On projected embeddings and desuspending the $\alpha$-invariant

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Abstract

A map $f : K \to L$ is called a projected embedding from $L \times B^s$ if there is an embedding $F : K \to L \times B^s$ such that $f = \pi \circ F$, where $\pi : L \times B^s \to L$ is the projection. A map $f : S^p \sqcup S^q \to S^m$ is a link map if $f(S^p) \cap f(S^q) = \emptyset$. We apply projected embeddings to desuspending the $\alpha$-invariant of link maps and to embeddings of double covers into Euclidean space.

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A map $f : K \to L$ is called a projected embedding from $L \times B^s$ if there is an embedding $F : K \to L \times B^s$ such that $f = \pi \circ F$, where $\pi : L \times B^s \to L$ is the projection. A map $f : X \sqcup Y \to Z$ is a link map if $f(X) \cap f(Y) = \emptyset$. In this paper we apply projected embeddings to desuspending the $\alpha$-invariant of link maps (Theorem 1) and to embeddings of double covers into Euclidean space (Theorem 3). For an introduction and motivation see [9,14,12], [16, Question on p. 152], [17, §6], [22,2].

We shall work in the smooth category. Let $EM_{pq}^m$ be the set of link maps $S^p \sqcup S^q \to S^m$ which embed $S^p$ standardly in the PL category (note that any embedding $S^p \to S^m$ is

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PL standard for \( m \geq p + 3 \) [7]). Let \( \lambda : EM^m_{pq} \to \pi_q(S^{m-p-1}) \) be the linking coefficient. A link concordance between link maps \( f_0, f_1 : S^p \sqcup S^q \to S^m \) is a link map

\[
F : S^p \times I \sqcup S^q \times I \to S^m \times I
\]
such that \( F(x, 0) = f_0(x) \) and \( F(x, 1) = f_1(x) \). The link concordance does not necessarily embed \( S^p \times I \).

**Theorem 1.** Denote \( k = 2p + 1 - m \). The mapping \( a = \Sigma^k \lambda : EM^m_{pq} \to \pi_{k+q}(S^p) \) is a link concordance invariant, provided \( \frac{4p}{2} + 1 \leq m \leq 2p \) and the binomial coefficient \( (m-k) \) is odd.

Clearly, \( \text{im} \ a = \Sigma^k \pi_q(S^{m-p-1}) \) and \( \Sigma^{q+3-m} a = \Sigma^\infty \lambda \) is the well-known \( \alpha \)-invariant [8,10], see also [19,21]. Thus Theorem 1 for \( q \geq m - 1 \) together with examples of non-surjectivity and non-injectivity of a non-stable suspension homomorphism gives examples of non-surjectivity and non-injectivity of the \( \alpha \)-invariant. Theorem 1 is not interesting for \( q \leq m - 2 \); for \( q \leq m - 3 \) the \( a \)-invariant is a suspension of the \( \alpha \)-invariant, and for \( q = m - 2 \) we have

\[
\text{im} \ a = \ker(h : \pi_{2p-1}(S^p) \to \mathbb{Z}(p)) \cong \pi_{2p}(S^{p+1}) \cong \pi_{p-1}^2,
\]
and \( a \) gives no more information than \( \alpha \).

Denote by \( LM^m_{pq} \) the set of link maps \( S^p \sqcup S^q \to S^m \), up to the link concordance. In [9, 14] an invariant \( a' : LM^m_{pq} \to \pi_{k+q+1}(S^{p+1}) \) was constructed such that \( \Sigma^{q+2-m} a' = \alpha \) (note that the concordance invariance of \( \Sigma a = a' \) follows analogously to Lemma 2 below, since \( S^p \times I \) embeds into \( S^m \times I \times \mathbb{R}^{k+1} \) by general position).

The desuspension of \( \alpha \) given by Theorem 1 is stronger in the sense that \( a' = \Sigma a \) but weaker in the sense that \( a \) is defined only on \( EM^m_{pq} \) not on \( LM^m_{pq} \). It would be interesting to know if \( EM^m_{pq} \) in Theorem 1 can be replaced by \( LM^m_{pq} \) (we can approximate the composition \( S^p \to S^m \to S^m \times \mathbb{R}^k \) by embeddings, but it remains to prove that our invariant will not depend on this approximation).

