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The first-order spectral flow of the odd signature operator on a manifold with boundary

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Abstract

In this paper we study spectral flow for paths of signature operators associated to analytic paths of flat connections on an odd-dimensional manifold with boundary. We provide a topological method for computing the "first order" spectral flow using cup products. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

1.1. This paper has as its goal the computation of spectral flow for paths of signature operators corresponding to analytic paths of flat connections on an odd-dimensional manifold with boundary.

Our approach to this problem is to apply the results of analytic perturbation theory. This theory asserts that an analytic path of self-adjoint operators (appropriately defined) has analytically varying eigenvectors and eigenvalues [14]. Thus one can study how the spectrum changes along an analytic path by studying the derivatives of the eigenvalues at a point. In particular, one can study the spectral flow near a particular parameter value by calculating the lowest order non-vanishing derivative of each eigenvalue passing through zero at that parameter value.

Flat connections on a principal G-bundle are parameterized by the variety of representations of the fundamental group, $R(X)=\mathrm{Hom}(\pi_1 X, G)$. In fact this parameterization can

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be taken to be analytic [11], so that analytic paths of representations give rise to analytic paths of flat connections. An analytic path of flat connections can be coupled with a geometric differential operator to define a path of differential operators. On a closed manifold the corresponding path of operators is analytic, and so analytic perturbation theory applies. For a manifold with boundary, the results of [17] show that imposing Atiyah–Patodi–Singer (APS) boundary conditions [2] yields an analytic path of self-adjoint operators in the sense of [14] and so analytic perturbation theory applies in this case as well.

In this way the odd signature operator (which is half the tangential operator of the signature operator) can be coupled to an analytic path of flat connections, providing a topologically interesting path of self-adjoint operators, which we will denote by D_t. For each of these flat connections, the Hodge theorem identifies the kernel of the corresponding twisted signature operator with the cohomology of X (with local coefficients corresponding to the connection). Thus the Hodge theorem allows one to detect when an eigenvalue crosses zero ($\lambda(t_0) = 0$) by examining when the cohomology jumps up in dimension. In this paper we show that first-order information about the spectrum, i.e., the derivative $\lambda'(t_0)$, can be computed from the cup product structure of these cohomology spaces (in the case in which $\lambda(t_0) = 0$).

Thus, the main results (Theorems 5.1 and 5.2) can be thought of as first order generalizations of the Hodge theorem for analytic paths of representations. In the case in which X is closed, it has been shown (see [16] and [9]) that the lowest order nonvanishing derivative of $\lambda(t)$ can always be expressed in terms of higher Massey products in the cohomology of X. Though there are strong indications that such a theorem should be true for manifolds with boundary, we have not as yet been able to prove it (see [18] for more on this subject).

1.2. We now outline the contents of this paper. In Section 2 we define a sequence of matrices associated to an analytic path of self-adjoint operators whose signatures determine the spectral flow. We then describe how an analytic path of $U(k)$ representations of the fundamental group of an odd-dimensional manifold with boundary X gives rise to a path of twisted odd signature operators, and by imposing APS boundary conditions an analytic path of self-adjoint operators. We also introduce related cohomology groups and cup products.

The technique of stretching the collar is introduced in Section 3. Fourier expansions along the collar of the boundary of X are used to relate the kernel of the odd signature operator with suitable APS boundary conditions to cohomology.

In Section 4 we introduce two Hermitian forms, B and $\widetilde{B}$, on cohomology. We prove that these forms have the same signatures, and relate their kernels to a "derived" cohomology. One of these forms, $\widetilde{B}$, can be defined entirely from the homotopy type of the manifold, while the other, B, depends on the diffeomorphism type and, indeed, on the Riemannian metric.

Finally, Section 5 contains the main theorems which state that the signatures of these forms tell us how many of those eigenvalues passing through zero have first derivatives which are positive, negative, or zero.

We emphasize two important assumptions in Theorem 5.2.

(a) The dimension of the kernel of the tangential operator $\widehat{D}_t$ is independent of t.

(b) The Lagrangian $\mathcal{L}_t$ (part of the global boundary conditions necessary in order to make D_t self-adjoint) will be chosen so that $\mathcal{L}_0$ is transverse to the limiting values of extended L^2 solutions to $D_0\phi = 0$.

Though Theorem 5.2 is quite useful as stated, eliminating the first of these assumptions would be a significant improvement and an interesting research project. The second is mostly for convenience; see the paragraph immediately preceding Theorem 5.2.

1.3. The results in this paper extend those in an earlier paper, *Computing Spectral Flow Via Cup Products* by the same authors [15]. In that paper, we proved the main theorems (5.1 and 5.2) for the case in which X is a 3-manifold with torus boundary, and we made a technical assumption on the arc of flat connections, that it be *fine* when restricted to the boundary. Moreover, in that paper we only defined the form B, not the form $\widetilde{B}$; hence it was not clear that the signature of B was an invariant of homotopy type. In the current paper, all of these technical problems are solved.

The results in this paper are also related to those in a recent article by Farber and Levine [8]. In the case of a closed odd-dimensional manifold, those authors define a sequence of forms on $H^l(X; V)$. Their forms are defined using a linking pairing on the cohomology of X with coefficients in a module over the ring of formal power series defined using an analytic path of flat connections. The first of their forms coincides with our reduced first order form $\widetilde{B}$. Farber and Levine then show that the sum of the signatures of their forms give the local contribution to the spectral flow along the analytic arc of flat connections. The main information in our paper which is not in Farber and Levine is that we consider the case of a manifold with boundary, for which boundary conditions and stretching arguments are required. In addition we write the first order form explicitly in terms of cup products.

If some of the eigenvalues passing through 0 at $t = 0$ have vanishing first derivatives then in order to compute their contribution to spectral flow one needs to calculate their first nonvanishing higher-order derivatives. There is a sequence of forms whose domains are subquotients of $H^{ev}(X; V)$ and whose signatures give the contributions of these higher-order derivatives to spectral flow. These forms were first defined by Farber and Levine [8] for closed manifolds as indicated in the previous paragraph. In subsequent work [16,18] we give an alternative definition of these forms (and of their domains) in terms of higher Massey products. Once again, for manifolds with boundary, we have not yet been able to prove a definitive theorem of this type.

1.4. Computing the spectral flow for the odd signature operator coupled to a path of flat connections has many important applications. Typically one is interested in computing the spectral flow of the odd signature operator coupled to a path of (possibly non-flat) connections starting and ending at flat connections on a *closed* manifold. Sample applications include computing the grading of Floer's Instanton homology for a homology 3-sphere [10], computing the Atiyah–Patodi–Singer ρ_α invariant [2], computing Casson's invariant and $SU(n)$-generalizations [1,25,3] and computing invariants of 3-manifolds coming from the perturbative expansion of Witten's Chern–Simons path integral [26].

The main obstacle to carrying out such computations is the fact that non-flat connections do not reflect the topology of the manifold; for example, spectral invariants of non-flat connections depend on the Riemannian metric. The crucial observation which motivates the present work is the following. A pair of flat connections A_0, A_1 on a closed manifold M may not lie on a path component of flat connections. However, if M is decomposed along a separating hypersurface, say $M = X \bigcup_\Sigma Y$, it may very well happen that the restrictions of A_0 and A_1 to X (respectively Y) lie on a path of *flat* connections on X (respectively Y).

More generally one may inductively decompose M by cutting it along a sequence of separating hypersurfaces until one reaches a stage where the restrictions of A_0 and A_1 lie on the same path component of the space of flat connections on each piece in the decomposition. Note that the space of flat connections modulo gauge transformations on any manifold X is homeomorphic to the space of conjugacy classes of representations of $\pi_1(X)$. In particular it is a purely topological question to decide if two flat connections lie on the same path component of the space of flat connections.

Thus to compute spectral flow on the closed manifold M one can use the following steps.

(1) Decompose M into pieces so that the restrictions of the connections A_0 and A_1 to each piece lie on the same path component of the space of flat connections.
(2) Compute the spectral flow of the odd signature operator along the paths of flat connections on each piece in the decomposition.
(3) Assemble the results using a "splitting formula" for spectral flow which relates the spectral flow on the pieces to the spectral flow on the closed manifold.

The second step of this program is the subject of the current article. As mentioned above, the first step is a purely topological problem. The third step involves using a splitting theorem. There are many articles in the literature which address this issue, starting with Taubes's article on the Casson invariant [25], and including [27,5,6,22,7].

Applications of this approach include the computations of spectral flow to compute Witten's 3-manifold invariants in [15]. In that article the relevant cup products which control the spectral flow were computed explicitly using group cohomology techniques. Another application in the literature is the computation of $SU(3)$ Casson invariants [4]. A particularly useful application of our technique is the computation of spectral flow along a path of flat connections with abelian holonomy on the complement of a knot in S^3. The relationship to cup products in this situation gives a formula for the spectral flow in terms of the Seifert matrix for the knot complement. This relationship was noticed using an indirect method in the article [19], and was exploited in the beautiful article of Herald [13] in which he identifies the "twisted" Casson invariant for knots with the Levine–Tristam signatures.

A long term potential application of the methods of the current paper is to gain a complete understanding of the topological meaning of the Atiyah–Patodi–Singer ρ_α invariant by combining the cohomological approach to computing spectral flow with cut-and-paste methods. The papers cited above contain partial results in this direction, mostly in dimension 3. We hope to eventually obtain general and comprehensive results.

2. First order deformations of the odd signature operator

2.1. We recall the notion of an analytic path of closed operators (see [14,17]). A path $D_t : H \to H$ of bounded operators on a Hilbert space H is called analytic if it has a power series expansion which converges with respect to the norm topology. If D_t is a path of closed, unbounded operators, then D_t is called analytic if there exists an analytic path $Q_t : K \to H$ of bounded operators from some Hilbert space K to H satisfying

(1) the image of Q_t is the domain of D_t,
(2) the composite $D_t \circ Q_t$ is an analytic path of bounded operators.

We can now define the signatures which arise as successive approximations to the spectral flow for any analytic path of self-adjoint operators. Suppose that $D_t : L^2 \to L^2$, $t \in (-\varepsilon, \varepsilon)$ is an analytic path of closed, self-adjoint Fredholm operators. Suppose that the kernel of D_t jumps up in dimension when $t = 0$. Analytic perturbation theory shows that one can find paths $\phi_i(t)$, for $i = 1, \dots, m$, of vectors and paths $\lambda_i(t)$ of real numbers so that

(1) $D_t\phi_i(t) = \lambda_i(t)\phi_i(t)$ for all t,
(2) $\{\phi_i(t)\}$ is an orthonormal set for all t,
(3) $\{\phi_i(0)\}$ span the kernel of D_0,
(4) $\phi_i(t) = \sum_{j=0}^{\infty} \phi_{ij} t^j$ for some $\phi_{ij} \in L^2$, the sum converges in L^2,
(5) $\lambda_i(t) = \sum_{j=1}^{\infty} \lambda_{ij} t^j$ for some λ_{ij}.

