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Behavior of Integral Curves of the Quasilinear Second Order Differential Equations

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Abstract

This paper deals with certain classes of Cauchy's solutions of quasilinear second order differential equations in general form, Van der Pol's differential equation, which is used in the theory of electric circuits, and Lagerstorm's differential equations, which is used in asymptotic treatment of viscous flow past a solid at low Reynolds number. Behaviour of integral curves in the neighbourhoods of an arbitrary or integral curve is considered. Obtained results establish sufficient conditions for the existence and asymptotic behaviour of the observed equations. The obtained results contain the answer to the question on approximation of solutions whose existence is established. The errors of the approximation are defined by functions that can be sufficiently small. The qualitative analysis theory and topological retraction methods were used.

Keywords: quasilinear differential equation; behavior of solutions; approximation of solutions

1. Introduction

Many processes in science and technique are described with the quasilinear differential equations whose solution is not always possible to find, [5]. Methods of qualitative analysis of differential equations allow to determine the existence and asymptotic behaviour of solutions of these equations, the stability of the solution, and if it is possible to approximately determine the requested solution, [3],[4]. For example, since introduced in the 1950s by P.A. Lagerstrom, the model of Lagerstrom's equation where studied by many authors with the help of variation techniques ([2],[9]). The authors Popovic and Szmolyan used in [9] geometrical approach. Here we shall use the qualitative analysis theory of differential equations and topological retraction method ([1],[6], [7], [8], [10], [11], [12]).

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Notation

$f \in C(I)$ f continuous function on interval $I=(a,\infty)$
$f \in C^1(I)$ f and f' continuous functions on interval I
$f \in C^2(I)$ f, f' and f'' continuous functions on interval I
$\psi_0=\psi(t_0)$, $t_0 \in I$
$y_0=y(t_0)$, $t_0 \in I$
$y'_0=y'(t_0)$, $t_0 \in I$
$S^p(I)$, $p \in \{1,2\}$ class of solutions defined on I, which depends on p parameters
$S^0(I)$ exists at least one solutions defined on I
L_i, $(i=1,2,3)$ Lipschitz's constants

The behaviour of the solutions of quasilinear second order differential equation in general form

$$y'' + P(y,t)\,y' + Q(y,t)\,y = F(y,t) \tag{1}$$

where P, Q, F continuous function on $\boldsymbol{R}\times I$, $I=(a,\infty)$, $a \in \boldsymbol{R}$, in the neighbourhood of an arbitrary curve are considered.
As special cases we consider Lagerstrom's equation and Van der Pol's equation, [5]. Let

$$\Gamma = \{(y,t) \in R \times I \,|\, y = \varphi(t),\, t \in I\},$$

where $\varphi \in C^2(I)$, is an arbitrary curve in $\boldsymbol{R}\times I$.

Let $r_1, r_2 \in C^1(I)$, $r_1>0$, $r_2>0$ on I and let the solutions $y(t)$ of equations (1) satisfy on I, either the conditions

$$|y_0 - \psi_0| \le r_2(t_0),\ \ |y'_0 - \psi'_0| \le r_1(t_0) \tag{2}$$

or

$$\frac{(y_0-\psi_0)^2}{r_2^2(t_0)} + \frac{(y'_0-\psi'_0)^2}{r_1^2(t_0)} \le 1. \tag{3}$$

Using substitution $y' = x$, where $x=x(t)$ is a new unknown function, equation (1) is transformed into a quasilinear system of equations

$$\begin{aligned} x' &= -P(y,t)x - Q(y,t)y + F(y,t), \\ y' &= x, \\ t' &= 1. \end{aligned} \tag{4}$$

