| = 1, q \neq 1$), the T-system becomes a finite set of difference equations. Thus the corresponding TBA equation becomes a finite set of integral equations. See, [8] for TBA analysis of integrable field theories related to $osp(1|2s)$. We also note that different types of NLIE with finite numbers of unknown functions for different algebras of arbitrary rank were considered in [9,10] in rather different contexts.

where we set $u = -\frac{J}{TN}$ (T: temperature); the Boltzmann constant is set to 1. One can obtain the eigenvalue formulae of the QTM (2.3) by replacing the vacuum part of the dressed vacuum form (DVF) for the row-to-row transfer matrix with that of the QTM. This DVF is imbedded into a DVF for a fusion hierarchy of the QTM. It reads as follows:

$$T_m^{(a)}(v) = \sum_{\{d_{jk}\}} \prod_{j=1}^{m} \prod_{k=1}^{a} z\left(d_{jk}; v - \frac{i}{2}(m-a-2j+2k)\right), \tag{2.5}$$

where the summation is taken over $d_{jk} \in \{1, 2, \ldots, s, 0, \bar{s}, \ldots, \bar{2}, \bar{1}\}$ such that $d_{jk} \preceq d_{j+1,k}$ and $d_{jk} \prec d_{j,k+1}$. The functions $\{z(a; v)\}$ are defined as

$$\begin{aligned}
z(a; v) &= \psi_a(v) \frac{Q_{a-1}\big(v + \frac{i}{2}(a+1)\big) Q_a\big(v + \frac{i}{2}(a-2)\big)}{Q_{a-1}\big(v + \frac{i}{2}(a-1)\big) Q_a\big(v + \frac{i}{2}a\big)} \quad \text{for } a \in \{1, 2, \ldots, s\}, \\
z(0; v) &= \psi_0(v) \frac{Q_s\big(v + \frac{i}{2}(s-1)\big) Q_s\big(v + \frac{i}{2}(s+2)\big)}{Q_s\big(v + \frac{i}{2}(s+1)\big) Q_s\big(v + \frac{i}{2}s\big)}, \\
z(\bar{a}; v) &= \psi_{\bar{a}}(v) \frac{Q_{a-1}\big(v - \frac{i}{2}(a-2s)\big) Q_a\big(v - \frac{i}{2}(a-2s-3)\big)}{Q_{a-1}\big(v - \frac{i}{2}(a-2s-2)\big) Q_a\big(v - \frac{i}{2}(a-2s-1)\big)} \quad \text{for } a \in \{1, 2, \ldots, s\},
\end{aligned} \tag{2.6}$$

where $Q_a(v) = \prod_{k=1}^{M_a}(v - v_k^{(a)})$; $M_a \in \mathbb{Z}_{\geqslant 0}$; $Q_0(v) := 1$. The vacuum parts are given as follows:

$$\psi_a(v) = \begin{cases}
\zeta_1 \dfrac{\phi_+(v)\phi_-(v+i)\phi_+\big(v - \frac{2s-1}{2}i\big)}{\phi_+\big(v - \frac{2s+1}{2}i\big)} & \text{for } a = 1, \\
\zeta_a \phi_+(v)\phi_-(v) & \text{for } 2 \preceq a \preceq \bar{2}, \\
\zeta_{\bar{1}} \dfrac{\phi_-(v)\phi_+(v-i)\phi_-\big(v + \frac{2s-1}{2}i\big)}{\phi_-\big(v + \frac{2s+1}{2}i\big)} & \text{for } a = \bar{1},
\end{cases} \tag{2.7}$$

where $\phi_\pm(v) = (v \pm iu)^{N/2}$; ζ_a is a phase factor. $\{v_k^{(a)}\}$ is a solution of the Bethe ansatz equation (BAE)

$$\left\{ \frac{\phi_-\big(v_k^{(a)} + \frac{i}{2}\big)\phi_+\big(v_k^{(a)} + \frac{i}{2} - \frac{ig}{2}\big)}{\phi_-\big(v_k^{(a)} - \frac{i}{2}\big)\phi_+\big(v_k^{(a)} - \frac{i}{2} - \frac{ig}{2}\big)} \right\}^{\delta_{a1}} = -\varepsilon_a \prod_{d=1}^{s+1} \frac{Q_d\big(v_k^{(a)} + \frac{i}{2}B_{ad}\big)}{Q_d\big(v_k^{(a)} - \frac{i}{2}B_{ad}\big)}, \tag{2.8}$$

where $k \in \{1, 2, \ldots, M_a\}$; $a \in \{1, 2, \ldots, s\}$; $Q_{s+1}(v) := Q_s(v)$; $B_{ad} = 2\delta_{ad} - \delta_{a,d+1} - \delta_{a,d-1}$; ε_a is a phase factor. For $a \in \{1, 2, \ldots, s\}$ and $m \in \mathbb{Z}_{\geqslant 1}$, we define a normalization function