For \( m = 2p \geq 6 \) and \( q \leq 3p - 6 \) Theorem 1 (with the invariant defined even on \( LM^m_{pq} \)) follows from [27, Proposition F], and was also stated without proof in [15]. Nezhinskij outlined a geometric proof of this simplest case of Theorem 1 (without the restriction \( q \leq 3p - 6 \) at the Alexandrov Session in 1999, but with the invariant defined on \( EM^m_{pq} \) not on \( LM^m_{pq} \)). Our proof of Theorem 1 extends his ideas.

**Proof of Theorem 1.** Suppose that

\[
F : S^p \times I \sqcup S^q \times I \to S^m \times I
\]
is a link concordance between \( F_0, F_1 : S^p \sqcup S^q \to S^m \) such that \( F|_{S^p \times [0,1]} \) is an embedding. Since there exists some proper framed immersion \( S^p \times I \sqcup S^q \times I \to S^m \times I \), we may assume by [6, 1.2.2], [1, Lemma 2] that \( F|_{S^p \times [0,1]} \) is a general position framed immersion.

By general position, \( F|_{S^p \times [0,1]} \) has no triple points. Therefore by Lemma 2 below for \( n = p + 1 \), there is an embedding

\[
\overline{F} : S^p \times I \to S^m \times I \times \mathbb{R}^k
\]
such that \( \pi \circ \overline{F} = F|_{S^p \times I} \), where

\[
\pi : S^m \times I \times \mathbb{R}^k \to S^m \times I
\]

is the projection.

We may assume that \( S^m \times I \times \mathbb{R}^k \subseteq \Sigma^k (S^m \times I) \) close to the base \( S^m \times I \subseteq \Sigma^k (S^m \times I) \). Let \( \overline{F}|_{\Sigma^k(S^p \times I)} = \Sigma^k F|_{S^p \times I} \). Since \( F(S^p \times I) \cap F(S^q \times I) = \emptyset \), it follows that

\[
\overline{F} : S^p \times I \sqcup \Sigma^k (S^q \times I) \to \Sigma^k (S^m \times I)
\]

is a link concordance, which embeds \( S^p \times I \), between \( \overline{F}_0 = \Sigma^k F_0 \) and \( \overline{F}_1 = \Sigma^k F_1 \). Therefore \( \Sigma^k \lambda(F_0) = \lambda(\overline{F}_0) = \lambda(\overline{F}_1) = \Sigma^k \lambda(F_1) \). \( \square \)

**Lemma 2.** If the binomial coefficient \( \binom{n-k}{k} \) is odd, \( N \) is an \( n \)-manifold and \( f : N \to B^{2n-k} \) is a proper general position framed immersion without triple points and such that \( f|_{\partial N} \) is an embedding, then \( f \) is a projected embedding from \( B^{2n-k} \times B^k \).

**Proof.** Let

\[
\Delta = \{ x \in B^{2n-k} : |f^{-1}x| \geq 2 \} \quad \text{and} \quad \tilde{\Delta} = \{ x \in N : |f^{-1}x| \geq 2 \}.
\]

Then \( \tilde{f} = f|_{\tilde{\Delta}} : \tilde{\Delta} \to \Delta \) is a double covering. Denote by \( \tilde{f} \) the line bundle associated with the double cover \( \tilde{f} \) and let \( w_1(\tilde{f}) \in H^k(\Delta, \mathbb{Z}_2) \) be the first Stiefel–Whitney class of this line bundle.

The normal bundle of \( \Delta \) in \( B^{2n-k} \) is isomorphic to \( (n-k) \oplus (n-k) \tilde{f} \). Hence

\[
\tilde{w}(\Delta) = (1 + w_1(\tilde{f}))^{n-k}, \quad \text{so} \quad 0 = \tilde{w}_k(\Delta) = \binom{n-k}{k} (w_1(\tilde{f}))^k = (w_1(\tilde{f}))^k
\]

cf. [3, proof of proposition].

By general position \( \dim \Delta = k \). Hence it follows by Theorem 3(a) below that \( \tilde{f} \) is a projected embedding from \( \Delta \times B^k \). This implies that \( f \) is a projected embedding from \( B^{2n-k} \times B^k \).