Our convention is to define the spectral flow of D_t for $t \in [a, b]$ by

$$SF\bigl(D_t;\ t \in [a, b]\bigr) \\ = \#\bigl\{i \mid \lambda_i(a) < 0 \text{ and } \lambda_i(b) > 0\bigr\} - \#\bigl\{i \mid \lambda_i(a) \geqslant 0 \text{ and } \lambda_i(b) \leqslant 0\bigr\}.$$

We define the spectral flow of D_t through $t = 0$ to be $SF(D_t; t \in [-\varepsilon, \varepsilon])$ where $\varepsilon > 0$ is chosen small enough that the dimension of the kernel of D_t is constant on the two subintervals $[-\varepsilon, 0)$ and $(0, \varepsilon]$.

The spectral flow through $t = 0$ can be computed once one knows the sign of the first non-vanishing derivative of $\lambda_i(t)$ at $t = 0$ for those eigenvalues satisfying $\lambda_i(0) = 0$. We make this assertion precise as follows:

Let $S_r = \{i \in \mathbb{Z} \mid \frac{\mathrm{d}^k\lambda_i}{\mathrm{d}t^k}|_{t=0} = 0 \text{ for all } k < r\}$. Let M_r be the diagonal matrix with entries:

$$\left\{ \frac{\mathrm{d}^r\lambda_i}{\mathrm{d}t^r}\bigg|_{t=0} \right\}_{i \in S_r}.$$

The following easy result is the basic principle on which our approach rests:

Theorem 2.1. *The signatures of the matrices M_{2r+1} determine the spectral flow of the family D_t through $t = 0$.*

Proof. Notice that the eigenvalues are varying analytically. Hence either $\lambda_i(t) = 0$ for all t, or else $\lambda_i(t) = \lambda_{i,r}t^r + \mathrm{o}(t^r)$, so that if r is odd $\lambda_i(t)$ contributes $\operatorname{sign}(\lambda_{i,r})$ to the spectral flow, and 0 if r is even. □

A similar analysis applies when studying paths of the form D_t, $t \in [0, \varepsilon)$. In this case, we define the spectral flow of D_t through $t = 0$ as the difference

$$\#\{i \mid \lambda_i(0) = 0 \text{ and } \lambda_i(t) > 0 \text{ for small } t > 0\}$$
$$- \#\{i \mid \lambda_i(0) = 0 \text{ and } \lambda_i(t) < 0 \text{ for small } t > 0\}$$

with the $\lambda_i(t)$ as above. Theorem 2.1 easily generalizes to this case, however notice that the signatures of *all* the M_r are needed to determine the spectral flow, not just those with r odd.

In this paper our goal is to compute the eigenvalues of the matrix M_1 for the case in which D_t is the path of signature operators associated to a path of flat connections on an odd-dimensional manifold with boundary. I.e., we wish to compute the first derivatives of those eigenvalues passing through 0 at time $t = 0$.

2.2. Let X^{2l-1} be an oriented, compact odd-dimensional manifold with possibly non-empty boundary. Assume that X has been given a Riemannian metric which is isometric to a product $[0, 1] \times \partial X$ on a collar of the boundary.

Throughout this paper we will let J denote either the parameter interval $(-\varepsilon, \varepsilon)$ or, occasionally, $[0, \varepsilon)$. Define an *analytic path of representations* to be a path $\alpha : J \to \mathrm{Hom}(\pi_1 X, U(k))$ so that for each $x \in \pi_1 X$, the path $t \mapsto \alpha_t(x) \in U(k) \subset \mathbf{C}^{k^2}$ is real-analytic. This is the same as saying that α_t is an analytic path in the real-algebraic variety $\mathrm{Hom}(\pi_1 X, U(k))$.

We next define the relevant cohomology groups and cup products. Let V be some Hermitian vector space and let $r : U(k) \to U(V)$ be a unitary representation. Then the composite $\pi_1 X \xrightarrow{\alpha_t} U(k) \xrightarrow{r} U(V)$ defines a system of local coefficients on X whose cohomology we denote by $H^*(X; V_t)$. By restricting the coefficients to the boundary we obtain a local coefficient system on Y giving cohomology $H^*(Y; V_t)$. Similarly we have a relative cohomology group $H^*(X, Y; V_t)$. Define $\overline{H}^*(X; V_t)$ to be the image of the relative cohomology in the absolute:

$$\overline{H}^*(X; V_t) = \mathrm{Im}\big(H^*(X, Y; V_t) \to H^*(X; V_t)\big).$$

Cup products on cohomology with local coefficients are constructed from equivariant bilinear forms on the coefficients. In what follows, we will use two different bilinear forms to define two types of cup products: a *dot product* arising from the positive definite Hermitian inner product on V, and a second type of cup product induced by the action of the Lie algebra $u(k)$ on V.

To be precise, let $K : V \times V \to \mathbf{C}$ be the Hermitian inner product on V. We will refer to any cup product defined using the inner product K on the coefficients as a *dot product*. For example, K induces dot products

$$H^p(X; V_t) \times H^q(X, Y; V_t) \to H^{p+q}(X, Y; \mathbf{C})$$

and

$$H^p(Y; V_t) \times H^q(Y; V_t) \to H^{p+q}(Y; \mathbf{C}).$$

We will denote these pairings by $(\phi, \tau) \mapsto \phi \cdot \tau$ and refer to them as dot products. The range of these products is the ordinary (untwisted) **C**-cohomology.

When p and q are complementary dimensions we will also call the composites

$$H^p(X; V_t) \times H^{2l-1-p}(X, Y; V_t) \to H^{2l-1}(X, Y; \mathbf{C}) = \mathbf{C}$$

and

$$H^p(Y; V_t) \times H^{2l-2l-p}(Y; V_t) \to H^{2l-2}(Y; \mathbf{C}) = \mathbf{C}$$

dot products, where the isomorphisms are given by capping with the fundamental class. Poincaré duality implies that the first of these induces a non-degenerate pairing

$$\overline{H}^p(X; V_t) \times \overline{H}^{2l-1-p}(X; V_t) \to H^{2l-1}(X, Y; \mathbf{C}) = \mathbf{C};$$

we use the "dot product" notation for this pairing also.

The other cup product we will need is obtained from the bilinear form coming from the action of $u(k)$, the Lie algebra of $U(k)$, on V. Composing the representation $\alpha_t : \pi_1 X \to U(k)$ with the adjoint representation $ad : U(k) \to GL(u(k))$, one obtains another system of local coefficients over X, with fiber the Lie algebra $u(k)$. It is traditional to denote the corresponding cohomology groups by $H^*(X; ad\,\alpha_t)$ and $H^*(Y; ad\,\alpha_t)$.

The differential of r, $dr : u(k) \to \mathrm{End}(V)$, gives V the structure of a module over $u(k)$, i.e., a bilinear form

$$r_* : u(k) \times V \to V.$$

This gives $H^*(X; V_t)$ the structure of a module over $H^*(X; ad\,\alpha_t)$; To distinguish this product from the dot product defined above we will denote it by

$$H^*(X; ad\,\alpha_t) \times H^*(X; V_t) \ni (\phi, \tau) \mapsto r(\phi)(\tau) \in H^*(X; V_t).$$

Since the action of r_* is skew-Hermitian the two products are related by the formula:

$$r(\phi)(x) \cdot y = (-1)^{|\phi||x|+1} x \cdot r(\phi)(y).$$

2.3. Let A be a flat connection on a principal bundle P over X with holonomy α_0. (Assume, as before, that $\alpha : J \to \mathrm{Hom}(\pi_1 X, U(k))$ is an analytic path of representations.) We fix forever an identification of the restriction of P to the collar with $\pi^*(\widehat{P})$, where $\widehat{P}$ denotes the restriction of P to the boundary of X and $\pi : [0,1] \times \partial X \to \{1\} \times \partial X$ denotes the projection. We assume that A is in *cylindrical form* on the collar, that is, A is the product of a flat connection $\widehat{A}$ on the boundary with the trivial connection in the normal direction. Any flat connection with holonomy α_0 is gauge equivalent to such a connection.

Corollary 4.3 of [11] shows that (perhaps after shortening the interval J) one can find a sequence $a_i \in \Omega^1_X(ad\,P)$, $i = 1, 2, \ldots$, of smooth 1-forms with values in the Lie algebra bundle $ad\,P = P \times_{ad} u(k)$ in cylindrical form so that

$$A_t = A + \sum_{i=1}^{\infty} a_i t^i$$

is a path of flat connections with holonomies given by the path α_t. The sum converges in the C^k norm for all k.

For each t, the connection A_t defines a covariant derivative $d_{A_t}:\Omega_X^p(F)\to\Omega_X^{p+1}(F)$ for any bundle F associated to P. Since A_t is flat, $d_{A_t}^2=0$ and so $(\Omega_X^*(F),d_{A_t})$ forms a complex for each t. In particular, for the bundle $ad\,P$ the cohomology of the complex $(\Omega_X^*(ad\,P),d_{A_t})$ is isomorphic to $H^*(X;ad\,\alpha_t)$ by the DeRham theorem. (In fact these are isomorphic as graded Lie algebras.) Similarly, the representation $r:U(n)\to U(V)$ defines a bundle $E=P\times_r V$ and the cohomology of $(\Omega_X^*(E),d_{A_t})$ is isomorphic to $H^*(X;V_t)$. Denote by $\widehat{E}$ the restriction of E to the boundary Y. Restricting the flat connection A_t to the boundary Y one obtains a flat connection $\widehat{A}_t$ on $\widehat{E}$ and hence complexes $(\Omega_Y^*(ad\,\widehat{P}),d_{\hat{A}_t})$ and $(\Omega_Y^*(\widehat{E}),d_{\hat{A}_t})$ with the appropriate cohomology. Since Y is closed, these last 2 complexes are elliptic and Hodge theory applies, so that we can identify the cohomology with the kernel of $d_{\hat{A}_t}+d_{\hat{A}_t}^*$. This is not true for X until appropriate boundary conditions are imposed.

The formula for the curvature of A_t is

$$F(A_t)=F(A)+d_A\Big(\sum a_it^i\Big)+\frac{1}{2}\Big[\sum a_it^i,\sum a_it^i\Big].$$

Since A_t is flat, $F(A_t)=0$, and expanding the right side each coefficient of t^i is zero. In particular:

(1) $d_Aa_1=0$, so that a_1 defines a 1-dimensional cohomology class in $H^1(X;ad\,\alpha_0)$,

(2) $[a_1,a_1]=-2d_Aa_2$, and so $[a_1,a_1]$ is zero in cohomology.

Notice that $[-,-]$ is the cup product on $H^*(X;ad\,\alpha_0)$ induced by the Lie bracket on the coefficients; this is a basic example of the second type of cup product which we defined in Section 2.2 (corresponding to $V=u(k)$).

Remark. Comparing higher coefficients of t gives the sequence of expressions:

$$-2d_Aa_n=\sum_{k=1}^{n-1}[a_k,a_{n-k}]. \tag{2.1}$$

This says that the "homogeneous Massey powers" of a_1, $\{a_1,\dots,a_1\}$ all vanish. For a definition of Massey products in a differential graded-commutative Lie algebra see [24]; see also [16] for applications of these Massey products to the closed manifold case.

We will let a_1 denote both the form and its cohomology class in $H^1(X;ad\,\alpha_0)$. The image of a_1 in the group cohomology $H^1(\pi_1(X),ad\,\alpha_0)$ is just the Zariski tangent vector to the path α_t of representations at $t=0$.