$$\begin{aligned}
\widetilde{\mathcal{N}}_m^{(a)}(v) = \frac{\phi_-\big(v + \frac{m+a}{2}i\big)\phi_+\big(v - \frac{m+a}{2}i\big)}{\phi_-\big(v - \frac{m-a}{2}i\big)\phi_+\big(v + \frac{m-a}{2}i\big)} \prod_{j=1}^{m}\prod_{k=1}^{a} \phi_-\left(v - \frac{m-a-2j+2k}{2}i\right) \\
\times \phi_+\left(v - \frac{m-a-2j+2k}{2}i\right),
\end{aligned} \tag{2.9}$$

and set $\widetilde{T}_m^{(a)}(v) = T_m^{(a)}(v)/\widetilde{\mathcal{N}}_m^{(a)}(v)$. For $s = 1$, $\widetilde{T}_1^{(1)}(v)$ coincides with Eq. (4.8) in [3]. The poles of $\widetilde{T}_m^{(a)}(v)$ from the functions $\{Q_b(v)\}$ (dress part) are spurious under the BAE (2.8). If one formally set the vacuum parts (2.7) and (2.9) to 1, then the remaining part (dress part) of the DVF $\widetilde{T}_m^{(a)}(v)$ is same as the row-to-row case. Thus $\widetilde{T}_m^{(a)}(v)$ satisfies the following functional relation, which has essentially the same form as the $osp(1|2s)$ T-system in [5]:

$$\widetilde{T}_m^{(a)}\left(v + \frac{i}{2}\right)\widetilde{T}_m^{(a)}\left(v - \frac{i}{2}\right) = \widetilde{T}_{m+1}^{(a)}(v)\widetilde{T}_{m-1}^{(a)}(v) + \widetilde{T}_m^{(a-1)}(v)\widetilde{T}_m^{(a+1)}(v),$$

$$\widetilde{T}_m^{(s)}\left(v+\frac{i}{2}\right)\widetilde{T}_m^{(s)}\left(v-\frac{i}{2}\right)=\widetilde{T}_{m+1}^{(s)}(v)\widetilde{T}_{m-1}^{(s)}(v)+\widetilde{T}_m^{(s-1)}(v)\widetilde{T}_m^{(s)}(v) \quad \text{for } a\in\{1,2,\ldots,s-1\},\ m\in\mathbb{Z}_{\geqslant 1}, \tag{2.10}$$

where

$$\widetilde{T}_0^{(a)}(v)=1 \quad \text{for } a\in\mathbb{Z}_{\geqslant 1},$$
$$\widetilde{T}_m^{(0)}(v)=\frac{\phi_-\left(v-\frac{m}{2}i\right)\phi_+\left(v+\frac{m}{2}i\right)}{\phi_-\left(v+\frac{m}{2}i\right)\phi_+\left(v-\frac{m}{2}i\right)} \quad \text{for } m\in\mathbb{Z}_{\geqslant 1}. \tag{2.11}$$

3. Nonlinear integral equations

In [22], Takahashi's NLIE for the *XXZ*-model [6] was derived from the T-system of the QTM. Here we derive our new NLIE from our T-system (2.10).

A numerical analysis for finite N, u, s indicates that a two-string solution (for every color) in the sector $N = M_1 = M_2 = \cdots = M_s$ of the BAE (2.8) provides the largest eigenvalue of the QTM (2.3) at $v = 0$ [1]. From now on, we consider only this two-string solution. In this case, the phase factors are $\varepsilon_a = \zeta_a = 1$ for any a. Moreover, we expect the following conjecture is valid for this two-string solution [1].

Conjecture 3.1. *For small u ($|u| \ll 1$), $a \in \{1, 2, \ldots, s\}$ and $m \in \mathbb{Z}_{\geqslant 1}$, every zero $\{\tilde{z}_m^{(a)}\}$ of $\widetilde{T}_m^{(a)}(v)$ is located near the lines* $\operatorname{Im} v = \pm\frac{m+a}{2}, \pm\frac{g+m-a}{2}$.