Let $(\varphi(t), \psi(t), t), t \in I$, where $\varphi(t) = \psi'(t)$, be an arbitrary integral curve of system (4), and let $\Omega=\boldsymbol{R}^2\times I$. We shall consider the behaviour of the integral curve $(x(t), y(t), t)$ of system (4) with respect to the sets:

$$\sigma = \{(x,y,t) \in \Omega : |x-\varphi(t)| < r_1^2(t),\ |y-\psi(t)| < r_2^2(t)\},$$

$$\omega = \left\{ (x, y, t) \in \Omega \,\middle|\, \frac{(x - \varphi(t))^2}{r_1^2(t)} + \frac{(y - \psi(t))^2}{r_2^2(t)} \le 1 \right\}.$$

The boundary surfaces of σ and ω are, respectively,

$$H_i^1(x, y, t) \equiv (-1)^i (x - \varphi(t)) - r_1(t) = 0, \quad i = 1,2,$$
$$H_i^2(x, y, t) \equiv (-1)^i (y - \psi(t)) - r_2(t) = 0, \quad i = 1,2,$$
$$H(x, y, t) \equiv \frac{(x - \varphi(t))^2}{r_1^2(t)} + \frac{(y - \psi(t))^2}{r_2^2(t)} - 1 = 0.$$

Let us denote the tangent vector field to an integral curve *(φ(t), ψ(t), t)* of system (4) by T.

$$T(x, y, t) = (-P(y,t)x - Q(y,t)y + F(y,t),\, x,\, 1)$$

The vectors $\nabla H_i^1, \nabla H_i^2$ and ∇H are the outer normals on surfaces H_i^1, H_i^2 and H, respectively:

$$\nabla H_i^1(t) \equiv \left((-1)^i,\, 0,\, (-1)^{i-1}\varphi' - r_1'\right), \quad i = 1,2,$$
$$\nabla H_i^2(t) \equiv \left(0,\, (-1)^i,\, (-1)^{i-1}\psi' - r_2'\right), \quad i = 1,2,$$
$$\frac{1}{2}\nabla H(x, y, t) \equiv \left(\frac{x - \varphi}{r_1^2}, \frac{y - \psi}{r_2^2}, -\frac{(x - \varphi)^2 r_1'}{r_1^3} - \frac{(y - \psi)^2 r_2'}{r_2^3} - \frac{(x - \varphi)\varphi'}{r_1^2} - \frac{(y - \psi)\psi'}{r_2^2} \right).$$

By means of scalar products:

$$\pi_i^1(x, y, t) = \left(\nabla H_i^1(t),\, T\right),$$
$$\pi_i^2(x, y, t) = \left(\nabla H_i^2(t),\, T\right),$$
$$\pi(x, y, t) = \left(\frac{1}{2}\nabla H(t),\, T\right),$$

on surfaces H_i^1, H_i^2 and H, respectively, we establish the existence and behaviour of integral curves of (4) with respect σ and ω.

The results of this paper are based on the following Lemmas (see [6]-[8]).

Lemma 1. If, for the system (4), the scalar product $\pi < 0$ *on* H ($\pi_i^k < 0$ *on* $H_1^1 \cup H_2^1 \cup H_1^2 \cup H_2^2,\ i = 1,2,\ k = 1,2$)*, then the system (4) has a class of solutions* $S^2(I)$ *belonging to the set ω for all tϵI, i.e.* $S^2(I) \subset \omega$ ($S^2(I) \subset \sigma$).

Lemma 2. If, for the system (4), the scalar product $\pi > 0$ *on* H ($\pi_i^k > 0$ *on* $H_1^1 \cup H_2^1 \cup H_1^2 \cup H_2^2,\ i = 1,2,\ k = 1,2$)*, then the system (4) has at least one solutionon I whose graph belongs to the set ω for all tϵI, i.e.* $S^0(I) \subset \omega$ ($S^0(I) \subset \sigma$).

Lemma 3. If, for the system (4), the scalar product $\pi_i^1 < 0$ *on* $H_1^1 \cup H_2^1$*, and* $\pi_i^2 > 0$ *on* $H_1^2 \cup H_2^2$ *(or reversely), then the system (4) has a class of solutions* $S^1(I)$ *belonging to the set σ for all* $t\epsilon I$*, i.e.* $S^1(I) \subset \sigma$.

