MATHEMATICS Proceedings A 89 (3), September 29, 1986

Orthogonality of p-adic characters

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Communicated by Prof. T.A. Springer at the meeting of January 27, 1986

ABSTRACT

For an abelian topological group G and a non-archimedean complete valued field K necessary and sufficient conditions are derived in order that the K-valued characters on G form an orthogonal set with respect to the supremum norm (Theorems 2.1, 2.2, 3.1, 4.3). Examples of groups satisfying these conditions (for example $\mathbb{Q}_p$) are considered in § 5.

NOTATIONS AND TERMINOLOGY

Throughout this note, K is a non-archimedean nontrivially valued complete field with valuation $|\,|$ and residue class field k, G is an additively written topological abelian group. For a prime number p the field of the p-adic numbers is $\mathbb{Q}_p$, with valuation $|\,|_p$; $\mathbb{Z}_p := \{x \in \mathbb{Q}_p : |x|_p \leq 1\}$; C_p is the group of p elements. The characteristic of a field L is denoted char L. Let H be an abelian topological group. Then Hom (G,H) is the group of all continuous homomorphisms $G \to H$; $G_K^\wedge := \text{Hom}\ (G, \{x \in K : |x| = 1\})$ is the group of the K-valued *characters*.

DEFINITION. Let p be a prime number. G is *p-finite* if there is no sequence of open subgroups $G = H_0 \supset H_1 \supset H_2 \supset \cdots$ such that for each n the index $[H_n : H_{n+1}]$ equals p.

DEFINITION. Let $0 < c \leq 1$. A subset X of $G_K^\wedge$ is a *c-orthogonal set* if for each finite number of distinct elements $\alpha_1, \ldots, \alpha_n$ of X and for all $\lambda_1, \ldots, \lambda_n \in K$

$$\| \sum_{i=1}^{n} \lambda_i \alpha_i \|_\infty := \sup_{x \in G} | \sum_{i=1}^{n} \lambda_i \alpha_i(x) | \geq c \max_{1 \leq i \leq n} |\lambda_i|$$

A 1-orthogonal set is *orthonormal.*

For elementary analysis in K e.g. the properties of the K-valued functions exp and log (defined if char $K=0$) we refer to [1].

§ 1. TWO GENERAL PROPOSITIONS ON ORTHOGONALITY

PROPOSITION 1.1. *Let $\alpha_1, \ldots, \alpha_n \in G_K^\wedge$. Then $\{\alpha_1, \ldots, \alpha_n\}$ is an orthogonal set if and only if $i \neq j$ implies $\|\alpha_i - \alpha_j\|_\infty = 1$.*

PROOF. It suffices to consider the induction step $n-1 \to n$ of the "if" part of the statement, so let $\lambda_1, \ldots, \lambda_n \in K$ and $f = \sum_{i=1}^n \lambda_i \alpha_i$. We have

$$\|f\|_\infty \geq \sup_{s,x \in G} |f(s+x) - \alpha_n(s) f(x)| = \sup_{s,x \in G} | \sum_{i=1}^{n-1} \lambda_i (\alpha_i(s) - \alpha_n(s)) \alpha_i(x) |.$$

By the induction hypothesis the right hand side equals

$$\sup_{s \in G} \max_{1 \leq i \leq n-1} |\lambda_i| \, |\alpha_i(s) - \alpha_n(s)| = \max_{1 \leq i \leq n-1} |\lambda_i|$$

so that

$$\|f\|_\infty \geq \max_{1 \leq i \leq n-1} |\lambda_i|.$$

But also

$$|\lambda_n| = \|f - \sum_{i=1}^{n-1} \lambda_i \alpha_i\|_\infty \leq \max (\|f\|_\infty, \| \sum_{i=1}^{n-1} \lambda_i \alpha_i \|_\infty) \leq$$

$$\leq \max (\|f\|_\infty, \max_{1 \leq i \leq n-1} |\lambda_i|) \leq \|f\|_\infty$$

which finishes the proof.