$\widetilde{T}_m^{(a)}(v)$ has poles only at $\pm\tilde{\beta}_{m,k}^{(a)}$ ($k = 1, 2$): $\tilde{\beta}_{m,1}^{(a)} = \frac{m+a}{2}i - iu$, $\tilde{\beta}_{m,2}^{(a)} = \frac{g+m-a}{2}i - iu$. These poles are of order $N/2$ at most. Moreover, $\lim_{|v|\to\infty}\widetilde{T}_m^{(a)}(v) = Q_m^{(a)}$ is a finite number

$$Q_m^{(a)}=\left(\frac{(m+g)!m!}{(m+a)!(m+g-a)!}\right)^m\prod_{k=1}^{m}\left\{\frac{(k+a)(k+g-a)}{k(k+g)}\right\}^k. \tag{3.1}$$

This is a solution of the Q-system [1]:

$$\left(Q_m^{(a)}\right)^2=Q_{m-1}^{(a)}Q_{m+1}^{(a)}+Q_m^{(a-1)}Q_m^{(a+1)} \quad \text{for } a\in\{1,2,\ldots,s-1\},$$
$$\left(Q_m^{(s)}\right)^2=Q_{m-1}^{(s)}Q_{m+1}^{(s)}+Q_m^{(s-1)}Q_m^{(s)}, \tag{3.2}$$

where $m \in \mathbb{Z}_{\geqslant 1}$; $Q_0^{(a)} = Q_m^{(0)} = 1$. We are considering the case without an external field. If an external field exists, (3.1) will be deformed.

We may assume

$$\widetilde{T}_m^{(a)}(v)=Q_m^{(a)}+\sum_{j=1}^{N/2}\sum_{k=1}^{2}\left\{\frac{b_{m,j,k}^{(a)}}{\left(v-\tilde{\beta}_{m,k}^{(a)}\right)^j}+\frac{\bar{b}_{m,j,k}^{(a)}}{\left(v+\tilde{\beta}_{m,k}^{(a)}\right)^j}\right\}, \tag{3.3}$$

where the coefficients $b_{m,j,k}^{(a)}, \bar{b}_{m,j,k}^{(a)} \in \mathbb{C}$ are given as follows:

$$b_{m,j,k}^{(a)}=\oint_{C_{m,k}^{(a)}}\frac{dv}{2\pi i}\widetilde{T}_m^{(a)}(v)\left(v-\tilde{\beta}_{m,k}^{(a)}\right)^{j-1}, \qquad \bar{b}_{m,j,k}^{(a)}=\oint_{\overline{C}_{m,k}^{(a)}}\frac{dv}{2\pi i}\widetilde{T}_m^{(a)}(v)\left(v+\tilde{\beta}_{m,k}^{(a)}\right)^{j-1}. \tag{3.4}$$

Here the contour $C_{m,k}^{(a)}$ must surround $\tilde{\beta}_{m,k}^{(a)}$ counterclockwise manner and must not contain $\tilde{\beta}_{m,p}^{(a)}$ ($p \neq k$), $-\tilde{\beta}_{m,1}^{(a)}, -\tilde{\beta}_{m,2}^{(a)}$; the contour $\overline{C}_{m,k}^{(a)}$ must surround $-\tilde{\beta}_{m,k}^{(a)}$ counterclockwise manner and must not contain $-\tilde{\beta}_{m,p}^{(a)}$

$(p \neq k)$, $\tilde{\beta}_{m,1}^{(a)}$, $\tilde{\beta}_{m,2}^{(a)}$. Using the T-system (2.10), we can modify (3.4) as

$$b_{m,j,k}^{(a)} = \oint_{C_{m,k}^{(a)}} \frac{\mathrm{d}v}{2\pi i} \left\{ \frac{\widetilde{T}_{m-1}^{(a)}(v - \frac{i}{2})\widetilde{T}_{m+1}^{(a)}(v - \frac{i}{2})}{\widetilde{T}_m^{(a)}(v - i)} + \frac{\widetilde{T}_m^{(a-1)}(v - \frac{i}{2})\widetilde{T}_m^{(a+1)}(v - \frac{i}{2})}{\widetilde{T}_m^{(a)}(v - i)} \right\} (v - \tilde{\beta}_{m,k}^{(a)})^{j-1},$$

$$\bar{b}_{m,j,k}^{(a)} = \oint_{\overline{C}_{m,k}^{(a)}} \frac{\mathrm{d}v}{2\pi i} \left\{ \frac{\widetilde{T}_{m-1}^{(a)}(v + \frac{i}{2})\widetilde{T}_{m+1}^{(a)}(v + \frac{i}{2})}{\widetilde{T}_m^{(a)}(v + i)} + \frac{\widetilde{T}_m^{(a-1)}(v + \frac{i}{2})\widetilde{T}_m^{(a+1)}(v + \frac{i}{2})}{\widetilde{T}_m^{(a)}(v + i)} \right\} (v + \tilde{\beta}_{m,k}^{(a)})^{j-1}, \quad (3.5)$$

where $\widetilde{T}_m^{(s+1)}(v) := \widetilde{T}_m^{(s)}(v)$. The first term and the second term in the first (respectively, second) bracket $\{\cdots\}$ in (3.5) have common poles at $\tilde{z}_m^{(a)} + i$ (respectively, $\tilde{z}_m^{(a)} - i$). However, these common poles are spurious since $\widetilde{T}_m^{(a)}(v)$ has no pole at these points. We also note that a subsidiary condition $Y_m^{(a)}(\tilde{z}_m^{(a)} \pm \frac{i}{2}) = -1$ for the excited state TBA equation (see [3, p. 2343] and (5.1)) follows from this observation. Substituting (3.5) into (3.3), we obtain