According to Lemma 1. the set H ($H_1^1 \cup H_2^1 \cup H_1^2 \cup H_2^2$) is a set of points of strict entrance of integral curves of the system (4) with respect to the sets ω (σ) and Ω. Hence, all solutions of system (4) which satisfy condition $|y_0 - \psi_0| \le r_2(t_0)$, $|y'_0 - \psi'_0| \le r_1(t_0)$ also satisfy conditions $|y(t)\text{-}\psi(t)| < r_2, |y'(t)\text{-}\psi'(t)| < r_1$, for every $t > t_0$, i.e. $S^2(I) \subset \omega$ $(S^2(I) \subset \sigma)$.

According to Lemma 2. the set H $(H_1^1 \cup H_2^1 \cup H_1^2 \cup H_2^2)$ is a set of points of strict exit of integral curves of the system (4) with respect to the sets ω (σ) and Ω. Hence, according to T.Wazewski's retraction method [12], system (4) has at least one solution belonging to set ω (σ) for $t > t_0$, i.e. $S^0(I) \subset \omega$ $(S^0(I) \subset \sigma)$.

According to Lemma 3. the set $H_1^1 \cup H_2^1$ is a set of points of strict entrance, and $H_1^2 \cup H_2^2$ is a set of strict exit (or reversely) of integral curves of (4) with respect to the sets σ and Ω. Hence, according to retraction method, system (4) has a one-parameter class of solutions belonging to set σ for $t > t_0$, i.e. $S^1(I) \subset \sigma$.

2. The main results

Theorem 1. *Let P(y,t),Q(y,t),F(y,t)ϵ(**R**×I) satisfy the conditions:*

$$\begin{aligned} &|P(y_1,t) - P(y_2,t)| < L_1|y_1 - y_2|, \\ &|Q(y_1,t) - Q(y_2,t)| < L_2|y_1 - y_2|, \\ &|F(y_1,t) - F(y_2,t)| < L_3|y_1 - y_2|. \end{aligned} \tag{5}$$

where $(y_1,t),(y_2,t)\,\epsilon(\mathbf{R}\times I)$ and $r_1,r_2 \epsilon C^1(I)$, $r_1>0$, $r_2>0$. Then:

a) If the conditions

$$\left(L_1|\varphi| + L_2|\psi| + L_3 + |Q|\right)r_2 < r_1' + P r_1 \tag{6}$$

$$r_1 < r_2' \tag{7}$$

are satisfied on $H_1^1 \cup H_2^1 \cup H_1^2 \cup H_2^2$, then all solutions y(t) of the problem (1),(2) satisfy the conditions

$$|y(t) - \psi(t)| < r_2, \quad |y'(t) - \psi'(t)| < r_1, \quad t > t_0 \tag{8}$$

b) If the conditions

$$\left(L_1|\varphi| + L_2|\psi| + L_3 + |Q|\right)r_2 < -r_1' - P r_1 \tag{9}$$

$$r_1 < -r_2' \tag{10}$$

are satisfied on $H_1^1 \cup H_2^1 \cup H_1^2 \cup H_2^2$, then at least one solutions of the problem (1),(2) satisfies the conditions (8).

c) If the conditions (6) *and* (10), *or* (7) *and* (9) *are satisfied on $H_1^1 \cup H_2^1 \cup H_1^2 \cup H_2^2$, then the problem (1),(2) has one-parameter class of solutions that satisfy the conditions* (8).