Indeed, take a map \( \tilde{g} : \tilde{\Delta} \to B^k \) such that \( \tilde{f} \times \tilde{g} : \tilde{\Delta} \to \Delta \times B^k \) is an embedding. Take a Riemannian metric on \( N \) such that 1-neighborhood \( U \) of \( \overline{A}(f) \) in \( N \) is a tubular neighborhood of \( \overline{\Delta} \) in \( N \). Let \( r : U \to \overline{\Delta} \) be the projection of the normal bundle. Define a map \( g : N \to B^k \) by \( g(x) = 0 \) for \( x \notin U \) and \( g(x) = (1 - \text{dist}(x, \overline{\Delta}))(\tilde{g}(r(x)) \) for \( x \in U \). Then \( f \times g : N \to N \times B^k \) is an embedding. \( \square \)

**Theorem 3.** Let \( \Delta \) be a \( k \)-manifold (closed or with boundary), \( \overline{\Delta} \) its double cover and \( \text{pr} : \overline{\Delta} \to \Delta \) the projection. Consider the following conditions:

1. \( \text{E} \) there exists an equivariant map \( g : \overline{\Delta} \to S^{n-1} \);
2. \( \text{P} \) \( \text{pr} \) is a projected embedding from \( \Delta \times B^k \);
3. \( \text{A} \) the composition \( \overline{\Delta} \xrightarrow{\text{pr}} \Delta \subset \Delta \times B^k \) is approximable by embeddings;
4. \( \text{W} \) \( (w_1(\text{pr}))^k = 0 \in H^k(\Delta, \mathbb{Z}_2) \).

Then \( \text{E} \iff \text{P} \implies \text{A} \implies \text{W} \). Moreover,
(a) if \( s = k \), then \( (E) \Leftrightarrow (P) \Leftrightarrow (A) \Leftrightarrow (W) \);

(b) if \( 2s \geq k + 3 \) and both \( \Delta \) and \( \tilde{\Delta} \) are parallelizable, then \( (E) \Leftrightarrow (P) \Leftrightarrow (A) \).

Note that in Theorem 3 (and below) \( \tilde{\Delta} \) and \( \Delta \) are arbitrary manifolds, not necessarily double point sets. By general position, all conditions of Theorem 3 hold for \( s > k \).

The implications \( (P) \Rightarrow (A) \) and \( (E) \Rightarrow (W) \) are obvious and well known. To prove \( (E) \Rightarrow (P) \), it suffices to observe that the map \( \text{pr} \times g : \tilde{\Delta} \to \Delta \times S^{s-1} \) is an embedding. To prove \( (P) \Rightarrow (E) \), take an embedding \( F = F \times F : \Delta \to \Delta \times B^s \) such that \( \pi \circ F = \text{pr} \) and define an equivariant map \( g : \tilde{\Delta} \to S^{s-1} \) by

\[
g(x) = \frac{F_2(x) - F_2(-x)}{|F_2(x) - F_2(-x)|}.
\]

To prove Theorem 3(a) it suffices to prove either \( (W) \Rightarrow (E) \) or \( (W) \Rightarrow (P) \). The implication \( (W) \Rightarrow (E) \) is a folklore result from obstruction theory. For completeness, we present below its proof which was kindly communicated to us by A. Volovikov. We also sketch a geometric proof of the implication \( (W) \Rightarrow (P) \). The proofs of \( (A) \Rightarrow (W) \), \( (W) \Rightarrow (P) \) and 5(b) below are based on the ideas of [26], [7, §11], [13], [1, proof of Lemma 3], [18, §5]. Theorem 3 should be compared to [5,24].

The following remark improves [16, Theorem 2], [17, Hacon’s remark in §6], see also [11,25].

**Remark 4.** The group \( \text{Spin}(r) \) embeds into Euclidean space with a trivial normal bundle in codimension

\[
s = \begin{cases} 
\lceil \frac{1}{2} - 1 + 2, & r = 2l (\dim \text{Spin}(r) = 2l^2 - l), \\
\lceil \frac{1}{2} + 1 + 2, & r = 2l - 1 (\dim \text{Spin}(r) = 2l^2 + l).
\end{cases}
\]

**Proof.** Let \( \Delta = SO(r) \) and \( \tilde{\Delta} = \text{Spin}(r) \). By [16, Theorem 1 and table on p. 154], \( \Delta \) embeds with trivial normal bundle in codimension \( \lceil \frac{1}{2} \rfloor \), and hence in any greater codimension.