2.4. One can couple an operator to the path A_t, giving an analytic path of operators. The operator which we will work with in what follows is the Atiyah–Patodi–Singer odd signature operator on X defined in [2]. Thus

$$D_t:\bigoplus_p\Omega_X^{2p}(E)\to\bigoplus_p\Omega_X^{2p}(E)$$

is defined by

$$D_t\omega = i^l(-1)^{p-1}(*d_{A_t} - d_{A_t}*)\omega$$

for $\omega \in \Omega_X^{2p}(E)$, where $*: \Omega_X^p(E) \to \Omega_X^{2l-1-p}(E)$ denotes the Hodge $*$ operator. Then D_t is formally self-adjoint for each t, and is half the tangential operator to the signature operator on a $2l$ manifold.

Notice that D_t has a power series expansion. In fact, $D_t = D_0 + \sum_{i=1}^{\infty} C_i t^i$ where

$$C_i\omega = i^l(-1)^{p-1}\big(*r(a_i) - r(a_i)*\big)\omega. \tag{2.2}$$

Since the series $\sum_i a_i t^i$ converges in C^k for any k, so does $\sum_i C_i t^i$. In the next section, we will apply the results of [17] to construct an analytic path of self-adjoint operators $D_t(\mathcal{L} + P_+)$ on X using APS boundary conditions.

2.5. The tangential operator of the odd signature operator is just the DeRham operator $\pm(d - d^*)$. The next technical lemma is needed to set conventions and signs. We omit the routine proof.

Lemma 2.2. *Let u denote the collar coordinate in the collar $I \times Y$. Identify $\bigwedge^p T^*X \otimes E|_Y$ with $\bigwedge^p T^*Y \otimes \widehat{E} \oplus \bigwedge^{p-1} T^*Y \otimes \widehat{E}$ by sending $\omega = \omega_1 + \omega_2\,\mathrm{d}u$ to the pair (ω_1, ω_2). This gives an isomorphism of bundles $(\bigoplus_p \bigwedge^{2p} T^*X \otimes E)|_Y$ with $\bigoplus_q \bigwedge^q T^*Y \otimes \widehat{E}$. Then on the cylinder $I \times Y$, D_t takes the form*

$$D_t = \sigma\left(\widehat{D}_t + \frac{\partial}{\partial u}\right)$$

where $\sigma : \bigwedge^* T^*Y \otimes \widehat{E} \to \bigwedge^* T^*Y \otimes \widehat{E}$ is the bundle isomorphism defined for $\phi_k \in \bigwedge^k T^*Y \otimes \widehat{E}$ by

$$\sigma(\phi_k) = \begin{cases} i^l(-1)^{p+1}\hat{*}\phi_k & \text{if } k = 2p; \\ i^l(-1)^p\hat{*}\phi_k & \text{if } k = 2p-1. \end{cases}$$

$\hat{*}$ is the Hodge star operator on $\bigoplus_p \Omega_Y^p(\widehat{E})$, and $\widehat{D}_t : \bigoplus_p \Omega_Y^p(\widehat{E}) \to \bigoplus_p \Omega_Y^p(\widehat{E})$ is the (self-adjoint) twisted DeRham operator given by the formula

$$\widehat{D}_t\phi_k = (-1)^{k+1}\big(\widehat{d}_{\hat{A}_t} - \widehat{d}^{\,*}_{\hat{A}_t}\big)\phi_k.$$

Note that $\sigma^2 = -1$, and $\sigma + \sigma^* = 0$. Using the L^2 inner product one obtains a symplectic structure on $\Omega_Y^*(\widehat{E})$ by the formula

$$\{\phi, \tau\} = \langle\sigma\phi, \tau\rangle_{L^2}.$$

This symplectic inner product is independent of the Riemannian metric (because the $*$ appearing in the definition of σ cancels with the $*$ in the definition of the L^2 inner product).

The kernel of $\widehat{D}_t$ is the set of $\widehat{d}_{\hat{A}_t}$-harmonic forms, which we denote by $\mathcal{H}_t$. By the DeRham and Hodge theorems $\mathcal{H}_t^p \cong H^p(Y; V_t)$. Moreover, σ preserves harmonic forms, and the induced symplectic structure on $H^*(Y; V_t)$ coincides (up to $\pm i$) with the dot product

$$\cdot : H^p(Y; V_t) \times H^{2l-2-p}(Y; V_t) \to \mathbf{C}$$

defined above.

We restate the important assumption made in the introduction.

Assumption. The kernel of $\widehat{D}_t$ is independent of t.

Since the connection $\widehat{A}_t$ is flat, this assumption is equivalent to assuming that the dimension of $H^*(Y; V_t) = \mathcal{H}_t$ is independent of t. With this assumption, the kernels of the operators $\widehat{D}_t$ form a finite dimensional symplectic vector subbundle $\mathcal{H} \subset \Omega^*_Y(\widehat{E}) \times J$ over the interval J whose fiber over t is $\mathcal{H}_t$. As a topological object, one can think of this as a symplectic bundle with fiber $H^*(Y; V_t)$. Notice however that $\mathcal{H}$ has more structure coming from the Riemannian metric on X: the involution σ induces a complex structure on $\mathcal{H}$, and $\mathcal{H}$ has a Hermitian metric induced by restricting the L^2 inner product on $\Omega^*_Y(\widehat{E})$.

We turn the path D_t into a path of self-adjoint operators by imposing Atiyah–Patodi–Singer boundary conditions. To do this, first fix an analytic path of Lagrangians $\mathcal{L}_t \subset \mathcal{H}_t$. What this means is that $\mathcal{L}_t$ is spanned by paths of vectors $e_i(t)$, $i = 1, \ldots, \dim(\mathcal{H}_t)/2$, which have an expansion $e_i(t) = \sum_j a_{i,j}(t)\psi_j(t)$ with $\psi_j(t)$, $j = 1, \ldots, \dim(\mathcal{H}_t)$, analytic paths of $\widehat{D}_t$-harmonic forms (which exist by analytic perturbation theory since Y is closed) and $a_{i,j}(t)$ are analytic functions.

Then use $\mathcal{L}_t$ to define the path of self-adjoint operators

$$D_t(\mathcal{L} + P_+) : L^2\left(\bigoplus_p \wedge^{2p} T^*X \otimes E; \mathcal{L}_t \oplus P_+(t)\right) \to L^2\left(\bigoplus_p \wedge^{2p} T^*X \otimes E\right),$$

where $L^2(\bigoplus_p \wedge^{2p} T^*X \otimes E; \mathcal{L}_t \oplus P_+(t))$ denotes the L^2-closure of the space of those sections of $\bigoplus_p \wedge^{2p} T^*X \otimes E$ whose restrictions to Y lie in the sum of $\mathcal{L}_t$ and the positive eigenspace $P_+(t)$ of $\widehat{D}_t$, and $D_t(\mathcal{L} + P_+)$ is the restriction of D_t.

The main theorem of [17] states that $D_t(\mathcal{L} + P_+)$ forms an analytic path of self-adjoint operators and hence one can find an L^2-basis of analytically varying eigenvectors and corresponding analytically varying eigenvalues for $D_t(\mathcal{L} + P_+)$.

Proposition 5.2 of [11] shows that the set of $t \in J$ where the kernel of $D_t(\mathcal{L} + P_+)$ jumps up is discrete, and so we may assume by shrinking the interval J if necessary that the kernel jumps up only at $t = 0$. In particular, the path $D_t(\mathcal{L} + P_+)$ defines the sequence of matrices M_r as in 2.1 corresponding to the jump at $t = 0$.

Thus the triple $(\alpha : J \to \mathrm{Hom}(\pi_1 X, U(k)), r : U(k) \to U(V), \mathcal{L}_t)$ *determines an integer, namely the spectral flow of the family* $D_t(\mathcal{L} + P_+)$ *through* $t = 0$*, and this spectral flow is determined by the signatures, dimension and kernels of the matrices* M_r.

3. Stretching the collar

3.1. The crudest approximation to the spectral flow is the dimension of the kernel of $D_t(\mathcal{L} + P_+)$ as t varies. We will now show how to identify this kernel with a certain cohomology group. In the next section we will show how the "first order part" of the spectral flow (i.e., the signature of the matrix M_1) can be understood in terms of cup products in the cohomology of X.

First, we will need a "stretched" version of X. Let

$$X(R) = X \cup_{[0,1]\times Y} \big([0, R] \times Y\big).$$

Thus $X(R)$ corresponds to X with a long tube added to the boundary. Similarly let $X(\infty)$ denote X with an infinitely long tube $[0, \infty) \times Y$ added to the boundary. Since the connection A and the forms a_i are cylindrical, there is an obvious way to extend the operator D_t to $X(R)$ and $X(\infty)$. We denote this operator by D_t^R. If W is any closed subspace of $\bigoplus_p \Omega_Y^p(\widehat{E})$, denote by $D_t^R(W)$ the restriction of D_t^R to those sections with boundary values in W. In particular, if $\mathcal{L}_t$ is a path of Lagrangian subspaces of $\mathcal{H}_t$ we have the important path of self-adjoint operators $D_t^R(\mathcal{L} + P_+)$ on $X(R)$.

We can motivate the introduction of the stretched manifold in the following way. Our goal is to relate cohomological invariants constructed from cup products to invariants constructed from differential forms and wedge products. Consider for example the intersection form on an oriented manifold X^d with boundary. There is a well defined non-degenerate cup product $\overline{H}^p(X; \mathbf{C}) \times \overline{H}^{d-p}(X; \mathbf{C}) \to H^d(X, \partial X, \mathbf{C}) = \mathbf{C}$ in singular cohomology (where as before $\overline{H}^p(X)$ means the image of the relative cohomology in the absolute cohomology). Suppose that a is a closed p-form representing a class in $\overline{H}^p(X; \mathbf{C})$ and b is a closed $(d - p)$-form representing a class in $\overline{H}^{d-p}(X; \mathbf{C})$. Then the wedge product of a and b gives a closed form, but

$$[a] \cup [b] \neq \int_X a \wedge b.$$

The right hand side is not a topological invariant; for example one can replace X by the complement of an open collar to change the right side, but the left depends only of the cohomology classes of a and b.

If, however, a and b extend to exponentially decaying forms on $X(\infty)$, then

$$[a] \cup [b] = \int_{X(\infty)} a \wedge b.$$

Thus the stretching procedure is a convenient way of relating cup products to wedge products on a manifold with boundary.

3.2. We use Fourier expansions in terms of eigenvectors of $\widehat{D}_t$, the tangential operator defined in Lemma 2.2. The results of [14] imply that there exists a complete system of analytically varying eigenvectors $\psi_i(t)$, $i \in \mathbb{Z} - \{0\}$, with analytically varying eigenvalues $\mu_i(t)$, $i \in \mathbb{Z} - \{0\}$, so that $\sigma(\psi_i) = \psi_{-i}$, $\mu_{-i} = -\mu_i$, and $\mu_i \leqslant \mu_{i+1}$. One may assume that

$\mathcal{H}_t = \operatorname{span}\{\psi_i\}_{i=-n}^{n}$, so that $\mu_i(t) = 0$ for $-n \leqslant i \leqslant n$ and that the Lagrangian $\mathcal{L}_t$ is the span of $\psi_i(t)$ for $i = 1, \ldots, n$ by a change of basis.