Theorem 2. *Let P(y,t),Q(y,t),F(y,t)ϵ(**R**×I) and let the conditions (5) be satisfied. Let $r_1,r_2 \epsilon C^1(I)$, $r_1>0$, $r_2>0$ and*

$$\left(\left(L_1|\varphi| + L_2|\psi| + L_3\right)r_2^2 + \left|r_1^2 - Qr_2^2\right|\right)^2 < 4r_1r_2\left(Pr_1 + r_1'\right)r_2' . \tag{11}$$

Then:
(i) If

$$r_2' > 0, \tag{12}$$

then all solutions y(t) of the problem (1),(3) *satisfy the condition*

$$\frac{\left(y(t) - \psi(t)\right)^2}{r_2^2(t)} + \frac{\left(y'(t) - \psi'(t)\right)^2}{r_1^2(t)} < 1, \quad t > t_0 . \tag{13}$$

(ii) If

$$r_2' < 0, \tag{14}$$

then at least one solution of the problem (1),(3) *satisfies the condition* (13).

Proof of Theorem 1. We shall consider the equations (1) through the equivalent system (4). Let us consider the integral curves of the system (4) with respect to the set σ.
For the scalar product $\pi_i^1(x, y, t) = (\nabla H_i^1(t), T)$ on H_i^1 and $\pi_i^2(x, y, t) = (\nabla H_i^2(t), T)$ on H_i^2 we have:

$$\pi_i^1 = (-1)^i\left[-Px - Qy + F\right] + (-1)^i\varphi' - r_1' = -Pr_1 + (-1)^i\left[-Q(y-\psi) + F - P\varphi - Q\psi - \varphi'\right] - r_1' ,$$

$$\pi_i^2 = (-1)^i x + (-1)^{i-1}\psi' - r_2' = (-1)^i(x - \varphi) - r_2' .$$

a) According to the conditions (5) and (9), the following estimates for π_i^1 on H_i^1 and π_i^2 on H_i^2 are valid, respectively:

$$\pi_i^1 \le -Pr_1 + |Q|r_2 + |F - P\varphi - Q\psi - \varphi'| - r_1' \le -Pr_1 + |Q|r_2 + \left(L_3 + L_1|\varphi| + L_2|\psi|\right)r_2 - r_1' < 0,$$

$$\pi_i^2 \le r_1 - r_2' < 0.$$

Accordingly, set $H_1^1 \cup H_2^1 \cup H_1^2 \cup H_2^2$ is a set of points of strict entrance of integral curves of the system (4) with respect to the sets σ and Ω. Hence, all solutions of the system (4) which satisfy the conditions

$$|x_0 - \varphi_0| \le r_1(t_0), \quad |y_0 - \psi_0| \le r_2(t_0),$$

also satisfy conditions

$$|x(t) - \varphi(t)| \le r_1(t), \quad |y(t) - \psi(t)| \le r_2(t), \quad t > t_0 .$$

Since, in view of $y' = x$, $x_0 - \varphi_0 = y_0' - \psi_0'$, all solutions of the problem (1),(2) satisfy the conditions (8).

b) According to the conditions (5), (9) and (10), the following estimates for π_i^1 on H_i^1 and π_i^2 on H_i^2 are valid, respectively:

$$\pi_i^2 \geq -Pr_1 + |Q|(-r_2) - |F - P\varphi - Q\psi - \varphi'| - r_1' \geq -Pr_1 - |Q|r_2 - (L_3 + L_1|\varphi| + L_2|\psi|)r_2 - r_1' > 0,$$

$$\pi_i^2 \geq -r_1 - r_2' > 0.$$

Accordingly, set $H_1^1 \cup H_2^1 \cup H_1^2 \cup H_2^2$ is a set of points of strict exit of integral curves of the system (4) with respect to the sets σ and Ω. Hence, according to T. Wazewski's retraction method [10], the system (4) has at least one solutions belonging to the set σ for all $t \in I$. Consequently, the problem (1), (2) has at least one solution which satisfies the conditions (8).

c) In this case $H_1^1 \cup H_2^1$ is a set of point of strict exit, and $H_1^2 \cup H_2^2$ is a set of points of strict entrance (or reversely) of integral curves of the system (4) with respect to the sets σ and Ω. According to the retraction method, the system (4) has one-parameter class of solutions belonging to the set σ for all $t \in I$. Hence, the problem (1), (2) also has one-parameter class of solutions which satisfy the conditions (8). ■