$$\begin{aligned}
\widetilde{T}_m^{(a)}(v) = Q_m^{(a)} &+ \sum_{k=1}^{2} \oint_{C_{m,k}^{(a)}} \frac{\mathrm{d}y}{2\pi i} \frac{1 - \left(\frac{y}{v - \tilde{\beta}_{m,k}^{(a)}}\right)^{N/2}}{v - y - \tilde{\beta}_{m,k}^{(a)}} \left\{ \frac{\widetilde{T}_{m-1}^{(a)}(y + \tilde{\beta}_{m,k}^{(a)} - \frac{i}{2})\widetilde{T}_{m+1}^{(a)}(y + \tilde{\beta}_{m,k}^{(a)} - \frac{i}{2})}{\widetilde{T}_m^{(a)}(y + \tilde{\beta}_{m,k}^{(a)} - i)} \right. \\
&\left. + \frac{\widetilde{T}_m^{(a-1)}(y + \tilde{\beta}_{m,k}^{(a)} - \frac{i}{2})\widetilde{T}_m^{(a+1)}(y + \tilde{\beta}_{m,k}^{(a)} - \frac{i}{2})}{\widetilde{T}_m^{(a)}(y + \tilde{\beta}_{m,k}^{(a)} - i)} \right\} \\
&+ \sum_{k=1}^{2} \oint_{\overline{C}_{m,k}^{(a)}} \frac{\mathrm{d}y}{2\pi i} \frac{1 - \left(\frac{y}{v + \tilde{\beta}_{m,k}^{(a)}}\right)^{N/2}}{v - y + \tilde{\beta}_{m,k}^{(a)}} \left\{ \frac{\widetilde{T}_{m-1}^{(a)}(y - \tilde{\beta}_{m,k}^{(a)} + \frac{i}{2})\widetilde{T}_{m+1}^{(a)}(y - \tilde{\beta}_{m,k}^{(a)} + \frac{i}{2})}{\widetilde{T}_m^{(a)}(y - \tilde{\beta}_{m,k}^{(a)} + i)} \right. \\
&\left. + \frac{\widetilde{T}_m^{(a-1)}(y - \tilde{\beta}_{m,k}^{(a)} + \frac{i}{2})\widetilde{T}_m^{(a+1)}(y - \tilde{\beta}_{m,k}^{(a)} + \frac{i}{2})}{\widetilde{T}_m^{(a)}(y - \tilde{\beta}_{m,k}^{(a)} + i)} \right\} \\
&\text{for } a \in \{1, 2, \ldots, s\},\ m \in \mathbb{Z}_{\geqslant 1},
\end{aligned} \quad (3.6)$$

where $\widetilde{T}_m^{(s+1)}(v) = \widetilde{T}_m^{(s)}(v)$. Here the contour $C_{m,k}^{(a)}$ (respectively, $\overline{C}_{m,k}^{(a)}$) must surround 0 counterclockwise manner and must not contain $\tilde{\beta}_{m,p}^{(a)} - \tilde{\beta}_{m,k}^{(a)}$ $(p \neq k)$, $-\tilde{\beta}_{m,1}^{(a)} - \tilde{\beta}_{m,k}^{(a)}$, $-\tilde{\beta}_{m,2}^{(a)} - \tilde{\beta}_{m,k}^{(a)}$ (respectively, $-\tilde{\beta}_{m,p}^{(a)} + \tilde{\beta}_{m,k}^{(a)}$ $(p \neq k)$, $\tilde{\beta}_{m,1}^{(a)} + \tilde{\beta}_{m,k}^{(a)}$, $\tilde{\beta}_{m,2}^{(a)} + \tilde{\beta}_{m,k}^{(a)}$). For $m \geqslant 2$, the first term and the second term in the first bracket $\{\cdots\}$ in (3.6) have a common singularity at 0 which contributes to the contour integral. On the other hand for $m = 1$, this singularity at 0 from the first term disappears since $\widetilde{T}_0^{(a)}(y) = 1$. Thus the contribution to the contour integral from the first term in the first bracket $\{\cdots\}$ in (3.6) vanishes as long as the contour $C_{1,k}^{(a)}$ does not contain the singularities at $\tilde{z}_1^{(a)} - \tilde{\beta}_{1,k}^{(a)} + i$ (cf. Conjecture 3.1). This situation is parallel with the second bracket $\{\cdots\}$ in (3.6). Therefore, for $m = 1$, (3.6) reduces to