We now turn to properties of the operators D_t which are independent of the parameter t. For notational ease we will therefore temporarily drop the subscript t. The following assertions hold for every value of t.

Every $\omega \in \Omega_X^{ev}(E)$ has an expansion on the collar $[0,1] \times Y$

$$\omega = \sum_{-\infty}^{\infty} c_i(u)\psi_i,$$

where $u \in [0,1]$. Forms in the kernel of D have expansions on the cylinder

$$\omega|_{[0,1]\times Y} = \sum_{i \in Z} c_i \mathrm{e}^{-\mu_i u}\psi_i.$$

If in addition ω satisfies the $\mathcal{H} + P_+$ boundary conditions, then

$$\omega|_{[0,1]\times Y} = \sum_{i \geqslant -n} c_i \mathrm{e}^{-\mu_i u}\psi_i.$$

In particular, one can extend $\omega \in \ker D$ to $X(R)$ by this formula for any R including $R = \infty$. The resulting expression converges and gives an element of $\ker D_t^R(\mathcal{L} + P_+)$.

Notice that the indexing is chosen so that if $\omega \in \ker D$ satisfies the P_+ boundary conditions, then $\omega|_{[0,1]\times Y} = \sum_{i>n} c_i \mathrm{e}^{-\mu_i u}\psi_i$. If ω satisfies the $P_+ + \mathcal{L}$ boundary conditions, then $\omega|_{[0,1]\times Y} = \sum_{i \geqslant 1} c_i \mathrm{e}^{-\mu_i u}\psi_i$.

We list two useful observations:

(1) If ω satisfies the P_+ boundary conditions, then it extends (as a kernel element) over $X(\infty)$ and is in $L^2(X(\infty))$. This follows from the estimate $\|\omega_{\{u\}\times Y}\|_{L^2(Y)} \leqslant K\mathrm{e}^{-\mu u}$ for some $\mu > 0$ smaller than the smallest positive eigenvalue of $\widehat{D}$. Conversely any L^2 solution to $D\omega = 0$ on $X(\infty)$ satisfies P_+ boundary conditions when restricted to $X(R)$ for any R. This gives a natural identification between the spaces $\ker D^R(P_+)$ for a given R and the L^2 kernel on $X(\infty)$.

(2) Let $i_R : \{R\} \times Y \to X$ denote the inclusion and i_R^* the corresponding restriction of sections. Let $p_{\mathcal{H}}$ denote the L^2 projection onto $\mathcal{H}$. Let

$$\mathcal{N}^R = p_{\mathcal{H}}\big(i_R^*\big(\ker D^R(P_+ + \mathcal{H})\big)\big) \subset \mathcal{H}.$$

Then $\mathcal{N}^R$ is the set of "limiting values of extended L^2 solutions to $D\omega = 0$" (see [2]) and is independent of R. We denote it by $\mathcal{N}$ (or by $\mathcal{N}_t$ when the parameter t is introduced). Explicitly, $\mathcal{N}$ is the set of $x \in \mathcal{H}$ such that the kernel of D contains an element of the form $\sum_{i=-n}^{\infty} c_i \mathrm{e}^{-\mu_i u}\psi_i$ on the cylinder with $x = \sum_{i=-n}^{n} c_i\psi_i$. (Recall that $\mu_i = 0$ for $-n \leqslant i \leqslant n$.)

3.3. The next lemma shows that elements of $\ker D$ with appropriate boundary conditions are both closed and co-closed, just as in the case of closed manifolds.

Lemma 3.1. *If $\omega \in \Omega_X^{ev}(E)$ has $\mathcal{H} + P_+$ boundary conditions and $D\omega = 0$, then $d_A\omega = 0 = d_A^*\omega$.*

Proof. Extend ω as above to $X(\infty)$. Then

$$0 = D\omega = i^l \sum_p (-1)^{p+1}(*d_A - d_A*)\omega_{2p}.$$

Taking homogeneous parts we see that $*d_A\omega_{2p} - d_A * \omega_{2p+2} = 0$. We claim that both terms are zero. Expand ω on the cylinder of $X(\infty)$ into the sum of its harmonic part and its exponentially decaying part:

$$\omega = \sum_{i=-n}^{n} c_i\psi_i + \sum_{i=n+1}^{\infty} c_i \mathrm{e}^{-\mu_i u}\psi_i.$$

The ψ_i for $i = -n, \ldots, n$ are in the kernel of $\widehat{D}_0$ and hence are harmonic since Y is closed. Thus $d_{\hat{A}}\psi_i = 0 = d^*_{\hat{A}}\psi_i$ for $i = -n, \ldots, n$.

On the cylinder, $d_A\omega = d_{\hat{A}}\omega + du \wedge (\partial\omega/\partial u)$.

Thus on the cylinder we have the expansion:

$$d_A\omega = \sum_{i=n+1}^{\infty} c_i \mathrm{e}^{-\mu_i u}(d_{\hat{A}}\psi_i - \mu_i\, du\psi_i).$$

Consequently, there is an estimate

$$\|d_A\omega|_{\{u\}\times Y}\|_{L^2(\{u\}\times Y)} \leqslant k_1 \mathrm{e}^{-\mu u}$$

for some constant k_1 depending only on the restriction of ω to $\{0\} \times Y$. Here $\mu > 0$ is smaller than the smallest positive eigenvalue of $\widehat{D}$.

A similar argument shows that $\|d_A * \omega\| \leqslant k_2 \mathrm{e}^{-\mu u}$ for some constant k_2 depending only on the restriction of ω to $\{0\} \times Y$. Thus both $d_A\omega$ and $d_A * \omega$ exponentially decay, and hence have extensions to $L^2(X(\infty), E)$. Furthermore these extensions have the property that their wedge product $d_A\omega \wedge d_A * \omega$ exponentially decays.

Similarly there is an estimate

$$\| * \omega|_{\{u\}\times Y}\|_{L^2(\{u\}\times Y)} \leqslant k_3$$

for some constant k_3 depending only on the restriction of ω to $\{0\} \times Y$. (Notice that the harmonic part does not vanish and so $*\omega|_{\{u\}\times Y}$ does not exponentially decay. However, it does exponentially limit to a harmonic form, i.e., it is an extended L^2 form.)

We can therefore integrate by parts:

$$\begin{aligned}
\left|\langle * d_A\omega_{2p}, d_A * \omega_{2p+2}\rangle_{X(\infty)}\right| &= \left|\lim_{R\to\infty} \int_{X(R)} K(d_A\omega_{2p} \wedge d_A * \omega_{2p+2})\right| \\
&= \left|\lim_{R\to\infty} \int_{\{R\}\times Y} K(d_A\omega_{2p} \wedge *\omega_{2p+2})\right| \\
&\leqslant \lim_{R\to\infty} k_1k_3 \mathrm{e}^{-\mu R} = 0.
\end{aligned}$$

Thus $*d_A\omega_{2p}$ is orthogonal to $d_A * \omega_{2p+2}$ in $L^2(X(\infty))$. Since their sum is zero, they are both zero on $X(\infty)$, and in particular, on $X(R)$ for any R. This shows that $d_A\omega = 0 = d^*_A\omega$. □

It follows from the previous lemma that the DeRham map induces a linear map from the kernel of $D^R(\mathcal{H}+P_+)$ to $H^{ev}(X;V_\alpha)$, since forms in the kernel of $D^R(\mathcal{H}+P_+)$ are closed. Also there is a map from the kernel of $D^R(\mathcal{H}+P_+)$ to the odd forms $H^{odd}(X;V_\alpha)$ since taking the Hodge $*$ of a form in the kernel of $D^R(\mathcal{H}+P_+)$ yields a closed form. This second map depends on the Riemannian metric in general.

From the Fourier expansion one sees that the first map takes the kernel of $D^R(P_+)$ to the image of the relative cohomology in the absolute, i.e., to $\overline{H}^{ev}(X;V_\alpha)$. Indeed, such a form is closed on $X(\infty)$ but its values on cycles in $\{R\}\times Y$ exponentially decay (as $R\to\infty$), and hence the restriction of the corresponding cohomology class to the boundary is 0. In [2] it is shown that this gives an isomorphism from $\ker D^R(P_+)$ with $\overline{H}^{ev}(X;V_\alpha)$. In particular, if $\mathcal{L}\cap\mathcal{N}=0$ in $\mathcal{H}$, then the kernels of $D^R(\mathcal{L}+P_+)$ and $D^R(P_+)$ coincide. Now $\mathcal{N}$ is a Lagrangian in $\mathcal{H}$ [21,22,27]. It follows from the previous lemma that under the identification of $\mathcal{H}$ with $H^*(Y;V_\alpha)$, $\mathcal{N}$ lies in the image $H^*(X;V_\alpha)\to H^*(Y;V_\alpha)$. Since this image is also Lagrangian, they are equal.

3.4. In many applications one can choose an analytic path of Lagrangians $\mathcal{L}_t$ so that for all t, $\mathcal{L}_t$ misses $\mathcal{N}_t$. In such a case one can tell where the kernel of $D_t(\mathcal{L}+P_+)$ jumps in dimension along the path by computing when the image of the relative cohomology in the absolute cohomology jumps up. In any case we remind the reader of our standing assumption:

Assumption. $\mathcal{L}_0$ is transverse to $\mathcal{N}_0$.

With this assumption the kernel of $D_0(\mathcal{L}+P_+)$ is isomorphic to

$$\overline{H}^{ev}(X;V_0)=\operatorname{Im}H^{ev}(X,Y;V_0)\to H^{ev}(X;V_0),$$

and in particular is a homotopy invariant of (X,α_0).

Poincaré duality implies that the pairing

$$\overline{H}^p(X;V_0)\times\overline{H}^{2l-1-p}(X;V_0)\to\mathbf{C},$$

taking (ϕ,τ) to $\phi\cdot\tau$, is non-degenerate. The Hodge $*$ operator preserves harmonic forms, and the L^2 inner product induces a non-degenerate positive definite inner product on the L^2 harmonic forms. Identifying L^2 harmonic forms with cohomology using the Hodge theorem (or Proposition 4.9 of [2] if $\partial X\neq 0$) we see that the Riemannian metric on X induces a positive definite inner product $\langle\,,\rangle$ on the cohomology of X, and an isometry $*: H^*(X;V_t)\to H^*(X;V_t)$. These relate to the dot product in the following way, whose proof we leave to the reader.

Lemma 3.2. *If ϕ^p and τ^{2l-1-p} are L^2 harmonic forms on $X(\infty)$ representing cohomology classes $[\phi]$ and $[\tau]$ in $\overline{H}^*(X(\infty);V_0)$, then the dot product of these classes as defined in Section 2 may be expressed as follows.*

$$[\phi]\cdot[\tau]=\int_{X(\infty)}K(\phi\wedge\tau),$$

where K denotes the Hermitian inner product on V. The L^2 inner product on harmonic forms induces an inner product $\langle\,,\,\rangle$ on $\overline{H}^p(X; V_0)$ and the formula

$$\big\langle[\phi], [\tau]\big\rangle = [\phi]\cdot[*\tau]$$

relates the inner product, the Hodge $*$ operator, and the dot product.