Proof of Theorem 2. Let us consider the integral curves of the system (4) with respect to the set *ω*. For the scalar product $\pi(x, y, t) = \left(\frac{1}{2}\nabla H, T\right)$ on the surface *H*, we have:

$$\pi = [-Px - Qy + F]\frac{x-\varphi}{r_1^2} + x\frac{y-\psi}{r_2^2} - \frac{(x-\varphi)^2 r_1'}{r_1^3} - \frac{(y-\psi)^2 r_2'}{r_2^3} - \frac{(x-\varphi)\varphi'}{r_1^2} - \frac{(y-\psi)\psi'}{r_2^2}.$$

If we introduce the notation

$$X = \frac{x-\varphi}{r_1}, \quad Y = \frac{y-\psi}{r_2},$$

we have:

$$\pi = \left[-P - \frac{r_1'}{r_1}\right]X^2 + \left[-Q\frac{r_2}{r_1} + \frac{r_1}{r_2}\right]XY - \frac{r_2'}{r_2}Y^2 + [-P\varphi - Q\psi + F - \varphi']\frac{X}{r_1}$$

In view of (5), the following estimates for π on H are valid:

$$\pi \leq \left[-P - \frac{r_1'}{r_1}\right]X^2 + \left[(L_1|\varphi| + L_2|\psi| + L_3)\frac{r_2}{r_1} + \left|\frac{r_1}{r_2} - Q\frac{r_2}{r_1}\right|\right]|X||Y| + \left[-\frac{r_2'}{r_2}\right]Y^2,$$

$$\pi \geq \left[-P - \frac{r_1'}{r_1}\right]X^2 - \left[(L_1|\varphi| + L_2|\psi| + L_3)\frac{r_2}{r_1} + \left|\frac{r_1}{r_2} - Q\frac{r_2}{r_1}\right|\right]|X||Y| + \left[-\frac{r_2'}{r_2}\right]Y^2,$$

The right-hand sides of the above inequalities are the quadratic symmetric forms

$$a_{11}X^2 \pm 2a_{12}|X||Y| + a_{22}Y^2,$$

where corresponding coefficients a_{11}, a_{12}, a_{22} are introduced.

(i) Conditions (13) and (14) imply

$$a_{22} < 0, a_{11}a_{22} - a_{12}^2 > 0,$$

which, according to Sylvester's criterion, means that $\pi(x,y,t) < 0$ on H. Consequently, set H is a set of points of strict entrance of integral curves of system (4) with respect to the sets ω and Ω. Hence, all solutions of the system (4) which satisfy the conditions

$$\frac{(x_0-\varphi_0)^2}{r_1^2(t_0)} + \frac{(y_0-\psi_0)^2}{r_2^2(t_0)} \le 1, \tag{15}$$

satisfy the inequality

$$\frac{(x(t)-\varphi(t))^2}{r_1^2(t)} + \frac{(y(t)-\psi(t))^2}{r_2^2(t)} < 1, \quad t > t_0. \tag{16}$$

Since $x_0 - \varphi_0 = y_0' - \psi_0'$, then all solutions of the problem (1),(3) satisfy condition (13).

(ii) Conditions (11), (14) imply

$$a_{22} > 0, a_{11}a_{22} - a_{12}^2 > 0,$$

which, according to Sylvester's criterion, means that $\pi(x,y,t) > 0$ on H. Consequently, H is a set of points of strict exit of integral curves of the system (4) with respect to the sets ω and Ω. Hence, according to the retraction method, the problem (4), (15) has at least one solution which satisfies condition (16). Consequently, the problem (1), (3) has at least one solution which satisfies condition (13). ■

3. The applications

3.1. Van der Pol equation

For the Van der Pol's equation, [2] :

$$y'' - \mu(1-\Phi(y))y' + y = 0, \ \mu > 0 \tag{17}$$

and condition

$$y^2(t_0) + y'^2(t_0) \le \ln^2 t_0, \tag{18}$$

we can prove the following:

If function $\Phi(y)>1$, then all solutions of the problem (17), (18) *satisfy the condition*

$$y^2(t)+y'^2(t)\le \ln^2 t,$$

for $t \in (t_0,\infty)$, $t_0>1$.

