Suns, Moons, and Quasi-Polyhedra

DAN AMIR*
Department of Mathematics, Tel-Aviv University, Tel-Aviv, Israel

AND

FRANK DEUTSCH†
Department of Mathematics, Pennsylvania State University
University Park, Pennsylvania, 16802

Communicated by Oved Shisha

Received November 6, 1970

DEDICATED TO PROFESSOR J. L. WALSH ON THE OCCASION OF HIS 75TH BIRTHDAY

1. INTRODUCTION

The important concept of a sun, which was first introduced by Efimov and Stečkin in [17], arises quite naturally in the general theory of approximation in normed linear spaces. We recall that a set $V$ is a sun iff whenever $v_0 \in V$ is a best approximation to some element $x$ ($\neq V$), then $v_0$ is a best approximation to every element on the ray from $v_0$ through $x$. Since every convex set has this property, a sun may be regarded as a generalization of a convex set. Vlasov [21] showed that in a smooth Banach space every proximinal sun is convex. (A brief proof of this will be given in Section 2). Perhaps the most famous unsolved problem in approximation theory is whether or not every Tchebycheff set in a Hilbert space is convex. In view of Vlasov’s result, this problem may be stated equivalently as “Is every Tchebycheff set in a Hilbert space a sun?” Brosowski [6] has shown that being a sun is equivalent to being a Kolmogorov set (cf. Theorem 2.4). Also, he and his colleagues have indicated a theory of approximation for such sets which closely parallels the known linear or convex theory (cf., e.g., [10]). In recent years, a number of writers have studied certain classes of suns (e.g., the so-called “regular sets” introduced by Brosowski [5]); these authors have tried to determine, among other things, those spaces in which every sun is a member of this class [3-7, 8, 10].

In the present work, we define the concept of a “moon”\(^1\), which is a

* The work of this author was performed at The Pennsylvania State University during the summer of 1969.
† This author was supported by a grant from the National Science Foundation.
1 Originally called “sign regular,” [9]. The present name was given on “Moonday,” July 21, 1969, for obvious reasons.
generalization of a sun. We are especially interested in determining those normed linear spaces in which every moon is a sun. Knowledge of such spaces is often quite useful in practice since it is generally much easier to verify that a given set is a moon than verify it is a sun. Our approach to this problem is via certain geometric properties of the points of the unit sphere, in particular being “nonlunar”, “strongly nonlunar”, or “quasi-polyhedral” (abbr. QP) (in order of decreasing generality).

Section 2 includes the basic definitions, notation, and a number of general results. The main result of that section (Theorem 2.18) states that if each point of the unit sphere is strongly nonlunar, then every moon is a sun. We observe (Theorem 2.22) that every point of the unit sphere is QP if the unit ball is a “convex polytope” in the sense of Maserick [19]. Further, the finite-dimensional spaces in which each point of the unit sphere is QP are precisely those whose unit ball is polyhedral (Theorem 2.19). In Section 3 we consider certain product spaces. We prove, for example (Theorem 3.2), that each point of the unit sphere of the c_0-product of normed spaces is strongly nonlunar (or QP) iff each of the component spaces has the same property. The space C_0(T), T locally compact Hausdorff, is studied in Section 4. The main results there (Theorems 4.1 and 4.4) may be summarized as follows: Each point of the unit sphere in C_0(T) is strongly nonlunar; each point is QP iff T is discrete. In Section 5 a similar study is made of the space L_1(T, Σ, μ), where (T, Σ, μ) is σ-finite. The main results there (Theorems 5.4 and 5.6) may be stated as: Each point of the unit sphere in L_1(T, Σ, μ) is strongly nonlunar iff T is purely atomic; each point is QP iff T is a finite union of atoms. In Section 6 we remark about certain related matters and pose some open problems. In particular, we observe a certain close relationship (Theorem 6.3) between the QP property, property (P) of Brown [12], and property Q of Deutsch and Lindahl [15].