4. The first order forms

4.1. We next define the bilinear form which will give the first order part of the spectral flow. Recall that a_1 is the element of $H^1(X, ad\,\alpha_0)$ which represents the tangent vector to the arc of representations α_t at $t = 0$.

Definition 4.1. Define the *Reduced First Order Form* to be the bilinear form:

$$\widetilde{B} : \overline{H}^{l-1}(X; V_0) \times \overline{H}^{l-1}(X; V_0) \to \mathbf{C},$$

given by the formula

$$(x, y) \mapsto i^l r(a_1)(x)\cdot y.$$

This form clearly depends only on the homotopy type of X. Notice that

$$\begin{aligned}\widetilde{B}(x, y) &= i^l\big(r(a_1)(x)\cdot y\big) = i^l(-1)^{(l-1)^2+1} x\cdot r(a_1)(y)\\ &= i^l(-1)^{l+l(l-1)}\overline{\big(r(a_1)(y)\cdot x\big)} = \overline{\widetilde{B}(y, x)}.\end{aligned}$$

Thus $\widetilde{B}$ is Hermitian and has a well-defined signature. Notice that if $x \in \overline{H}^{l-1}(X; V_0)$ satisfies $\widetilde{B}(x, y) = 0$ for all $y \in \overline{H}^{l-1}(X; V_0)$, then non-degeneracy of the dot product implies that $r(a_1)(x) = 0$. We summarize:

Proposition 4.2. *The signature of the reduced first order form $\widetilde{B}$ is a homotopy invariant of X. Moreover, the kernel of $\widetilde{B}$ is*

$$\big\{x \in \overline{H}^{l-1}(X; V_0) \mid r(a_1)(x) = 0\big\}.$$

We next relate the reduced first order form $\widetilde{B}$ to a larger form which has the same signature as $\widetilde{B}$. This larger form is defined on all the even cohomology, and it is the form which will arise in the proof of the main theorem in the next section. We will show that this larger form is the direct sum of the reduced first order form and a hyperbolic form, and so the signatures of B and $\widetilde{B}$ coincide.

We define the *Total First Order Form*

$$B : \overline{H}^{ev}(X; V_0) \times \overline{H}^{ev}(X; V_0) \to \mathbf{C}$$

by the formula

$$B(x, y) = i^l \sum_{p=0}^{l-1} (-1)^{p+1}\big(r(a_1)\big(x_{2(l-p-1)}\big)\cdot y_{2p} + r(a_1)\big(*x_{2(l-p)}\big)\cdot *y_{2p}\big).$$

Where $x = \sum x_{2p}$ and $y = \sum y_{2p}$ is the decomposition into homogeneous parts. In this formula we have identified cohomology with harmonic forms. It is easy to check that B is Hermitian, and hence has a well-defined signature. The form B depends on the Riemannian metric in general.

The form B arises by considering the first variation of the path of operators D_t in the following way. Recall that $D_t = D_0 + \sum_i C_i t^i$ where C_i is given by formula (2.2). So $\dot{D}_0 = C_1$. Hence if x, y are harmonic forms (with respect to d_{A_0}) and $x \in \Omega^{2p}_X(E)$, then

$$\begin{aligned}\langle \dot{D}_0 x, y\rangle &= i^l(-1)^{p-1}\langle *r(a_1)(x) - r(a_1)(*x), y\rangle\\ &= i^l(-1)^{p-1}\big(r(a_1)(x)\cdot y - r(a_1)(*x)\cdot *y\big)\\ &= B(x,y).\end{aligned} \tag{4.1}$$

4.2. The next theorem relates B to $\widetilde{B}$. It also identifies the kernel of B. This is done by introducing a chain complex whose chain groups are the twisted cohomology groups $\overline{H}^p(X; V_0)$, and whose differential is the cup product $r(a_1): \overline{H}^p(X; V_0) \to \overline{H}^{p+1}(X; V_0)$ given by $x \mapsto r(a_1)(x)$.

Notice that $r(a_1)$ satisfies $r(a_1)^2 = 0$. Indeed,

$$r(a_1)r(a_1)(x) = \tfrac{1}{2} r\big([a_1, a_1]\big)(x)$$

which equals 0 since $[a_1, a_1] = -2d_{A_0} a_2$ (by Formula 2.1). Therefore

$$\big(\overline{H}^*(X; V_0), r(a_1)\big)$$

defines a (co-)chain complex.

Theorem 4.3. *The total first order form B has signature equal to the signature of $\widetilde{B}$. The signature of B does not depend on the Riemannian metric and is a homotopy invariant. The kernel of B is isomorphic to the even-dimensional cohomology of the complex $(\overline{H}^*(X; V_0), r(a_1))$.*

Proof. Assume first that l is odd, say $l = 2k+1$. Then split $\overline{H}^{l-1}(X; V_0)$ into the kernel of $r(a_1): \overline{H}^{l-1}(X; V_0) \to \overline{H}^l(X; V_0)$, which we denote by K, and its orthogonal complement in $\overline{H}^{l-1}(X; V_0)$ (with respect to the L^2 inner product defined in Lemma 3.2), $K^\perp$. Then write

$$\overline{H}^{ev}(X; V_0) = K^\perp \oplus \Bigg(K \bigoplus_{2p \neq l-1} \overline{H}^{2p}(X; V_0)\Bigg).$$

If $x, y \in K^\perp$, then $B(x,y) = i^l(-1)^{k+1}(r(a_1)(x)\cdot y)$. If $x \in K \bigoplus_{2p\neq l-1} \overline{H}^{2p}(X; V_0)$ and $y \in K^\perp$, then

$$B(x,y) = i^l\big(\pm r(a_1)(x_{l-1})\cdot y_{l-1} \pm r(a_1)(*x_{l+1})\cdot *y_{l-1}\big).$$

The first term vanishes since $x_{l-1} \in K$ and the second term is, up to sign, equal to $\langle r(a_1)(*x_{l+1}), y_{l-1}\rangle$. But $r(a_1)(*x_{l+1}) \in K$ since $r(a_1)^2 = 0$, and so this term vanishes also. Thus the splitting $K^\perp \oplus (K \bigoplus_{2p\neq l-1} \overline{H}^{2p}(X; V_0))$ is orthogonal with respect to the form B. Moreover, the restriction to $K^{\perp}$ is equal to the restriction of $\widetilde{B}$ to $K^{\perp}$. Clearly $\widetilde{B}$ vanishes on K and the signature of $\widetilde{B}$ equals the signature of the restriction of $\widetilde{B}$ to $K^{\perp}$.

An easy argument shows that if $F : V \times V \to \mathbf{C}$ is a Hermitian form, and $j : V \to V$ is an involution so that $F(j(x), j(y)) = -F(x, y)$ for all $x, y \in V$, then the signature of V is zero (even if F is degenerate).

Define $j : K \bigoplus_{2p \neq l-1} \overline{H}^{2p}(X; V_0) \to K \bigoplus_{2p \neq l-1} \overline{H}^{2p}(X; V_0)$ by the formula

$$j\left(\sum_{p=0}^{l-1} x_{2p}\right) = -\sum_{p=0}^{(l-1)/2} x_{2p} + \sum_{p=(l+1)/2}^{l-1} x_{2p}.$$

A routine calculation shows that $B(x, y) = -B(jx, jy)$. The fact that $x_{(l-1)/2} \in K$ implies that the term $r(a_1)(x_{(l-1)/2}) \cdot y_{(l-1)/2}$ vanishes, and this is the only term which does not change signs. It follows that the signature of B equals the signature of $\widetilde{B}$.

A similar argument works if l is even. In this case one decomposes $\overline{H}^l(X; V_0)$ into the kernel of $x \mapsto r(a_1)(*x)$ and its orthogonal complement. In addition one observes that the forms

$$\overline{H}^l(X; V_0) \times \overline{H}^l(X; V_0) \to \mathbf{C}, \quad (x, y) \mapsto r(a_1)(*x) \cdot *y$$

and

$$\overline{H}^{l-1}(X; V_0) \times \overline{H}^{l-1}(X; V_0) \to \mathbf{C}, \quad (x, y) \mapsto r(a_1)(x) \cdot y$$

are isomorphic (via the Hodge $*$) and hence have the same signature.

Finally we compute the kernel of B. If $x \in \overline{H}^{ev}(X; V_0)$ satisfies $B(x, y) = 0$ for all $y \in \overline{H}^{ev}(X; V_0)$, then $r(a_1)(x_{2(l-p-1)}) \cdot y_{2p} + r(a_1)(*x_{2(l-p)}) \cdot *y_{2p} = 0$ for any y. Since $e \cdot *f = \pm * e \cdot f$ it follows that $r(a_1)(x_{2(l-p-1)}) \pm *r(a_1)(*x_{2(l-p)}) = 0$ for some appropriate sign. But

$$\langle r(a_1)(e), *r(a_1)(f)\rangle = \pm r(a_1)(e) \cdot r(a_1)(f) = \pm r(a_1)^2(e) \cdot f = 0$$

and so $r(a_1)(x_{2(l-p-1)})$ and $*r(a_1)(*x_{2(l-p)})$ are orthogonal. Hence

$$r(a_1)\big(x_{2(l-p-1)}\big) \quad \text{and} \quad *r(a_1)\big(*x_{2(l-p)}\big)$$

both equal 0, and so $r(a_1)(x) = 0 = r(a_1)(*x)$.

Now

$$\langle r(a_1)e, f\rangle = r(a_1)(e) \cdot *f = \pm e \cdot r(a_1)(*f) = \pm\langle e, *r(a_1)(*f)\rangle.$$

Therefore $*r(a_1)*$ is the adjoint of $r(a_1)$ up to sign. It follows in the usual way that the cohomology of the complex $(\overline{H}^*(X; V_0), r(a_1))$ is isomorphic to the kernel of $r(a_1) + *r(a_1)*$, which in turn is isomorphic to the intersection of the kernels of $r(a_1)$ and $*r(a_1)*$.

We conclude that the set of $x \in \overline{H}^{ev}(X; V_0)$ satisfying $B(x, y) = 0$ for all $y \in \overline{H}^{ev}(X; V_0)$ coincides with the even dimensional cohomology of the complex $(\overline{H}^{ev}(X; V_0), r(a_1))$. $\square$

4.3. We finish this section by calculating explicit expressions for B and $\widetilde{B}$ on a 3-manifold. Consider first a connected 3-manifold X with non-empty boundary Y. Then $H^0(X; V_0)$

can be identified with the invariants $\{v \in V \mid (1-\alpha_0(g))v = 0, \text{ for all } g \in \pi_1 X\}$ and similarly for $H^0(Y; V_0)$. Thus $H^0(X; V_0)$ injects into $H^0(Y; V_0)$ and so $\overline{H}^0(X; V_0) = 0$. By Poincaré duality $\overline{H}^3(X; V_0) = 0$ also.

Therefore, the total form B is defined on $\overline{H}^2(X; V_0)$ by the formula

$$B(x, y) = -r(a_1)(*x) \cdot *y$$

and has kernel equal to the kernel of $r(a_1)* : \overline{H}^2(X; V_0) \to \overline{H}^2(X; V_0)$. The orthogonal complement of the kernel is the image of $r(a_1) : \overline{H}^1(X; V_0) \to \overline{H}^2(X; V_0)$. Moreover, the hyperbolic part of the form (as in the proof of the Theorem 4.3) is zero.