PROPOSITION 1.2. *Let H be a closed subgroup of G, let $c_1, c_2 \in (0, 1]$. Suppose that $H_K^\wedge$ is a c_1-orthogonal set and that $(G/H)_K^\wedge$ is a c_2-orthogonal set. Then $G_K^\wedge$ is a $c_1 c_2$-orthogonal set.*

PROOF. Let $\pi: G \to G/H$ be the canonical surjection and let $\pi^\wedge: (G/H)_K^\wedge \to G_K^\wedge$ be defined by the formula $\pi^\wedge(\beta) = \beta \circ \pi$. Choose a full set R of representatives modulo H in G and a full set S of representatives modulo $\pi^\wedge((G/H)_K^\wedge)$ in $G_K^\wedge$. Then we have trivially

(i) $\sigma_1, \sigma_2 \in S$, $\sigma_1 \neq \sigma_2 \Rightarrow \sigma_1 \neq \sigma_2$ on H,
(ii) $\beta_1, \beta_2 \in (G/H)_K^\wedge$, $\beta_1 \neq \beta_2 \Rightarrow \beta_1 \circ \pi \neq \beta_2 \circ \pi$,
(iii) Each $x \in G$ has a unique representation $x = r + h$ ($r \in R$, $h \in H$),
(iv) Each $\alpha \in G_K^\wedge$ has a unique representation $\alpha = \sigma \cdot \pi^\wedge(\beta)$ ($\sigma \in S$, $\beta \in (G/H)_K^\wedge$).

Now let $f = \sum_{\alpha \in G_K^\wedge} \lambda_\alpha \alpha$ be a finite K-linear combination of characters of G.

We shall prove that $\|f\|_\infty \geq c_1 c_2 \max_\alpha |\lambda_\alpha|$. We have

$$f = \sum_{\sigma \in S} \sum_{\beta \in (G/H)_K^\wedge} \lambda_{\sigma,\beta} \sigma \cdot \pi^\wedge(\beta)$$

where $\lambda_{\sigma,\beta} := \lambda_{\sigma\pi^\wedge(\beta)}$. For each $r \in R$ we have, by (i) and the c_1-orthogonality of $H_K^\wedge$,

$$\begin{aligned}
\|f\|_\infty &\geq \sup_{h \in H} |\sum_\sigma \sum_\beta \lambda_{\sigma,\beta}\sigma(r+h)\beta(\pi(r+h))| \\
&= \sup_{h \in H} |\sum_\sigma (\sum_\beta \lambda_{\sigma,\beta}\sigma(r)\beta(\pi(r)))\sigma(h)| \\
&\geq c_1 \sup_\sigma |\sum_\beta \lambda_{\sigma,\beta}\sigma(r)\beta(\pi(r))| \\
&= c_1 \sup_\sigma |\sum_\beta \lambda_{\sigma,\beta}\beta(\pi(r))|.
\end{aligned}$$

By the c_2-orthogonality of $(G/H)_K^\wedge$ *and (ii) we have the further estimate*

$$\begin{aligned}
\|f\|_\infty &\geq c_1 \sup_\sigma \|\sum_\beta \lambda_{\sigma,\beta}\beta\|_\infty \\
&\geq c_1 c_2 \sup_\sigma \sup_\beta |\lambda_{\sigma,\beta}| = c_1 c_2 \max_{\alpha \in G^\wedge} |\lambda_\alpha|.
\end{aligned}$$

§ 2. THE CASE OF MIXED CHARACTERISTICS

Throughout § 2 we assume char $K=0$, char $k=p\neq 0$. Without loss of generality, $K \supset \mathbb{Q}_p$.

THEOREM 2.1. (Compare Theorem 4.3). *Suppose that K does not contain p th roots of unity except* 1. *Then the following are equivalent.*
(α) *$G_K^\wedge$ is orthonormal.*
(β) *$G_K^\wedge$ is c-orthogonal for some $c \in (0,1)$.*
(γ) Hom $(G, \mathbb{Z}_p) = (0)$.