2. Notation, Definitions, and Some General Results

Let X be a real normed linear space, X* its dual space,

\[ B(x, r) = \{ y \in X : \| x - y \| < r \}, \text{ and } S(X) = \{ x \in X : \| x \| = 1 \} \]

For any x ∈ X, we define the peak set of x by

\[ P(x) = \{ x^* \in S(X^*) : x^*(x) = \| x \| \} \]

Given v_0, x ∈ X, we define the (open) cone of support at v_0 in the direction x, by

\[ K(v_0, x) = \{ v \in X : x^*(v - v_0) < 0 \ \forall \ x^* \in P(v_0 - x) \} \]

\[ = \{ v \in X : x^*(v - x) < \| v_0 - x \| \ \forall \ x^* \in P(v_0 - x) \}^c \]

\[ \text{Observet that } K(v_0, x) = \{ v \in X : x^*(v - v_0) > 0 \ \forall x^* \in P(x - v_0) \}, \text{ a fact which is sometimes useful.} \]
Since $P(v_0 - x)$ is a weak* compact convex extremal subset of $S(X^*)$, we can restrict ourselves, in the definition of $K(v_0, x)$, to those $x^* \in \text{ext } P(v_0 - x)$. (Here, and in the sequel, "ext" is an abbreviation for "the set of extreme points of.") In dealing with more than one normed linear space, we shall often use subscripts to emphasize the space in which we consider the ball, cone, etc.; e.g., $B_x(x, r)$, $K_x(v_0, x)$, etc.

There is a useful alternate representation for $K(v_0, x)$.

**Lemma 2.1.** $K(v_0, x) = \bigcup_{\lambda > 0} B(v_0 + \lambda(x - v_0), \lambda \| v_0 - x \|)$.

**Proof.** If $\| v - v_0 - \lambda(x - v_0) \| < \lambda \| v_0 - x \|$, then for any $x^* \in P(v_0 - x)$,

$$\lambda \| v_0 - x \| \geq x^*[v - v_0 - \lambda(x - v_0)]$$

$$= x^*(v - v_0) + \lambda \| v_0 - x \| ;$$

so $x^*(v - v_0) < 0$ and $v \in K(v_0, x)$.

Conversely, let $v \in K(v_0, x)$. The open line segment $(v_0, v)$ must intersect $B(x, \| v_0 - x \|)$ for, otherwise, by the Eidelheit separation theorem, we could find an $x^* \in P(v_0 - x)$ with $x^*(v - v_0) \geq 0$, which contradicts the choice of $v$. Choose $0 < \lambda < 1$ such that $z = \lambda v_0 + (1 - \lambda) v$ satisfies $\| z - x \| < \| v_0 - x \|$. Taking $0 < \alpha < 1/(1 - \lambda)$, we obtain

$$\| v - [v_0 + \alpha(x - v_0)] \| - \frac{1}{1 - \lambda} \| z - x \| < \alpha \| v_0 - x \| .$$

Thus $v \in B(v_0 + \alpha(x - v_0), \alpha \| v_0 - x \|)$.

**Corollary 2.2.** If $x_1 = v_0 + \lambda(x - v_0)$ for some $\lambda > 0$, then $K(v_0, x_1) = K(v_0, x)$.

**Definitions 2.3.** A set $V \subset X$ is called a Kolmogorov set iff whenever $v_0 \in V$ is a best approximation to $x \in X$, then

$$\min_{x^* \in P(x - v_0)} x^*(v - v_0) \leq 0 \quad \forall v \in V .$$

The set $V$ is called a sun iff whenever $v_0 \in V$ is a best approximation to $x \in X$, then $v_0$ is also a best approximation to $v_0 + \lambda(x - v_0) \forall \lambda \geq 0$, i.e. (if $x \neq v_0$), to each point on the ray from $v_0$ through $x$.

An interesting exposition on Kolmogorov sets was given by Brosowski [8]. It is easy to show that the condition (K) is always sufficient for $v_0$ to be a best approximation to $x$. The necessity of condition (K) was recently discussed by Brosowski and Wegmann [10]. The concept of a sun was introduced by Efimov and Stečkin [17] and further developed by Vlasov [21] (cf. also the encyclopedic monograph of Singer [20]).
THEOREM 2.4. Let $V \subseteq X$. The following are equivalent.