The reduced form $\widetilde{B}$ is defined on $\overline{H}^1(X; V_0)$ by

$$\widetilde{B}(x, y) = -r(a_1)(x) \cdot y$$

and has kernel equal to $\ker r(a_1) : \overline{H}^1(X; V_0) \to \overline{H}^2(X; V_0)$. Both forms are non-degenerate exactly when the first (or second) cohomology of the complex $(\overline{H}^*(X; V_0), r(a_1))$ is zero.

The same facts apply to a closed 3-manifold if $H^0(X; V_0) = 0$. This holds, for example, if the representation $r \circ \alpha_0 : \pi_1 X \to U(V)$ is irreducible.

If X is closed and $H^0(X; V_0) \neq 0$, then the total form B is metric dependent; it is given by the formula

$$B(x, y) = -r(a_1)x_0 \cdot y_2 - r(a_1)(*x_2) \cdot *y_2 + r(a_1)(x_2) \cdot y_0.$$

It splits orthogonally into the hyperbolic part

$$B(x, y) = -r(a_1)(x_0) \cdot y_2 + r(a_1)(x_2) \cdot y_0$$

on $K \oplus H^0(X; V_0)$ and the non-hyperbolic part

$$B(x, y) = -r(a_1)(*x) \cdot *y$$

on $K^\perp$. Here

$$K = \ker r(a_1)* : H^2(X; V_0) \to H^2(X; V_0)$$

and so

$$K^\perp = \operatorname{Im} r(a_1) : H^1(X; V_0) \to H^2(X; V_0).$$

The kernel of B is the subspace

$$\{(x_0, x_2) \mid 0 = r(a_1)(x_0) = r(a_1)(x_2) = r(a_1)(*x_2)\},$$

which is isomorphic to the even cohomology of the complex $(H^*(X; V_0), r(a_1))$.

The form $\widetilde{B} : H^1(X; V_0) \times H^1(X; V_0) \to \mathbf{C}$ is defined by

$$\widetilde{B}(x, y) = -r(a_1)(x) \cdot y.$$

Its kernel is the set $\{x \mid r(a_1)(x) = 0\}$. Notice that the signature of B is metric independent since it equals the signature of $\widetilde{B}$. Moreover, the dimension of the kernel of B is also metric independent since it equals the dimension of the even cohomology of the complex $(H^*(X; V_0), r(a_1))$.

The reduced form has a much simpler and, in particular, metric independent expression than the total form. In the next section we will see that the signature and kernel of the total form provide precisely the information we need to calculate the first order spectral flow. (For specific computations, using these forms, of spectral flow on 3-manifolds, see [15].)

5. The main theorems

5.1. In this section we show how the form B gives the first order part of the spectral flow. Again the main technicalities come from working on a manifold with boundary. It turns out that by stretching the the collar of X is sufficiently, B gives information about the first order part of the spectral flow.

This result can be thought of as a generalization of the Hodge theorem, which identifies the kernel of $D_0(\mathcal{L}+P_+)$ (i.e., the 0th order part of the spectral flow along a path of representations) with cohomology. Theorems 5.1 and 5.2 identify the first order part of the spectral flow in terms of the cup product.

5.2. We begin with the theorem for a closed manifold. The proof is just a standard argument working with the first variation of eigenvalues:

Theorem 5.1. *Let X be a closed manifold and $\alpha : J \to \operatorname{Hom}(\pi_1 X, U(k))$ an analytic path of representations, $J=[0,\varepsilon)$ or $(-\varepsilon,\varepsilon)$. Let $r : U(k)\to U(V)$ be a representation and let D_t denote the Atiyah–Patodi–Singer odd signature operators obtained from the flat connection A_t with holonomy $r\circ\alpha_t$ as described above. Suppose the dimension of the kernel of D_t jumps up at $t=0$.*

Then the signature of the reduced first-order form $\widetilde{B}$ is equal to the sum of the signs of the derivatives of the eigenvalues of D_t which pass through 0 at $t=0$. This is what we call the "first order spectral flow" of D_t at $t=0$.

Moreover, if the cohomology of $(H^*(X;V_0), r(a_1))$ is zero, then the signature of $\widetilde{B}$ equals the spectral flow through $t=0$ of the family D_t.

Proof. Choose an analytic path of flat connections A_t using the main results of [11]. Let $\phi_i(t)$, $i=1,\dots,m$, be an analytically varying orthonormal family of eigenvectors for D_t, with analytically varying eigenvalues $\lambda_i(t)$, so that $\{\phi_i(0)\}_{i=1}^m$ spans the kernel of D_t. The existence of such a family follows from the main results of [14]. The important point is that on a closed manifold the domain of D_t is independent of t. (See also Section 2.)

Differentiating the expression $D_t\phi_i(t)=\lambda_i(t)\phi_i(t)$ at $t=0$ one obtains:

$$D_0\dot{\phi}_i(0)+\dot{D}_0\phi_i(0)=\dot{\lambda}_i(0)\phi_i(0).$$

Taking the inner product with $\phi_j(0)$ and using the fact that D_0 is self adjoint and $D_0\phi_j(0)=0$ one gets:

$$\dot{\lambda}_i\delta_{ij}=\langle\dot{D}_0\phi_i(0),\phi_j(0)\rangle.$$

Formula (4.1) shows that $\langle \dot{D}_0 x, y\rangle = B(x, y)$. Therefore, in the basis $\{\phi_i(0)\}$ the form B is diagonal with entries $\dot{\lambda}_i(0)$. Hence the signature of B is just the sum of the signs of the derivatives of the $\lambda_i(t)$ at $t = 0$. Theorem 4.3 shows that the signature of B equals the signature of $\widetilde{B}$ and that B is nondegenerate if the cohomology of the complex $(H^*(X; V_0), r(a_1))$ vanishes. □

For sample computations using this theorem, see Theorems 7.9 and 7.10 of [15]. For extensions of this result (for closed manifolds) to higher derivatives of the eigenvalues, see [16,9].

5.3. We now turn to the case of a manifold with boundary. The results of [17] as explained in Section 2 show that one can find an appropriate family of analytically varying eigenvectors and eigenvalues of $D_t(\mathcal{L} + P_+)$ and so one can try to repeat the proof of Theorem 5.1. However, the argument fails in two ways. First, the derivative $\dot{\phi}_i(0)$ need no longer satisfy the boundary conditions since the boundary conditions are varying, and so $\langle D_0\dot{\phi}_i(0), \phi_j(0)\rangle$ need not vanish; in fact it equals $\{\dot{\phi}_i(0), \phi_j(0)\}|_{\partial X}$. Moreover, the derivative of D_t at $t = 0$ is not given by the bilinear form B, i.e., the difference $\langle \dot{D}_0 x, y\rangle - B(x, y)$ is non-zero in general. However, both of these difficulties can be overcome by stretching the collar of X; we will prove that both $\{\dot{\phi}_i(0), \phi_j(0)\}|_{\partial X}$ and $\langle \dot{D}_0 x, y\rangle - B(x, y)$ approach zero as the metric on X is deformed so that the collar becomes increasingly long. Hence, the eigenvalues of the form B give the limiting values (as the collar becomes infinitely long) of the derivatives of the eigenvalues of D_t which pass through 0.

If the first order form B is degenerate, then a further difficulty seems to arise: what if the time-derivative of an eigenvalue of D_t is, say, positive for each finite collar length, but its limit as the collar becomes infinite is 0? Then the forms defined above would simply tell us that 0 is the limiting value of this derivative, which would not tell us the first order spectral flow of the compact manifold (i.e., with finite collar) in which we were originally interested. The solution to this problem lies in the main theorems of [18], in which we show that this phenomenon cannot occur. In other words, if the time derivative of an eigenvalue of of D_t approaches 0 as the collar becomes infinitely long, then that derivative must already have been 0 for all finite collar lengths. Thus the theorems in this section really do give the first order spectral flow. We will give a more detailed discussion of this phenomenon in the statement and proof of Theorem 5.3, below.

We assemble our notation and assumptions: $\alpha : J \to \mathrm{Hom}(\pi_1(X), U(k))$ is an analytic path of representations on a compact manifold X with collar isometric to $[0, 1] \times Y$. We are given a representation $r : U(k) \to U(V)$ of $U(k)$ on a Hermitian vector space V. For each t, V_t is the system of local coefficients given by the composite $r \circ \alpha_t$. We assume that $\dim H^*(Y; V_t)$ is independent of $t \in J$. We let D_t denote the odd signature operator on X coupled to the flat connection with holonomy $r \circ \alpha_t$. We choose an analytic path $\mathcal{L}_t$ of Lagrangians in $H^*(Y; V_t)$ so that at $t = 0$, $\mathcal{L}_0$ is transverse to $\mathcal{N}_0$, the limiting values of extended L^2 solutions to $D_0\omega = 0$. Hence the kernel of $D_0(\mathcal{L}_0 + P_+)$ is isomorphic to $\overline{H}^{ev}(X; V_0)$. Finally, $X(R)$ denotes the manifold X with a collar $[0, R] \times Y$ glued to X

along $[0,1]\times Y$, and D_t^R denotes the obvious extension of D_t to $X(R)$. Notice that the kernel of $D_0^R(\mathcal{L}+P_+)$ is independent of R, although the other eigenvalues do (in general) depend on R.

The assumption in the last paragraph that $\mathcal{L}_0$ is transverse to the limiting values of extended L^2 solutions does not restrict the usefulness of this theorem in the calculation of spectral flow. Calculations of spectral flow when $\mathcal{L}_0$ is not transverse to the limiting values of extended L^2 solutions can be broken down into two problems: one when this assumption holds, and another involving a *fixed* operator but with varying boundary condition $\mathcal{L}_t$. This latter situation has been extensively studied, and is easily understood in terms of the Maslov index of the family $\mathcal{L}_t$ with respect to the limiting values of extended L^2 solutions. See [22] and [20].

Theorem 5.2. *Given any $\varepsilon>0$, there exists an $R_\varepsilon\gg 0$ so that for all $R>R_\varepsilon$, there is a* 1–1 *correspondence between the eigenvalues $\tau_i(B)$ of B and the first derivatives of the eigenvalues $\lambda_i(t)$ of $D_t^R(\mathcal{L}+P_+)$ passing through* 0 *at $t=0$, denoted by $\tau_i(B)\leftrightarrow\dot\lambda_i^R(0)$, so that*

$$\left|\tau_i(B)-\dot\lambda_i^R(0)\right|<\varepsilon.$$

In particular, if the cohomology of $(\overline{H}^*(X;V_0), r(a_1))$ is zero, then the signature of the reduced first order form $\widetilde{B}$ equals the spectral flow of $D_t^R(\mathcal{L}+P_+)$ through $t=0$ for $R>R_\varepsilon$, where $\varepsilon<\frac{1}{2}\inf|\tau_i(B)|$.

Before giving the proof of Theorem 5.2, we will state and prove an addendum to this theorem (stated as Theorem 5.3) which sharpens the results of Theorem 5.2.