$$\widetilde{T}_1^{(a)}(v) = Q_1^{(a)} + \sum_{k=1}^{2} \oint_{C_{1,k}^{(a)}} \frac{\mathrm{d}y}{2\pi i} \frac{1 - \left(\frac{y}{v - \tilde{\beta}_{1,k}^{(a)}}\right)^{N/2}}{v - y - \tilde{\beta}_{1,k}^{(a)}} \frac{\widetilde{T}_1^{(a-1)}(y + \tilde{\beta}_{1,k}^{(a)} - \frac{i}{2})\widetilde{T}_1^{(a+1)}(y + \tilde{\beta}_{1,k}^{(a)} - \frac{i}{2})}{\widetilde{T}_1^{(a)}(y + \tilde{\beta}_{1,k}^{(a)} - i)}$$

$$+\sum_{k=1}^{2}\oint_{\overline{C}^{(a)}_{1,k}}\frac{\mathrm{d}y}{2\pi i}\frac{1-\left(\frac{y}{v+\tilde{\beta}^{(a)}_{1,k}}\right)^{N/2}}{v-y+\tilde{\beta}^{(a)}_{1,k}}\frac{\widetilde{T}^{(a-1)}_{1}(y-\tilde{\beta}^{(a)}_{1,k}+\frac{i}{2})\widetilde{T}^{(a+1)}_{1}(y-\tilde{\beta}^{(a)}_{1,k}+\frac{i}{2})}{\widetilde{T}^{(a)}_{1}(y-\tilde{\beta}^{(a)}_{1,k}+i)}$$
$$\text{for } a\in\{1,2,\ldots,s\}, \tag{3.7}$$

where $\widetilde{T}^{(s+1)}_1(v)=\widetilde{T}^{(s)}_1(v)$. Here the contour $C^{(a)}_{1,k}$ (respectively, $\overline{C}^{(a)}_{1,k}$) must surround 0 counterclockwise manner and must not contain $\tilde{z}^{(a)}_1-\tilde{\beta}^{(a)}_{1,k}+i$, $\tilde{\beta}^{(a)}_{1,p}-\tilde{\beta}^{(a)}_{1,k}$ $(p\neq k)$, $-\tilde{\beta}^{(a)}_{1,1}-\tilde{\beta}^{(a)}_{1,k}$, $-\tilde{\beta}^{(a)}_{1,2}-\tilde{\beta}^{(a)}_{1,k}$ (respectively, $\tilde{z}^{(a)}_1+\tilde{\beta}^{(a)}_{1,k}-i$, $-\tilde{\beta}^{(a)}_{1,p}+\tilde{\beta}^{(a)}_{1,k}$ $(p\neq k)$, $\tilde{\beta}^{(a)}_{1,1}+\tilde{\beta}^{(a)}_{1,k}$, $\tilde{\beta}^{(a)}_{1,2}+\tilde{\beta}^{(a)}_{1,k}$). Now we shall take the Trotter limit $N\to\infty$ in (3.6).

$$\mathcal{T}^{(a)}_m(v)=\mathcal{Q}^{(a)}_m+\sum_{k=1}^{2}\oint_{C^{(a)}_{m,k}}\frac{\mathrm{d}y}{2\pi i}\frac{1}{v-y-\beta^{(a)}_{m,k}}\left\{\frac{\mathcal{T}^{(a)}_{m-1}(y+\beta^{(a)}_{m,k}-\frac{i}{2})\mathcal{T}^{(a)}_{m+1}(y+\beta^{(a)}_{m,k}-\frac{i}{2})}{\mathcal{T}^{(a)}_m(y+\beta^{(a)}_{m,k}-i)}\right.$$
$$\left.+\frac{\mathcal{T}^{(a-1)}_m(y+\beta^{(a)}_{m,k}-\frac{i}{2})\mathcal{T}^{(a+1)}_m(y+\beta^{(a)}_{m,k}-\frac{i}{2})}{\mathcal{T}^{(a)}_m(y+\beta^{(a)}_{m,k}-i)}\right\}$$
$$+\sum_{k=1}^{2}\oint_{\overline{C}^{(a)}_{m,k}}\frac{\mathrm{d}y}{2\pi i}\frac{1}{v-y+\beta^{(a)}_{m,k}}\left\{\frac{\mathcal{T}^{(a)}_{m-1}(y-\beta^{(a)}_{m,k}+\frac{i}{2})\mathcal{T}^{(a)}_{m+1}(y-\beta^{(a)}_{m,k}+\frac{i}{2})}{\mathcal{T}^{(a)}_m(y-\beta^{(a)}_{m,k}+i)}\right.$$
$$\left.+\frac{\mathcal{T}^{(a-1)}_m(y-\beta^{(a)}_{m,k}+\frac{i}{2})\mathcal{T}^{(a+1)}_m(y-\beta^{(a)}_{m,k}+\frac{i}{2})}{\mathcal{T}^{(a)}_m(y-\beta^{(a)}_{m,k}+i)}\right\}$$
$$\text{for } a\in\{1,2,\ldots,s\},\ m\in\mathbb{Z}_{\geqslant 1}, \tag{3.8}$$