(1) $V$ is a Kolmogorov set.

(2) $V \cap K(v_0, x) = \emptyset$ whenever $v_0 \in V$ is a best approximation to $x$.

(3) $V$ is a sun.

Proof. (1) $\Rightarrow$ (2). Let $v_0 \in V$ be a best approximation to $x$. By hypothesis,
\[
\max_{x^* \in P(v_0 - x)} x^*(v - v_0) > 0 \quad \forall v \in V.
\]
On the other hand,
\[
K(v_0, x) = \{v : x^*(v - v_0) < 0 \ \forall x^* \in P(v_0 - x)\}
\]
and so $V \cap K(v_0, x) = \emptyset$.

(2) $\Rightarrow$ (3). Let $v_0 \in V$ be a best approximation to $x$ and let $\lambda > 0$. If
\[
x_1 = v_0 + \lambda(x - v_0)
\]
then $K(v_0, x_1) = K(v_0, x)$ by Corollary 2.2 and so $K(v_0, x_1) \cap V = \emptyset$. From Lemma 2.1 we obtain, in particular, that
\[
V \cap B(x_1, \| x_1 - v_0 \|) = \emptyset
\]
and so $v_0$ is a best approximation to $x_1$.

(3) $\Rightarrow$ (1). Let $v_0 \in V$ be a best approximation to $x$ and let $v \in V$. If
\[
x^*(v - v_0) > 0 \ \forall x^* \in P(x - v_0),
\]
then $v \in K(v_0, x)$, and so
\[
v \in B(v_0 + \lambda(x - v_0), \lambda \| x - v_0 \|)
\]
for some $\lambda > 0$. Thus
\[
\| v_0 + \lambda(x - v_0) - v \| < \lambda \| x - v_0 \| = \| v_0 + \lambda(x - v_0) - v_0 \|,
\]
which contradicts the hypothesis that $V$ is a sun. Hence
\[
\min_{x^* \in P(x - v_0)} x^*(v - v_0) \leq 0. \quad \Box
\]

The equivalence of (1) and (3) in Theorem 2.4 had been proved earlier by Brosowski [6] by a different method.

A normed linear space $X$ is called smooth if there is a unique supporting hyperplane to the unit sphere at each point, i.e., if $P(v_0)$ is a singleton for each $v_0 \in S(X)$. A subset $V$ of $X$ is called proximinal if each $x \in X$ has at least one best approximation in $V$. We can now give a new short proof of a well-known result of Vlasov (cf. [21; or 20, p. 344]).
Theorem 2.5. Let $X$ be a smooth normed linear space. Then each proximinal sun is convex.

Proof. Let $V$ be a proximinal sun. If $V$ is not convex, there exist $v_1, v_2 \in V$ such that $x = \lambda v_1 + (1 - \lambda) v_2 \notin V$ for some $0 < \lambda < 1$. Let $v_0 \in V$ be a best approximation to $x$. Let $\{x^*\} = P(v_0 - x)$. By Theorem 2.4, $V \cap K(v_0, x) = \emptyset$, and so $x^*(v_i - v_0) \geq 0$ for $i = 1, 2$. Thus

$$0 < \| v_0 - x \| = x^*(v_0 - x) = \lambda x^*(v_0 - v_1) + (1 - \lambda) x^*(v_0 - v_2) \leq 0,$$

a contradiction. □

Using Theorem 2.4, one can also easily verify the known fact that every convex set is a sun. (It is easy to construct examples of nonconvex suns.)

Theorem 2.4 suggests (at least) one way of generalizing the concept of a sun.

Definition 2.6. Let $V \subset X$. A point $v_0 \in V$ is called a lunar point if $x \in X$ and $V \cap K(v_0, x) \neq \emptyset$ imply $v_0 \in V \cap K(v_0, x)$. (As a consequence of the next lemma, we may assume in this definition that $x$ has $v_0$ as a best approximation from $V$.) $V$ is called a moon if each of its points is lunar.