By [16, lemma on p. 166], there is an equivariant map \( g : \tilde{\Delta} \to S^{s-1} \). Now Remark 4 follows from the implication \( (E) \Rightarrow (P) \) of Theorem 3 (since the embedding obtained there has a trivial normal bundle). \( \square \)

**Proof of \( (A) \Rightarrow (W) \) in Theorem 3.** We need the following two facts. For a general position immersion \( F : \tilde{\Delta} \to \Delta \times B^s \), \( \varepsilon \)-close to \( i \circ \text{pr} \), let

\[
\Sigma(F) = \left\{ x \in \Delta \times B^s \mid \text{there are } y, z \in \tilde{\Delta} \text{ such that } |y, z| > 5\varepsilon, \ F_y = F_z = x \right\}
\]

be the ‘far away double points’ immersed submanifold.

It is proved analogously to [26], [7, §11] that the class \( [\Sigma(F)] \in H_{k-3}(\Delta, \mathbb{Z}_2) \) does not depend on homotopy of \( F \) through maps, \( \varepsilon \)-close to \( i \circ \text{pr} \). It is proved analogously to [13] that this class is dual to \( (w_1(\text{pr}))^2 \) (it suffices to prove this for the case when \( \pi \circ F = \text{pr} \)). This implies \( (A) \Rightarrow (W) \). \( \square \)
Sketch of the proof of \((W) \Rightarrow (P)\) in Theorem 3(a). For \(s = 1\) the proof is obvious so assume that \(s \geq 2\). We may assume that \(\Delta\) is connected. If \(w_1(pr) = 0\), then there exists an equivariant map \(\tilde{\Delta} \to S^0\), hence \((E)\) and \((P)\) are true.

If \(w_1(pr) \neq 0\), then \(\Delta\) is connected. Take a general position immersion \(F: \tilde{\Delta} \to \Delta \times B^s\) such that \(\pi \circ F = pr\). Since \([\Sigma(F)] = (w_1(pr))^k = 0\), it follows that the number of double points of \(F\) is even. If \(k\) is even and \(\Delta\) is orientable, then the algebraic number of double points of \(F\) is zero by \([20, \text{Lemma 5}]\). Therefore, as in \([1, \text{proof of Lemma 3}]\), we can apply ‘projected version’ of the Whitney trick to eliminate double points of \(F\) and obtain an embedding \(F': \tilde{\Delta} \to \Delta \times B^s\) such that \(\pi \circ F' = pr\). \(\square\)

Proof of \((W) \Rightarrow (E)\) in Theorem 3(a). (A. Volovikov) We can assume without loss of generality that \(\tilde{\Delta}\) is connected. Let \(Z_2\) act on \(R^k\) by multiplication with \(-1\). An equivariant map \(\tilde{\Delta} \to S^{k-1}\) exists if and only if there exists a non-zero section of the bundle \(\tilde{\Delta} \times Z_2R^k \to \Delta\). We will show that the unique obstruction class to defining a non-zero section of this bundle is trivial and hence this bundle has a non-zero section.

If \(\tilde{\Delta}\) has nonempty boundary, then it is easy to see that the obstruction class lies in the obstruction class lies in the zero group. Suppose further that \(\tilde{\Delta}\) is closed.

First case: \(k\) is even. The unique obstruction class to defining a non-zero section lies in \(H^k(\Delta; Z_2)\) (coefficients in cohomology are not twisted since \(k\) is even). This obstruction class is zero by \([20, \text{Lemma 5}]\). This implies that the unique obstruction class also vanishes.