where $\mathcal{T}^{(a)}_m(v):=\lim_{N\to\infty}\widetilde{T}^{(a)}_m(v)$; $\beta^{(a)}_{m,k}:=\lim_{N\to\infty}\tilde{\beta}^{(a)}_{m,k}$ ($\beta^{(a)}_{m,1}=\frac{m+a}{2}i$, $\beta^{(a)}_{m,2}=\frac{g+m-a}{2}i$); $\mathcal{T}^{(s+1)}_m(v)=\mathcal{T}^{(s)}_m(v)$; $\mathcal{T}^{(a)}_0(v)=1$; $\mathcal{T}^{(0)}_m(v)$ is a known function:

$$\mathcal{T}^{(0)}_m(v)=\lim_{N\to\infty}\widetilde{T}^{(0)}_m(v)=\exp\left(-\frac{mJ}{(v^2+\frac{m^2}{4})T}\right). \tag{3.9}$$

Here the contour $C^{(a)}_{m,k}$ (respectively, $\overline{C}^{(a)}_{m,k}$) must surround 0 counterclockwise manner with the condition $|y|<|v-\beta^{(a)}_{m,k}|$ (respectively, $|y|<|v+\beta^{(a)}_{m,k}|$) and must not contain $\beta^{(a)}_{m,p}-\beta^{(a)}_{m,k}$ $(p\neq k)$, $-\beta^{(a)}_{m,1}-\beta^{(a)}_{m,k}$, $-\beta^{(a)}_{m,2}-\beta^{(a)}_{m,k}$ (respectively, $-\beta^{(a)}_{m,p}+\beta^{(a)}_{m,k}$ $(p\neq k)$, $\beta^{(a)}_{m,1}+\beta^{(a)}_{m,k}$, $\beta^{(a)}_{m,2}+\beta^{(a)}_{m,k}$). In particular, for $m=1$ we have

$$\mathcal{T}^{(a)}_1(v)=\mathcal{Q}^{(a)}_1+\sum_{k=1}^{2}\oint_{C^{(a)}_{1,k}}\frac{\mathrm{d}y}{2\pi i}\frac{\mathcal{T}^{(a-1)}_1(y+\beta^{(a)}_{1,k}-\frac{i}{2})\mathcal{T}^{(a+1)}_1(y+\beta^{(a)}_{1,k}-\frac{i}{2})}{(v-y-\beta^{(a)}_{1,k})\mathcal{T}^{(a)}_1(y+\beta^{(a)}_{1,k}-i)}$$
$$+\sum_{k=1}^{2}\oint_{\overline{C}^{(a)}_{1,k}}\frac{\mathrm{d}y}{2\pi i}\frac{\mathcal{T}^{(a-1)}_1(y-\beta^{(a)}_{1,k}+\frac{i}{2})\mathcal{T}^{(a+1)}_1(y-\beta^{(a)}_{1,k}+\frac{i}{2})}{(v-y+\beta^{(a)}_{1,k})\mathcal{T}^{(a)}_1(y-\beta^{(a)}_{1,k}+i)}\quad\text{for } a\in\{1,2,\ldots,s\}, \tag{3.10}$$

where $\mathcal{T}^{(s+1)}_1(v)=\mathcal{T}^{(s)}_1(v)$. Here the contour $C^{(a)}_{1,k}$ (respectively, $\overline{C}^{(a)}_{1,k}$) must surround 0 counterclockwise manner with the condition $|y|<|v-\beta^{(a)}_{1,k}|$ (respectively, $|y|<|v+\beta^{(a)}_{1,k}|$) and must not contain $z^{(a)}_1-\beta^{(a)}_{1,k}+i$, $\beta^{(a)}_{1,p}-\beta^{(a)}_{1,k}$ $(p\neq k)$, $-\beta^{(a)}_{1,1}-\beta^{(a)}_{1,k}$, $-\beta^{(a)}_{1,2}-\beta^{(a)}_{1,k}$ (respectively, $z^{(a)}_1+\beta^{(a)}_{1,k}-i$, $-\beta^{(a)}_{1,p}+\beta^{(a)}_{1,k}$ $(p\neq k)$, $\beta^{(a)}_{1,1}+\beta^{(a)}_{1,k}$, $\beta^{(a)}_{1,2}+\beta^{(a)}_{1,k}$):