Theorem 5.3. *The correspondences of Theorem* 5.2 *may be set up in such a way that they satisfy the following additional condition*: *For each i, the sign of $\dot\lambda_i^R(0)$ equals the sign of $\tau_i(B)$* (*where, of course, the three possible* "*signs*" *are* $+$, $-$, *and* 0).

Note that this theorem implies that if the limiting value of $\dot\lambda_i^R(0)$ as $R\to\infty$ is 0, then for finite R we must already have $\dot\lambda_i^R(0)=0$.

Proof of Theorem 5.3. In Definition 6.6 of [18], we define forms B_m for all $m>0$; in the proof of Theorem 6.7 of the same paper, we show that B_1 coincides with the "total form" B of the current paper, i.e.,

$$B_1(v,w)=\langle\dot{D}v,w\rangle.$$

To conclude, Theorem 7.1 of [18] shows that the eigenvalues of B_1 (called B in the current paper) have the same signs as the set $\{\dot\lambda_i^R(0)\}$ for all R. □

Proof of Theorem 5.2. Choose the path of flat connections A_t on X with holonomy $r\circ\alpha_t$ to vary analytically and be in cylindrical form, using the result in [11]. These extend in the obvious way to $X(R)$ and $X(\infty)$.

Notice that by our assumption on the Lagrangian $\mathcal{L}_0$ the kernel of $D_0^R(\mathcal{L}+P_+)$ is independent of R, and in fact equals the L^2-kernel of D_0 on $X(\infty)$. (Recall that each element of the kernel of $D_0^R(\mathcal{L}+P_+)$ has a Fourier expansion $\sum_{i>n} c_i \mathrm{e}^{-\mu_i u}\psi_i$ on the collar.)

Given any $R>0$, the main result of [17] implies that we can find analytic paths of eigenvectors $\phi_i^R(t)$, $i=1,\ldots,m$, for $D_t^R(\mathcal{L}+P_+)$ with corresponding paths of eigenvalues $\lambda_i^R(t)$ so that $\{\phi_i^R(t)\}$ are orthonormal for each t and $\{\phi_i^R(0)\}$ spans the kernel of $D_0^R(\mathcal{L}+P_+)$.

The $\phi_i^R(t)$ have Fourier expansions on the collar

$$\phi_i^R(t)=\sum_{p=-\infty}^{\infty} a_{i,p}^R(t,u)\psi_p(t).$$

Here u denotes the collar coordinate. We recall that $\psi_p(t)$ are the orthonormal set of eigenvectors for the tangential operator $\widehat{D}_t$, and $\mu_p(t)$ their corresponding eigenvalues; the indexing is chosen so that $\mu_{-p}(t)=-\mu_p(t)$ and $\mu_p(t)=0$ for $p=-n,\ldots,-1,1,\ldots,n$, moreover $\mathcal{L}_t=\operatorname{span}\{\psi_p(t)|1\leqslant p\leqslant n\}$. (Also $\psi_0(t)=0$).

Since the $\phi_i^R(t)$ satisfy $P_+(t)+\mathcal{L}_t$ boundary conditions on $X(R)$, we know that $a_{i,p}^R(t,R)=0$ if $p<0$. Moreover, since $\phi_i^R(0)$ lies in the kernel of $D_0^R(\mathcal{L}+P_+)$, we know that

$$a_{i,p}^R(0,u)=\begin{cases}\alpha_{i,p}^R\mathrm{e}^{-\mu_p u}\psi_p(0) & \text{if } p>n,\\ 0 & \text{if } p\leqslant n\end{cases}$$

for some constants α_{ip}^R (recall that $\mathcal{L}_t$ is transverse to the limiting values of extended L^2 solutions, and therefore $a_{i,p}^R(0,u)=0$ for $1\leqslant p\leqslant n$).

Notice that each $\phi_i^R(0)$ exponentially decays on $[0,R]\times Y$, and so the Fourier expansion gives a canonical extension of $\phi_i^R(0)$ to $X(\infty)$. Moreover, a simple calculation shows that

$$\left\|\phi_i^R(0)\right\|^2_{L^2([0,\infty)\times Y)}\leqslant\frac{2}{1-\mathrm{e}^{-2\mu_{n+1}}}\left\|\phi_i^R(0)\right\|^2_{L^2([0,1]\times Y)}$$

and so since $\|\phi_i^R(0)\|^2_{L^2([0,1]\times Y)}<\|\phi_i^R(0)\|^2_{L^2(X(R))}=1$, the extension of $\phi_R^i(0)$ to $X(\infty)$ is in $L^2(X(\infty))$ and its L^2 norm is bounded by a constant K_0 independent of i and R. Let $W\subset L^2(X(\infty))$ denote the L^2 kernel of D_0, which is canonically identified with $\ker D_0^R(\mathcal{L}+P_+)$ for any R. Of course, W is isomorphic to $\overline{H}^{ev}(X;V_0)$ via the DeRham map.

For any i or R the vector $\phi_i^R(0)$ lies in W and has finite $L^2(X(\infty))$ norm. Also, the usual regularity theorems show that W consists only of smooth functions.

A few lemmas will be needed.

Lemma 5.4. *Let $\widehat{C}_1: L^2\Omega^*(Y,\widehat{E})\to L^2\Omega^*(Y,\widehat{E})$ denote the derivative at $t=0$ of the tangential operator $\widehat{D}_t$. Then*:

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\psi_p(t)=\sum_{q,\mu_q(0)\neq\mu_p(0)}\frac{1}{\mu_p(0)-\mu_q(0)}\langle\widehat{C}_1\psi_p(0),\psi_q(0)\rangle\psi_q(0)$$

$$+ \sum_{q,\mu_q(0)=\mu_p(0)} \left\langle \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \psi_p(t), \psi_q(0) \right\rangle \psi_q(0).$$

Proof. Differentiate the eigenvalue equation $\widehat{D}_t \psi_p(t) = \mu_p(t)\psi_p(t)$ with respect to t to obtain

$$\widehat{C}_1 \psi_p(0) + \widehat{D}_0 \dot{\psi}_p(0) = \dot{\mu}_p(0)\psi_p(0) + \mu_p(0)\dot{\psi}_p(0).$$

Suppose that $\mu_q(0) \neq \mu_p(0)$. Taking the inner product of the previous line with $\psi_q(0)$ and using the facts that $\widehat{D}_0$ is self adjoint and $\psi_q(0)$ is orthogonal to $\psi_p(0)$, one obtains:

$$\langle \dot{\psi}_p(0), \psi_q(0) \rangle = \frac{1}{\mu_p(0) - \mu_q(0)} \langle \widehat{C}_1 \psi_p(0), \psi_q(0) \rangle.$$

The lemma follows from the fact that $x = \sum_q \langle x, \psi_q(0) \rangle \psi_q(0)$ for any $x \in L^2$. □

We will also need the following estimate.

Lemma 5.5. *There exists a constant K independent of $i = 1, \ldots, m$ and R so that*

$$\sum_{p>n} |\alpha_{i,p}^R| < K.$$

Proof. Recall that on the cylinder, $\phi_i^R(0) = \sum_{p>n} \alpha_{i,p}^R \mathrm{e}^{-\mu_p u} \psi_p(0)$. Each $\phi_i^R(0)$ is smooth, and so its restriction to $Y \times 0$ is also smooth. Hence

$$\sum_{p>n} \alpha_{i,p}^R \psi_p(0)$$

lies in $L_s^2(Y \times 0)$ for any $s \geqslant 0$.

Since the $\psi_p(0)$ are eigenfunctions of the elliptic operator $\widehat{D}_0$, it follows that $\{(1 + \mu_p^2)^{-s/2}\psi_p(0)\}$ is an orthonormal basis for an admissible norm on $L_s^2(Y)$ (see [23]). Thus

$$\sum_{p>n} (\alpha_{i,p}^R)^2 (1 + \mu_p^2)^s = \| \phi_i^R(0)|_{\{0\}\times Y} \|_{L_s^2} < \infty.$$

We can say more. Fix an $s > (\dim Y)/2$. Each $\phi_i^R(0)$ has norm 1 in $L^2(X(R))$, and in particular

$$\| \phi_i^R(0) \|_{L^2([0,1]\times Y)} < 1.$$

Moreover each $\phi_i^R(0)$ lies in W which consists only of smooth functions. It is easy to see that the norm $\|x\|_s = \|x\|_{L_s^2([0,1]\times Y)}$ is a norm on W, and each $\phi_i^R(0)$ has $\|\phi_i^R(0)\|_0 < 1$. Since any two norms on W are equivalent, for each $r \geqslant 0$ there is some constant C_r independent of i or R so that $\|\phi_i^R(0)\|_r$ is bounded by C_r.

Choosing r large enough and applying the restriction theorem it follows that there is a constant K_1 independent of i or R so that the restriction of $\phi_i^R(0)$ to $\{0\} \times Y$ has $L_s^2(Y)$ norm less than K_1, i.e.,

$$\sum_{p>n} (\alpha_{i,p}^R)^2 (1 + \mu_p^2)^s = \| \phi_i^R(0)|_{\{0\}\times Y} \|_{L_s^2} < K_1.$$

On the other hand, the eigenvalues μ_p grow like $p^{1/(\dim Y)}$ [12]. Thus

$$\sum_{p>n}\left(1+\mu_p^2\right)^{-s} < K_2$$

for some constant K_2.

Partition the spectrum of $\widehat{D}_0$ into $S_1 \cup S_2$, where

$$S_1 = \left\{p \mid \left(1+\mu_p^2\right)^s\left(\alpha_{i,p}^R\right)^2 \geqslant \left|\alpha_{i,p}^R\right|\right\}, \quad S_2 = \operatorname{Spec}\left(\widehat{D}_0\right) - S_1.$$

Then

$$\sum_{p\in S_1}\left|\alpha_{i,p}^R\right| \leqslant \sum_p \left(1+\mu_p^2\right)^s\left(\alpha_{i,p}^R\right)^2 < K_1.$$

If $p \in S_2$ then $|\alpha_{i,p}^R| < (1+\mu_p^2)^{-s}$ and so

$$\sum_{p\in S_2}\left|\alpha_{i,p}^R\right| \leqslant \sum_p \left(1+\mu_p^2\right)^{-s} < K_2.$$

The lemma follows by setting $K = K_1 + K_2$. □

We can now proceed with the proof of Theorem 5.2. We compare the diagonal form M_1 whose diagonal entries are the derivatives $\dot{\lambda}_i^R(0)$ to the total first order form B.