Second case: \(k\) is odd. In this case coefficients are twisted and we have the following Smith–Richardson sequence

\[
\cdots \to H^k(\Delta; Z_2) \to H^k(\tilde{\Delta}; Z_2) \to H^k(\Delta; \tilde{Z}_2) \to 0.
\]

This Smith–Richardson sequence (one of the two Smith–Richardson sequences) is induced by the short coefficient sequence \(0 \to Z \to pr_{\ast}Z \to \tilde{Z} \to 0\) of sheaves over \(\Delta\). Here \(Z_2\) is the constant sheaf over \(\Delta\) (with \(Z\) as a fiber), \(pr_{\ast}Z\) is the direct image of the constant sheaf \(Z\) over \(\tilde{\Delta}\) and \(\tilde{Z}\) is a subsequence of sheaf where the inclusion is defined on a fiber as \(m \to (m, m), m \in Z_2\). Note that \(H^i(\Delta; pr_{\ast}Z) = H^i(\tilde{\Delta}; Z_2)\).

It follows from this sequence that \(H^k(\Delta; \tilde{Z}_2)\) can be one of \(0, Z_2\) or \(Z\). Indeed, if \(\tilde{\Delta}\) is not orientable, then \(H^k(\tilde{\Delta}; Z_2)\) is either \(0\) or \(Z_2\). If \(\tilde{\Delta}\) and \(\Delta\) are orientable, then \(H^k(\Delta; \tilde{Z}_2) = Z_2\) because \(\tilde{\Delta} \to \Delta\) is a double cover. In the remaining case when \(\tilde{\Delta}\) is orientable and \(\Delta\) is not orientable we have \(H^k(\Delta; \tilde{Z}_2) = Z\).

The obstruction class obviously vanishes if \(H^k(\Delta; \tilde{Z}_2) = 0\). If \(H^k(\Delta; \tilde{Z}_2) = Z_2\), then \(H^k(\Delta; \tilde{Z}_2) \to H^k(\Delta; Z_2)\) is an isomorphism and we see that the obstruction class also vanishes. Finally, if \(H^k(\Delta; \tilde{Z}_2) = Z\), then the obstruction class again vanishes since the nonzero obstruction class has order 2 by \([20, \text{Lemma 5}]\). \(\square\)
Proof Theorem 3(b). It suffices to prove \((A) \Rightarrow (E)\). We shall construct an equivariant map \(\Sigma^k \widetilde{\Delta} \to S^{k+s-1}\). If \(k \leq 2(s-1) - 1\), then Theorem 2.5 of [4] implies \((E)\). Consider the natural action of \(\mathbb{Z}_2\) on \(\widetilde{\Delta}\) and denote it by \(x \mapsto -x\). Since both \(\Delta\) and \(\widetilde{\Delta}\) are parallelizable, there is a continuous family \([h_x : D^k \to \Delta]_{x \in \Delta}\) of embeddings such that \(h_x 0 = x\) and \(h_{-x} = -h_x\).

Denote by \(i : \Delta \to \Delta \times B^s\) the inclusion. Let \(F = F_1 \times F_2 : \widetilde{\Delta} \to \Delta \times B^s\) be an embedding sufficiently close to \(i \circ pr\). Since \(F\) is close to \(i \circ pr\), we may assume that \(F_1 h_x(\frac{t}{|t|}) \subset h_x(D^k)\). Therefore a map \(\phi : \widetilde{\Delta} \times D^k \to \Delta \times B^s\) is well-defined by the formula \(\phi(x, t) = (h_{-x}^{-1} F_1 h_x(t), F_2 h_x(t))\) see [18, Fig. 4]. Since \(F\) is an embedding, it follows that \(\phi\) does not identify antipodes \((x, t)\) and \((-x, -t)\). Extend \(\phi\) to \(\Sigma^k \Delta \cong \Delta \times D^k / \{\Delta \times \{t \mid t \in \partial D^k\}\}\) by

\[
\phi[x, t] = \begin{cases} 
\phi(x, t), & |t| \leq \frac{1}{2}, \\
(2 - \frac{2t}{|t|})\phi(x, \frac{t}{2|t|}) + \left(2 - \frac{t}{|t|} - 1\right)(t, 0), & |t| \geq \frac{1}{2}.
\end{cases}
\]

Since \(\phi(x, \frac{t}{2|t|})\) is close to \((\frac{t}{2|t|}, 0)\), it follows that the new map \(\phi\) does not identify antipodes. Hence we can obtain an equivariant map \(\Sigma^k \widetilde{\Delta} \to S^{k+s-1}\). \(\Box\)

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