$z_1^{(a)} = \lim_{N\to\infty} \tilde{z}_1^{(a)}$. In (3.10), (3.1) is written as a binomial coefficient: $Q_1^{(a)} = \binom{2s+1}{a}$. We note the fact that this set of NLIE (3.10) contains only a *finite* number of unknown functions $\{\mathcal{T}_1^{(a)}(v)\}_{1\leqslant a\leqslant s}$. Moreover, we can express $\{\mathcal{T}_m^{(a)}(v)\}_{1\leqslant a\leqslant s; m\in\mathbb{Z}_{\geqslant 1}}$ in terms of the solution of (3.10) $\{\mathcal{T}_1^{(a)}(v)\}_{1\leqslant a\leqslant s}$ by using a Jacobi–Trudi determinant formula [5]. In this sense, we need not consider (3.8) for $m\in\mathbb{Z}_{\geqslant 2}$ in practical calculations. However, consideration on (3.8) for $m\in\mathbb{Z}_{\geqslant 2}$ manifests the relation between our new NLIE and the traditional TBA equations (see Section 5). The free energy per site is given as

$$f = -J - T\log \mathcal{T}_1^{(1)}(0). \tag{3.11}$$

Solving (3.10) iteratively, we can obtain the free energy through (3.11). For $s=1$ case, we find (3.10) converges numerically at least for $|T/J| > 0.37$.

4. High temperature expansion: *osp*(1|2) case

It was pointed out [7] that Takahashi's NLIE for the *XXX*-model [6] is very useful to calculate the high temperature expansion of the free energy. In this section, we shall calculate the high temperature expansion of the free energy (3.11) for $s=1$ by our new NLIE (3.10). We assume the following expansion for large T:

$$\mathcal{T}_1^{(1)}(v) = \exp\left(\sum_{n=0}^{\infty} a_n(v)\left(\frac{J}{T}\right)^n\right). \tag{4.1}$$

Due to the shift of the argument y of $\mathcal{T}_1^{(1)}(y)$ in (3.10), we have to take into account the residues from the coefficients $\{a_n(v)\}$, which contrasts with the *XXX*-model case [7]. Thus the derivation is not so easy as the *XXX*-model case. But we still expect that it is easier than to use the traditional TBA equation which contains an infinite number of unknown functions (as for the high temperature expansion for the *XXX*-model from the traditional TBA equation, see, for example, [29, p. 124]). To calculate the coefficients $\{a_n(v)\}$ efficiently, we need further assumption for $n\in\mathbb{Z}_{\geqslant 1}$:

$$a_n(v) = \sum_{j=0}^{n-1}\left\{\frac{b_{n,j}v^{2j}}{(v^2+1)^n} + \frac{c_{n,j}v^{2j}}{(v^2+\frac{9}{4})^n}\right\}, \tag{4.2}$$

where $b_{n,j}, c_{n,j}\in\mathbb{C}$ are independent of v. Substituting (4.1) into (3.10), we obtain the coefficients $\{a_n(v)\}$ up to order 12. For example, we have

$$\begin{aligned}
&a_0(v) = \log 3, \qquad a_1(v) = -\frac{2}{3(v^2+1)} - \frac{1}{3(v^2+\frac{9}{4})}, \qquad a_2(v) = \frac{4(77+32v^2)}{405(v^2+1)^2} - \frac{4(27+32v^2)}{405(v^2+\frac{9}{4})^2},\\
&a_3(v) = -\frac{8(727+1040v^2+448v^4)}{10\,935(v^2+1)^3} + \frac{28(837+720v^2+128v^4)}{10\,935(v^2+\frac{9}{4})^3},\\
&a_4(v) = \frac{2(-9097+781\,314v^2+783\,969v^4+175\,808v^6)}{7\,381\,125(v^2+1)^4}\\
&\qquad - \frac{144\,376\,263+138\,298\,104v^2+376\,92144v^4+2\,812\,928v^6}{59\,049\,000\,(v^2+\frac{9}{4})^4}.
\end{aligned} \tag{4.3}$$

Using $\{a_n(0)\}$, we obtain

$$\frac{f}{T} = -\log 3 - \frac{5J}{27T} - \frac{172J^2}{243T^2} + \frac{20\,296J^3}{59\,049T^3} + \frac{52\,010J^4}{531\,441T^4} - \frac{466\,964J^5}{1\,594\,323T^5} + \frac{252\,697\,291J^6}{1\,937\,102\,445T^6}$$