Differentiating the expression $D_t^R\phi_i^R(t) = \lambda_i^R(t)\phi_i^R(t)$ at $t = 0$ and taking the inner product with $\phi_j^R(0)$ gives

$$\begin{aligned}\dot{\lambda}_j^R\delta_{ij} &= \left\langle \dot{D}_0^R\phi_i^R(0), \phi_j^R(0)\right\rangle\big|_{X(R)} + \left\langle D_0^R\dot{\phi}_i^R(0), \phi_j^R(0)\right\rangle\big|_{X(R)}\\ &= \left\langle \dot{D}_0^R\phi_i^R(0), \phi_j^R(0)\right\rangle\big|_{X(R)} + \left\langle \dot{\phi}_i^R(0), \sigma\left(\phi_j^R(0)\right)\right\rangle_{\{R\}\times Y}.\end{aligned}$$

(The second line follows by integrating by parts.) Formula (4.1) shows that

$$B\left(\phi_i^R(0), \phi_j^R(0)\right) = \left\langle \dot{D}_0^R\left(\phi_i^R(0)\right), \phi_j^R(0)\right\rangle_{X(\infty)}.$$

Hence if we subtract $B(\phi_i^R(0), \phi_j^R(0))$ from both sides of the preceding formula we obtain

$$\begin{aligned}\dot{\lambda}_j^R\delta_{ij} - B\left(\phi_i^R(0), \phi_j^R(0)\right) &= -\left\langle \dot{D}_0^R\phi_i^R(0), \phi_j^R(0)\right\rangle\big|_{[R,\infty)\times Y}\\ &\quad + \left\langle \dot{\phi}_i^R(0), \sigma\left(\phi_j^R(0)\right)\right\rangle_{\{R\}\times Y}.\end{aligned} \tag{5.1}$$

We first estimate $\langle \dot{D}_0^R\phi_i^R(0), \phi_j^R(0)\rangle|_{[R,\infty)\times Y}$. With $\widehat{C}_1$ as above,

$$\begin{aligned}&\|\widehat{C}_1\|\frac{e^{-\mu_{n+1}R}}{2\mu_{n+1}}\left(\sum_{p>n}\left|\alpha_{i,p}^R\right|\right)\left(\sum_{q>n}\left|\alpha_{j,q}^R\right|\right)\\ &\quad\geqslant \sum_{p>n}\sum_{q>n}\left|\alpha_{i,p}^R\right|\left|\alpha_{j,q}^R\right|\,\|\widehat{C}_1\|\left(\int_R^\infty e^{-(\mu_p+\mu_q)u}\,du\right)\\ &\quad\geqslant \left|\int_R^\infty \alpha_{i,p}^R\alpha_{j,q}^R e^{-(\mu_p+\mu_q)u}\left\langle \widehat{C}_1\psi_p, \psi_q\right\rangle du\right|\end{aligned}$$

$$= \left| \left\langle \sum_{p>n} \alpha_{i,p}^R \mathrm{e}^{-\mu_p U} \widehat{C}_1 \psi_p, \sum_{q>n} \alpha_{j,q}^R \mathrm{e}^{-\mu_q u} \psi_q \right\rangle \Big|_{Y\times[0,\infty)} \right|$$
$$= \left| \left\langle \dot{D}_0^R \phi_i^R(0), \phi_j^R(0) \right\rangle \big|_{[R,\infty)\times Y} \right|.$$

The last line follows from the fact that $D_t = \sigma(\mathrm{d}/\mathrm{d}u + \widehat{D}_t)$ on the collar and so $\dot{D}_0 = \sigma(\widehat{C}_1)$ on the collar. Using Lemma 5.4 we find that there exists a constant K independent of i or R so that

$$\left| \left\langle \dot{D}_0^R \phi_i^R(0), \phi_j^R(0) \right\rangle \big|_{[R,\infty)\times Y} \right| < K \mathrm{e}^{-2\mu_{n+1} R}.$$

We turn to the second term in (5.1). Since $\phi_i^R(t)$ satisfies $P_+(t) + \mathcal{L}_t$ boundary conditions, its restriction to the boundary of $X(R)$ equals

$$\sum_{p>0} a_{i,p}^R(t, R)\, \psi_p(t).$$

Therefore, the restriction of the derivative of $\phi_i^R(t)$ at $t = 0$ to $\{R\} \times Y$ equals $\sum_{p>0} \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}(a_{i,p}^R(t, R)\ \psi_p(t))$.

Suppose $q > n$. Then (taking inner products in $L^2(\{R\} \times Y)$):

$$\begin{aligned} \left\langle \dot{\phi}_i^R(0), \psi_{-q}(0) \right\rangle &= \left\langle \sum_{p>0} \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \left(a_{i,p}^R(t, R)\, \psi_p(t)\right), \psi_{-q}(0) \right\rangle \\ &= \sum_{p>0} \left\langle \dot{a}_{i,p}^R(0, R)\psi_p(0) + a_{i,p}^R(0, R)\dot{\psi}_p(0), \psi_{-q}(0) \right\rangle \\ &= \sum_{p>0} \left\langle a_{i,p}^R(0, R)\dot{\psi}_p(0), \psi_{-q}(0) \right\rangle \\ &= \sum_{p>n} \left\langle \alpha_{i,p}^R \mathrm{e}^{-\mu_p R} \dot{\psi}_p(0), \psi_{-q}(0) \right\rangle. \end{aligned}$$

This last sum converges absolutely, since

$$\begin{aligned} &\left| \left\langle \alpha_{i,p}^R \mathrm{e}^{-\mu_p R} \dot{\psi}_p(0), \psi_{-q}(0) \right\rangle \right| \\ &\quad = \mathrm{e}^{-\mu_p R} \left|\alpha_{i,p}^R\right| \left| \left\langle \dot{\psi}_p(0), \psi_{-q}(0) \right\rangle \right| \\ &\quad = \mathrm{e}^{-\mu_p R} \left|\alpha_{i,p}^R\right| \left| \frac{1}{\mu_p(0) + \mu_q(0)} \left\langle \widehat{C}_1 \psi_p(0), \psi_{-q}(0) \right\rangle \right| \\ &\quad \leqslant \mathrm{e}^{-\mu_{n+1} R} \frac{1}{2\mu_{n+1}} \left\| \widehat{C}_1 \right\| \left|\alpha_{i,p}^R\right| \end{aligned}$$

(note that μ_{n+1} is the smallest non-zero eigenvalue of $\widehat{D}_0$) and so the sum converges absolutely by Lemma 5.4.

The sum

$$\sum_{q>n} \sum_{p>n} \left\langle \alpha_{i,p}^R \mathrm{e}^{-\mu_p R} \dot{\psi}_p(0), \alpha_{l,q}^j \mathrm{e}^{-\mu_q R_j} \psi_{-q}(0) \right\rangle \tag{5.2}$$

also converges absolutely. In fact, using the same calculation as the previous estimate we can bound the sum of the absolute values of the terms by

$$\mathrm{e}^{-2\mu_{n+1}R}\frac{1}{2\mu_{n+1}}\,\|\widehat{C}_1\|\sum_{q>n}\sum_{p>n}\left|\alpha_{i,p}^R\right|\left|\alpha_{l,p}^R\right|.$$

Using Lemma 5.4 we conclude that there exists a constant K which is independent of i, l, and j so that the sum (5.2) is bounded by $K\mathrm{e}^{-2\mu_{n+1}R}$.

Notice that

$$\begin{aligned}&\sum_{q>n}\sum_{p>n}\left\langle\alpha_{i,p}^R\mathrm{e}^{-\mu_p R}\dot{\psi}_p(0),\alpha_{j,q}^R\mathrm{e}^{-\mu_q R}\psi_{-q}(0)\right\rangle\\&\quad=\left\langle\sum_{p>n}\alpha_{i,p}^R\mathrm{e}^{-\mu_p R}\dot{\psi}_p(0),\sum_{q>n}\alpha_{j,q}^R\mathrm{e}^{-\mu_q R}\psi_{-q}(0)\right\rangle\\&\quad=\left\langle\dot{\phi}_i^R(0),\sigma\left(\phi_j^R(0)\right)\right\rangle.\end{aligned}$$

Hence

$$\left|\left\langle\dot{\phi}_i^R(0),\sigma\left(\phi_j^R(0)\right)\right\rangle\right|\leqslant K\mathrm{e}^{-2\mu_{n+1}R}$$

for some constant K independent of i, j or R.

To simplify notation let ϕ_i^R denote $\phi_i^R(0)$ for the rest of the proof. Returning to equation (5.1) we see that

$$\left|\dot{\lambda}_i^R\delta_{i,j}-B\left(\phi_i^R,\phi_j^R\right)\right|\leqslant K\mathrm{e}^{-2\mu_{n+1}R}.$$

This estimate suggests that the $\dot{\lambda}_i^R$ are approaching the eigenvalues of B.

To prove this fact, we first show that the ϕ_i^R are almost orthonormal on $X(\infty)$. Indeed,

$$\begin{aligned}\left\langle\phi_i^R,\phi_j^R\right\rangle\big|_{X(\infty)}&=\left\langle\phi_i^R,\phi_j^R\right\rangle\big|_{X(R)}+\left\langle\phi_i^R,\phi_j^R\right\rangle\big|_{[R,\infty)\times Y}\\&=\delta_{i,j}+\left\langle\phi_i^R,\phi_j^R\right\rangle\big|_{[R,\infty)\times Y}.\end{aligned}$$

Again we can easily estimate $|\langle\phi_i^R,\phi_j^R\rangle|_{[R,\infty)\times Y}|<K\mathrm{e}^{-\mu_{n+1}R}$ for some K independent of i, j, or R.

Theorem 5.2 now follows from the next lemma.

Lemma 5.6. *For each $\varepsilon>0$ there exists an $R_\varepsilon>1$ so that for all $R>R_\varepsilon$,*

$$\inf_{s\in S_m}\left(\sup_i\left|\tau_i(B)-\dot{\lambda}_{s(i)}^R\right|\right)<\varepsilon.$$

Remark. Here S_m is just the permutation group. Hence this lemma says that for R large enough, the set $\{\dot{\lambda}_i^R\}$ is arbitrarily close to the set of eigenvalues of B.

Proof. Suppose not. Then there exists an $\varepsilon>0$ and an unbounded sequence $R_1<R_2<\cdots$ so that

$$\inf_{s\in S_m}\left(\sup_i\left|\tau_i(B)-\dot{\lambda}_{s(i)}^{R_j}\right|\right)>\varepsilon$$

for all j. Choose a subsequence so that $\phi_i^{R_c}$ converges for each c, say to ϕ_i^∞. Choose a further subsequence so that the infimum is realized by the same permutation, which we may assume is the identity by reindexing.

The limit $\lim_{c\to\infty} B(\phi_i^{R_c}, \phi_j^{R_c})$ equals $B(\phi_i^\infty, \phi_j^\infty)$, since

$$B(x, y) = \langle \dot{D}_0 x, y\rangle_{X(\infty)}.$$

On the other hand this limit equals the limit

$$\lim_{c\to\infty} \dot{\lambda}_i^{R_c}\delta_{i,j} + f_{ij}(R_c)$$

for some exponentially decreasing function $f_{ij}(R)$. Hence the limit $\lim_{c\to\infty} \dot{\lambda}_i^{R_c}$ exists; call it ζ_i. Then:

$$B\left(\phi_i^\infty, \phi_j^\infty\right) = \zeta_i\,\delta_{i,j}.$$

A similar argument shows that the $\{\phi_i^\infty\}$ are an orthonormal basis, and hence the eigenvalues of B are just the ζ_i. But then

$$0 = \sup_i \left|\zeta_i - \tau_i(B)\right| = \lim_{c\to\infty} \sup_i \left|\dot{\lambda}_i^{R_c} - \tau_i(B)\right| > \varepsilon,$$

a contradiction.

This concludes the proof of the lemma, and of Theorem 5.2. □

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