PROOF. $(\alpha) \Rightarrow (\beta)$ is trivial. Assume (β). Let $\phi \in \text{Hom}\,(G,\mathbb{Z}_p)$. Choose $s \in K$ such that $0<|s|<\min\,(p^{1/(1-p)}, c)$. Then $\alpha: x \mapsto \exp\,(s\phi(x))$ $(x \in G)$ is a well defined element of $G_K^\wedge$. For $x \in G$ we have $|\alpha(x)-1| = |s\phi(x)| \leq |s| < c$ so that $\|\alpha - 1\|_\infty < c$. By (β) we then have $\alpha = 1$ implying $\phi(x)=0$ for all $x \in G$ and (γ) follows. Finally we prove $(\gamma) \Rightarrow (\alpha)$. Suppose (α) is not true. Then by Proposition 1.1 there exists an $\alpha \in G_K^\wedge$ with $0 < \|1-\alpha\|_\infty < 1$. Our assumption on K implies that the function log maps $\{x \in K : |1-x|<1\}$ injectively into the additive group K. By analyticity log is bounded on $\alpha(G)$. The ultrametric Hahn Banach Theorem ([1] A.8) yields a continuous $\mathbb{Q}_p$-linear map $\phi: K \to \mathbb{Q}_p$ that does not vanish on $(\log \circ \alpha)(G)$. Then $\phi \circ \log \circ \alpha$ is a continuous nontrivial homomorphism of G into a bounded subgroup of $\mathbb{Q}_p$. It follows that Hom $(G, \mathbb{Z}_p) \neq (0)$.

THEOREM 2.2. *Suppose that K contains the p th roots of unity. Then the following are equivalent.*

(α) $G_K^\wedge$ *is orthonormal.*
(β) Hom $(G, C_p) = (0)$.

PROOF. $(\alpha) \Rightarrow (\beta)$. It suffices to prove that $\alpha \in G_K^\wedge$, $\alpha^p = 1$ implies $\alpha = 1$. For each $x \in G$ we have $|1 - \alpha(x)| \leq p^{1/(1-p)} < 1$. Hence, by ($\alpha$), $\alpha = 1$. We proceed to prove $(\beta) \Rightarrow (\alpha)$. If ($\alpha$) is not true then by Proposition 1.1 there is an $\alpha \in G_K^\wedge$ such that $\tau := \|1 - \alpha\|_\infty$ is strictly between 0 and 1. We shall prove that Hom $(G, C_p) \neq (0)$. In fact, set $\tau' := \tau \max(\tau, 1/p)$. Then $H := \{x \in G: |\alpha(x) - 1| \leq \tau'\}$ is a proper open subgroup of G. We have $x \in G \Rightarrow |\alpha(x) - 1| \leq \tau \Rightarrow$ $\Rightarrow |\alpha(px) - 1| \leq \max(\tau, 1/p) |\alpha(x) - 1| \leq \tau'$. It follows that each nonzero element of the nontrivial discrete group G/H has order p. One easily obtains a homomorphism of G/H onto C_p. We see that Hom $(G, C_p) \neq (0)$.

THEOREM 2.3. *Suppose that, for each* $n \in \mathbb{N}$, *K contains the* p^n *th roots of unity. Then the following are equivalent.*
(α) $G_K^\wedge$ *is c-orthogonal for some* $c \in (0, 1)$.
(β) G *is p-finite.*

PROOF. $(\alpha) \Rightarrow (\beta)$. Suppose ($\alpha$) and G is not p-finite; we derive a contradiction. Let $n \in \mathbb{N}$ be such that $p^{-n+1} < c$. There is an open subgroup H of G for which $[G:H] = p^{-n}$. Then $F := G/H$ has p^n elements and so has $F_K^\wedge$ by our assumption on K. From

$$\sum_{\alpha \in F_K^\wedge} \alpha(x) = \begin{cases} 0 & \text{if } x \in F,\ x \neq 0 \\ p^n & \text{if } x \in F,\ x = 0 \end{cases}$$

it follows directly that $F_K^\wedge$ is not p^{-n+1}-orthogonal, hence not c-orthogonal. But then $G_K^\wedge$ is not c-orthogonal.