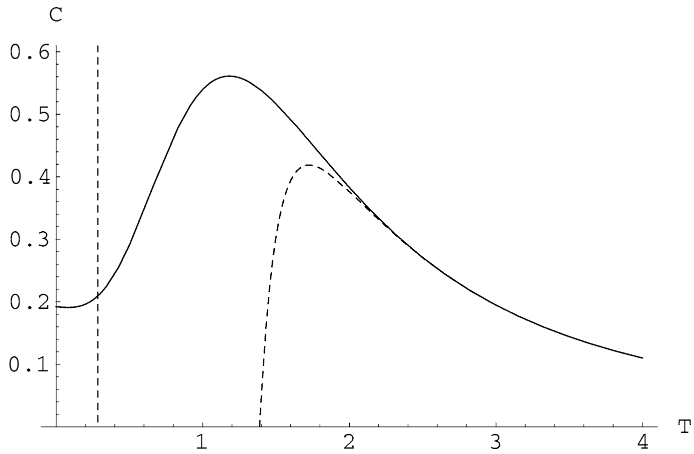

Fig. 1. Temperature dependence of the high temperature expansion of the specific heat for $s = 1$ and $J = -1$: the broken line denotes the plain series (4.5); the smooth line denotes its Padé approximation.

$$+\frac{2\,867\,981\,638J^7}{17\,433\,922\,005T^7}-\frac{319\,036\,008\,559J^8}{1\,255\,242\,384\,360T^8}+\frac{68\,608\,132\,529\,023J^9}{1\,921\,650\,566\,216\,724T^9}+\frac{69\,872\,025\,931\,694\,239J^{10}}{288\,247\,584\,932\,508\,600T^{10}}-\frac{157\,451\,559\,799\,196\,839J^{11}}{648\,557\,066\,098\,144\,350T^{11}}-\frac{48\,290\,843\,858\,722\,808\,551J^{12}}{693\,437\,215\,072\,135\,939\,020T^{12}}+O\big((J/T)^{13}\big). \tag{4.4}$$

We also calculate the specific heat $C = -T\frac{\partial^2 f}{\partial T^2}$:

$$C=\frac{344J^2}{243T^2}-\frac{40\,592J^3}{19\,683T^3}-\frac{208\,040J^4}{177\,147T^4}+\frac{9\,339\,280J^5}{159\,4323T^5}-\frac{505\,394\,582J^6}{129\,140\,163T^6}-\frac{40\,151\,742\,932J^7}{5\,811\,307\,335T^7}+\frac{2\,233\,252\,059\,913J^8}{156\,905\,298\,045T^8}-\frac{137\,216\,265\,058\,046J^9}{53\,379\,182\,394\,909T^9}-\frac{69\,872\,025\,931\,694\,239J^{10}}{3\,202\,750\,943\,694\,540T^{10}}+\frac{1\,731\,967\,157\,791\,165\,229J^{11}}{64\,855\,706\,609\,814\,435T^{11}}+\frac{48\,290\,843\,858\,722\,808\,551J^{12}}{5\,253\,312\,235\,394\,969\,235T^{12}}+O\big((J/T)^{13}\big). \tag{4.5}$$

We have plotted the high temperature expansion of the specific heat (4.5) in Fig. 1. For comparison, we also plotted a Padé approximation for (4.5). For large T, (4.5) agrees with our TBA analysis (see Fig. 1 in [4]). This indicates the validity of our new NLIE (3.10).

5. Discussion

In this Letter, we have derived a system of NLIE with a *finite* number of unknown functions, which describes thermodynamics of the $osp(1|2s)$ integrable spin chain. This type of NLIE for *arbitrary* rank is derived for the first time. We shall point out a relation between our new NLIE and usual TBA equations. The functions $\{\widetilde{T}^{(a)}_m(v)\}$ are related to the dependent variables $\{Y^{(a)}_m(v)\}$ of the Y-system (or TBA equations) [1] as

$$Y^{(a)}_m(v)=\frac{\widetilde{T}^{(a)}_{m+1}(v)\widetilde{T}^{(a)}_{m-1}(v)}{\widetilde{T}^{(a-1)}_m(v)\widetilde{T}^{(a+1)}_m(v)},\qquad Y^{(s)}_m(v)=\frac{\widetilde{T}^{(s)}_{m+1}(v)\widetilde{T}^{(s)}_{m-1}(v)}{\widetilde{T}^{(s-1)}_m(v)\widetilde{T}^{(s)}_m(v)}\quad \text{for } a\in\{1,2,\dots,s-1\},\ m\in\mathbb{Z}_{\geqslant 1}. \tag{5.1}$$

This relation is also valid in the Trotter limit.

In closing this Letter, we shall enumerate future problems:

- There are T-systems for other algebras [5,23–28]. Using these T-systems, we can derive NLIE similar to the ones in this Letter, which will be reported elsewhere [30].
- In this Letter, we have considered the case without an external field. When an external field exists, a fugacity expansion of the free energy can be calculated recursively through (3.10) (as for the fugacity expansion of the *XXX*-model from the traditional TBA equation, see, for example, [29, p. 127]).
- One will be able to extend (3.10) to trigonometric or elliptic case. In this case, one must take into account the periodicity in the summation in (3.3).

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