This result follows from *Theorem 2.* with $r_1(t)=r_2(t)=\ln t$.

3.2. Lagerstrom's equation

In general form Lagerstrom's equation is given by the non-autonomous second order differential equation:

$$y''+\left(\frac{n-1}{t}+y\right)y'=0,\ \ n\in N,\ \ n\ge 1 \tag{19}$$

The cases $n=2$ and $n=3$ represent the physically relevant settings of flow in two and three dimensions, respectively. For the Lagerstrom's equation we can prove the following:

Let Γ be an arbitrary curve and r_1, $r_2 \in C^1(I,\boldsymbol{R}^+)$.

a) If

$$\left|\left(\frac{n-1}{t}+\psi\right)\varphi+\varphi'\right|<\left(\frac{n-1}{t}+\psi\right)r_1+r_1'-\left(|\varphi|+r_1\right)r_2, \tag{20}$$

$$r_1(t)<r_2'(t) \tag{21}$$

on $H_1^1\cup H_2^1\cup H_1^2\cup H_2^2$, then all solutions of the problem (19),(2) satisfy the conditions

$$|y(t)-\psi(t)|<r_2(t),\quad |y'(t)-\psi'(t)|<r_2(t),\quad t>t_0. \tag{22}$$

b) If

$$\left|\left(\frac{n-1}{t}+\psi\right)\varphi+\varphi'\right|>\left(\frac{n-1}{t}+\psi\right)r_1+r_1'+\left(|\varphi|+r_1\right)r_2, \tag{23}$$

$$r_1(t)<-r_2'(t) \tag{24}$$

on $H_1^1\cup H_2^1\cup H_1^2\cup H_2^2$, then at least one solution of the problem (17),(2) satisfy the conditions (22).

c) If conditions (20) i (24) or (21) i (23) are satisfied, then the problem (17), (2) has a one-parameter class of solutions that satisfy the conditions (22).

4. Conclusion

This paper deals with existence and behaviour of integral curves of second order quasilinear differential equations in general form. As special cases Van der Pol's differential equation and Lagerstorm's differential equations are considered. The obtained results establish sufficient conditions for the existence and asymptotic behaviour of the observed equations in neighbourhoods of an arbitrary (or integral) curve in definition domain.

Also in this paper, is presented a topological retraction method as a very useful method of qualitative analysis of differential equations.

The results also contain an answer to the question on approximation of solutions $y(t)$ whose existence is established. For example, the errors of approximation for solutions $y(t)$ and derivative $y'(t)$ in Theorem 1. are defined by the function $r_1(t)$, and $r_2(t)$, which tend to zero as as $t\to\infty$ and $r_i'(t) < 0$, $(i=1,2)$, $t \epsilon I$. For example, we can use $r_1(t) = \alpha e^{-st}$ and $r_1(t) = \beta e^{-pt}$, $s > 0$, $p > 0$ and with parameters α and β that can be arbitrary small. In that case curve Γ represents a good approximation of solutions $y(t)$ in σ.

The obtained results also give the possibility to discuss the stability (instability) of solutions of the system (4). For example, under the conditions of Theorem 1. a), every solutions of (4) with initial value in ω is r-stable (stable with the functions of stability $r_i(t)$, $(i=1,2)$), if $r_i(t)$ tends to zero as $t\to\infty$ and $r_i'(t) < 0$, $(i=1,2)$, $t \epsilon I$. However, if we consider the case b), then it is established solutions in ω is r-unstable in case where $r_i'(t) > 0$, $t \epsilon I$.

The next step would be a numerical simulation of solutions in observed domains and comparison with obtained results.

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