Now suppose (β). There is an open subgroup H of finite index such that Hom $(H, C_p) = (0)$. By Theorem 2.2 $H_K^\wedge$ is orthonormal. By finiteness and linear independence $(G/H)_K^\wedge$ is c-orthogonal for some $c \in (0, 1)$. Then, by Proposition 1.2, $G_K^\wedge$ is c-orthogonal.

PROBLEM. Is $(\alpha) \Rightarrow (\beta)$ true if we assume only that K contains the p th roots of unity?

§ 3. THE CASE char $k = 0$

THEOREM 3.1. *Let* char $k = 0$. *The following are equivalent.*
(α) $G_K^\wedge$ *is orthonormal.*
(β) $G_K^\wedge$ *is c-orthogonal for some* $c \in (0, 1)$.
(γ) Hom $(G, \mathbb{Q}) = (0)$ (*where* $\mathbb{Q}$ *carries the discrete topology*).

PROOF. $(\alpha) \Rightarrow (\beta)$ is trivial. Assume (β). Let $\phi \in$ Hom $(G, \mathbb{Q})$. Choose $s \in K$, $0 < |s| < c$. Then $\alpha: x \mapsto \exp(s\phi(x))$ $(x \in G)$ is in $G_K^\wedge$ and $\|1 - \alpha\|_\infty \leq |s| < c$. By ($\beta$) we have $\alpha = 1$ implying $\phi = 0$ and (γ) is proved. To prove $(\gamma) \Rightarrow (\alpha)$, suppose ($\alpha$)

is false. By Proposition 1.1 there is an $\alpha \in G_K^{\wedge}$ with $0 < \|1-\alpha\|_\infty < 1$. Then $T := (\log \circ \alpha)(G)$ is a nontrivial additive subgroup of K. Choose $t \in T$, $t \neq 0$ and let $\pi : K \to K/\{x \in K : |x| < |t|\}$ be the quotient map. Then $\pi(nt) \neq 0$ for all $n \in \mathbb{N}$. Thanks to the divisibility of $\mathbb{Q}$ there is a homomorphism $\phi : K/\{x \in K : |x| < |t|\} \to \mathbb{Q}$ mapping $\pi(t)$ into 1. We see that $\phi \circ \pi \circ \log \circ \alpha$ is a continuous nonzero homomorphism $G \to \mathbb{Q}$ so that $\text{Hom}\,(G, \mathbb{Q}) \neq (0)$.

REMARK. It is easy to see that condition (γ) is equivalent to the following. For each open subgroup H the quotient G/H is a torsion group.

§ 4. THE CASE char $K = p \neq 0$

This case has to be treated in a way different from the previous ones as we do not have a K-valued logarithm or exponential.

Let K be algebraically closed, char $K = p \neq 0$. The group $K^+ := \{x \in K : |1-x| < 1\}$ does not contain roots of unity except 1. For each $n \in \mathbb{N}$ and $a \in K^+$ there is a unique $b \in K^+$ for which $b^n = a$. We write $b = a^{1/n}$. In an obvious way we obtain a homomorphism

$$r \mapsto a^r$$

of $\mathbb{Q}$ into K^+ which is uniformly continuous with respect to the p-adic metric on $\mathbb{Q}$ since for $r_1, r_2 \in \mathbb{Q}$, $r_1 \neq r_2$

$$|a^{r_1} - a^{r_2}| = |a^{r_1 - r_2} - 1| = |a-1|^{|r_1 - r_2|_p^{-1}}$$

and therefore extends to a continuous homomorphism $\lambda \mapsto a^\lambda$ of $\mathbb{Q}_p$ into K. The easy proof of the following proposition is left to the reader.

PROPOSITION 4.1. *Let K be algebraically closed, let* char $K = p \neq 0$. *Then* $K^+ := \{x \in K : |x-1| < 1\}$ *has the structure of a Banach space over* $\mathbb{Q}_p$ *with respect to addition, scalar multiplication and norm defined respectively by*

$$(x, y) \mapsto xy \quad (x, y) \in K^+)$$

$$(\lambda, x) \mapsto x^\lambda \quad (\lambda \in \mathbb{Q}_p, x \in K^+)$$

$$\|x\| = \begin{cases} -(\log|1-x|)^{-1} & \text{if } x \in K,\ x \neq 1 \\ 0 & \text{if } x \in K,\ x = 1 \end{cases}$$

Furthermore, the norm topology equals the initial topology on K^+.

COROLLARY 4.2. *Let* char $K = p \neq 0$, *let* $s \in K$, $0 < |s-1| < 1$. *Then there exists a continuous homomorphism*

$$\{x \in K : |1-x| \leq |1-s|\} \to \mathbb{Z}_p$$

that maps s into 1.

PROOF. We may assume that K is algebraically closed. By Proposition 4.1 and the ultrametric Hahn Banach theorem there is a homomorphism

$\phi: K^+ \to \mathbb{Q}_p$ such that $\phi(s)=1$ and $|\phi(x)|_p \leq \|s\|^{-1}\|x\|$ for all $x \in K^+$, where $\| \ \|$ is the norm defined above. Then ϕ is continuous. If $x \in K$, $|1-x| \leq |1-s|$ then $\|x\| \leq \|s\|$ so $|\phi(x)|_p \leq \|s\|^{-1}\|s\| = 1$ i.e. $\phi(x) \in \mathbb{Z}_p$.

THEOREM 4.3. *Let* char $K = p \neq 0$. *The following are equivalent.*
(α) $G_K^\wedge$ *is orthonormal.*
(β) $G_K^\wedge$ *is c-orthogonal for some* $c \in (0,1)$.
(γ) Hom $(G, \mathbb{Z}_p) = (0)$.

PROOF. $(\beta) \Rightarrow (\gamma)$. Choose $a \in K$, $0 < |1-a| < c$. It is easily seen that the map $n \mapsto a^n$ extends continuously to an injection $x \mapsto a^x$ of $\mathbb{Z}_p$ into $\{x \in K: |1-x| \leq |1-a|\}$. Let $\phi \in$ Hom $(G, \mathbb{Z}_p)$. Then $\alpha: x \mapsto a^{\phi(x)}$ $(x \in G)$ is in $G_K^\wedge$ and $\|1-\alpha\|_\infty \leq |1-a| < c$. By ($\beta$) we have $\alpha = 1$, whence $\phi = 0$. To prove $(\gamma) \Rightarrow (\alpha)$ we may assume that K is algebraically closed. Suppose (α) is not true. By Proposition 1.1 there is an $\alpha \in G_K^\wedge$ such that $\tau := \|1-\alpha\|_\infty$ is strictly between 0 and 1. There is an $x \in G$ for which $|\alpha(x)-1| \leq \tau < |\alpha(x)-1|^{1/p}$; let $s \in K$ be such that $s^p = \alpha(x)$. For each $y \in G$ we have $|\alpha(y)-1| \leq \tau \leq |\alpha(x)-1|^{1/p} = |s^p - 1|^{1/p} = |s-1|$. By Corollary 4.2 there is a continuous homomorphism $\phi: \{x \in K: |1-x| \leq |1-s|\} \to \mathbb{Z}_p$ mapping s into 1. Then $\phi \circ \alpha \in$ Hom $(G, \mathbb{Z}_p)$ and since $\phi(\alpha(x)) = \phi(s^p) = p \neq 0$ we conclude that Hom $(G, \mathbb{Z}_p) \neq (0)$.

§ 5. COROLLARIES

The next two theorems can easily be obtained by modifying the proofs of the previous theorems in an obvious way. $(\mathbb{Z}_p)_d$ stands for the group $\mathbb{Z}_p$ with the discrete topology.

THEOREM 5.1. (Compare Theorems 2.1 and 4.3). *Let* char $k = p \neq 0$ *and suppose that K does not contain p th roots of unity except* 1. *Then the locally constant K-valued characters on G form an orthonormal set if and only if* Hom $(G, (\mathbb{Z}_p)_d) = (0)$.

THEOREM 5.2. (Compare Theorems 2.2 and 3.1). *Suppose either* char $k = 0$, *or* char $k = p \neq 0$ *and K contains the p th roots of unity. Then the locally constant K-valued characters form an orthonormal set if and only if all K-valued characters form an orthonormal set.*

Let us consider the class $\mathscr{C}$ of all G for which the K-valued characters form an orthonormal set for *any* choice of K. It is easily seen that each one of the following conditions $(\alpha)-(\gamma)$ is equivalent to $G \in \mathscr{C}$.
(α) *For each K the locally constant K-valued characters form an orthonormal set.*
(β) Hom $(G, \mathbb{Q}) = (0)$, Hom $(G, C_p) = (0)$ *for each prime p.*
(γ) *For each open subgroup* $H \neq G$ *of G the quotient* G/H *is an infinite torsion group.*

The class $\mathscr{C}$ is closed for products. If ϕ is a continuous homomorphism of G into an abelian group and $G \in \mathscr{C}$ then $\phi(G) \in \mathscr{C}$. If G is a dense subgroup of an abelian topological group G' then $G \in \mathscr{C}$ if and only if $G' \in \mathscr{C}$.

It follows that $\mathbb{Q}_p$, $\prod_{p \text{ prime}} \mathbb{Q}_p$, $\mathbb{Q}_p/\mathbb{Z}_p$, $\mathbb{Q}/\mathbb{Z}$ are in $\mathscr{C}$. (In [3] it is proved in a different way that the $\mathbb{C}_p$-valued characters of $\mathbb{Q}_p/\mathbb{Z}_p$ are orthonormal.) No compact zerodimensional group, except (0), is in $\mathscr{C}$.

We present some further examples.

PROPOSITION 5.3. *Let E be a locally convex space over $\mathbb{Q}_p$. Then for any K the K-valued characters on E form an orthogonal set.*

PROOF. Let H be an open subgroup of E, let $x \in E \setminus H$. There is an $n \in \mathbb{N}$ such that $p^n x \notin H$, $p^{n+1}x \in H$. Hence E/H is a torsion group. Set $y := p^n x$. Then $y, p^{-1}y, p^{-2}y, \ldots$ are mutually distinct modulo H. It follows that E/H is infinite.

From 5.3 we may conclude that the additive group of any valued field extension of $\mathbb{Q}_p$ is in $\mathscr{C}$. This leads to a question that is solved in the next proposition.

PROPOSITION 5.4. *Let L be a non-archimedean complete value field with residue class field l.*
(i) *If* char $L =$ char l *then no subgroup of L, except* (0), *is in $\mathscr{C}$.*
(ii) *If $L \supset \mathbb{Q}_p$ for some prime p then a closed subgroup of L is in $\mathscr{C}$ if and only if it is a vector space over $\mathbb{Q}_p$.*

PROOF. (i) Let H be a subgroup of L, let $s \in H$, $s \neq 0$. Set $H_1 := H/\{x \in H : |x| < |s|\}$. One easily establishes a nontrivial homomorphism of H_1 into $\mathbb{Q}$ if char $l = 0$, into C_p if char $l = p \neq 0$. Then $H \notin \mathscr{C}$.

(ii) By Proposition 5.3 it suffices to prove that a closed subgroup G of L that is not a vector space over $\mathbb{Q}_p$ is not in $\mathscr{C}$. By continuity G is a $\mathbb{Z}_p$-module i.e. G is a closed convex set. There is an $x \in L$ such that $x \notin G$, but $\lambda \in G$ for some $\lambda \in \mathbb{Q}_p$, $\lambda \neq 0$. By the geometric form of the ultrametric Hahn Banach Theorem (see, for example, [2]) there is a continuous $\mathbb{Q}_p$-linear function $\phi : L \to \mathbb{Q}_p$ such that $|\phi(G)|_p < 1$, $|\phi(x)| > 1$. Observe that $\phi(G)$ is a nontrivial bounded subgroup of $\mathbb{Q}_p$. We see that Hom $(G, \mathbb{Z}_p) \neq (0)$ so that $G \notin \mathscr{C}$.

We now turn to multiplicative groups in L. First we consider $L^+ := \{x \in L : |1 - x| < 1\}$.

PROPOSITION 5.5. *Let L be as in 5.4, L algebraically closed.*
(i) *If* char $L = p \neq 0$ *then $L^+ \in \mathscr{C}$.*
(ii) *If* char $l = 0$ *then $L^+ \notin \mathscr{C}$.*
(iii) *If $L \supset \mathbb{Q}_p$ then $L^+ \in \mathscr{C}$.*

PROOF. (i) This is a direct consequence of Proposition 4.1 and Proposition 5.3.

(ii) The log function maps L^+ homeomorphically into a bounded additive subgroup of L which is not in $\mathscr{C}$ by Proposition 5.4.

(iii) The function exp and log can, since the additive group L and the multiplicative group L^+ are divisible, be extended to continuous homomorphisms $\mathrm{EXP}: L \to L^+$ and $\mathrm{LOG}: L^+ \to L$ respectively. Set $\phi(x) := \mathrm{EXPLOG}\, x$ $(x \in L^+)$, set $G := \{\mathrm{EXP}\, x : x \in L\}$, $C_{p\infty} := \{x \in L^+ : x^{p^n} = 1$ for some $n \in \mathbb{N}\}$. For each $x \in L^+$, $x/\phi(x)$ is in $C_{p\infty}$. The formula

$$x = (x/\phi(x)) \cdot \phi(x)$$

yields a decomposition of L^+ as a direct product of $C_{p\infty}$ and G. But $C_{p\infty} \cong \mathbb{Q}_p/\mathbb{Z}_p \in \mathscr{C}$ and $G \simeq L \in \mathscr{C}$ (EXP is injective). Hence $L^+ \in \mathscr{C}$.

Finally we have (observe that a subgroup of $L^\times$ that belongs to $\mathscr{C}$ must lie in $\{x \in L : |x| = 1\}$ as $L^\times / \{x \in L : |x| = 1\}$ is isomorphic to the value group of L which is not in $\mathscr{C}$)

PROPOSITION 5.6. *Let L be as in* 5.5.

(i) *If* char $l = 0$ *then* $\{x \in L : |x| = 1\} \notin \mathscr{C}$.

(ii) *If* char $l = p \neq 0$ *then* $\{x \in L : |x| = 1\} \in \mathscr{C}$ *if and only if l is algebraic over the field of p elements.*

PROOF. (i) $\{x \in L : |x| = 1\}/L^+$ is the multiplicative group of a field with characteristic 0, which is not in $\mathscr{C}$ (as a discrete group).

(ii) If $\{x \in L : |x| = 1\} \in \mathscr{C}$ then $l^\times := \{x \in l : x \neq 0\}$, being a quotient of $\{x \in L : |x| = 1\}$, must be a torsion group so that $l^\times$ is algebraic over the prime field. Conversely, if $l^\times$ is algebraic it is easily seen that $l^\times \in \mathscr{C}$. By Proposition 5.5 we have $L^+ \in \mathscr{C}$. As $\{x \in L : |x| = 1\}/L^+ \simeq l^\times$ we have $\{x \in L : |x| = 1\} \in \mathscr{C}$ by Proposition 1.2.

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