Lemma 2.7. Let $V \subset X$ and $v_0 \in V$. The following are equivalent:

1. $v_0$ is a lunar point.
2. Whenever $v_0$ is a best approximation to an $x \in X$ with $V \cap K(v_0, x) \neq \emptyset$, then $v_0 \in \overline{V \cap K(v_0, x)}$.

Proof. (1) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (1). Let $x \in X$ and $V \cap K(v_0, x) \neq \emptyset$. We have to show $v_0 \in \overline{V \cap K(v_0, x)}$.

If $v_0$ is not a local best approximation to $x$ (i.e., if $\forall \epsilon > 0$ there is a $v_\epsilon \in V$ such that $\| v_\epsilon - v_0 \| < \epsilon$ and $\| v_\epsilon - x \| < \| v_0 - x \|$), then

$$v_\epsilon \in B(x, \| v_0 - x \|) \subset K(v_0, x),$$

so $v_0 \in \overline{V \cap K(v_0, x)}$. Thus we can assume $v_0$ is a best approximation to $x$ from $V \cap B(v_0, \epsilon)$ for some $\epsilon > 0$. Let $y = v_0 + \lambda(x - v_0)$ where $0 < \lambda < \epsilon/2 \| v_0 - x \|$.

Then $K(v_0, y) = K(v_0, x), \| y - v_0 \| < \epsilon/2$, and $v_0$ is a best approximation to $y$ from $V$. Thus $v_0 \in V \cap K(v_0, y) = \overline{V \cap K(v_0, x)}$. □

Corollary 2.8. Every sun is a moon.
This follows from Theorem 2.4 and Lemma 2.7.

In the important special case $V = S(X)$, the definition of a lunar point of $V$ can be somewhat simplified. Indeed, $v_0 \in S(X)$ is a lunar point iff for each $x \in B(0, 1)$ having $v_0$ as a best approximation from $S(X)$, $v_0 \in K(v_0, x) \cap S(X)$. To shorten the writing, we define, for each $v_0 \in S(X)$,

$$
\mathcal{A}(v_0) = \{x \in B(0, 1) : v_0 \text{ is a best approximation to } x \text{ from } S(X)\}
$$

$$
= \{x \in B(0, 1) : \|v_0 - x\| = 1 - \|x\|\}
$$

$$
= \{x \in B(0, 1) : v_0 = x + (1 - \|x\|)u \text{ for some } u \in S(X)\}.
$$

Thus $v_0 \in S(X)$ is a lunar point iff $v_0 \in K(v_0, x) \cap S(X) \forall x \in \mathcal{A}(v_0)$.

**DEFINITIONS 2.9.** Let $v_0 \in S(X)$.

(a) $v_0$ is called a nonlunar point of $S(X)$ if it is not a lunar point, i.e., if there is some $x \in B(0, 1)$ such that $v_0 \notin K(v_0, x) \cap S(X)$.

(b) $v_0$ is called a strongly nonlunar point of $S(X)$ if for each $u \in K(v_0, 0)$ there is an $x \in B(0, 1)$ such that $u \in K(v_0, x)$ and $v_0 \notin K(v_0, x) \cap S(X)$. The space $X$ is called strongly nonlunar if each $v_0 \in S(X)$ is strongly nonlunar.

(c) $v_0$ is called a quasi-polyhedral (abbr. $QP$) point of $S(X)$ if

$$
v_0 \notin K(v_0, 0) \cap S(X).
$$

$X$ is called a $QP$-space if each $v_0 \in S(X)$ is $QP$.

It should be noted that (by an argument similar to that used in the proof of Lemma 2.7) the $x \in B(0, 1)$ which appears in the definitions of nonlunar and strongly nonlunar points may be restricted to lie in $\mathcal{A}(v_0)$. We leave to the reader the straightforward task of verifying that the $QP$ property is hereditary (i.e., if $X$ is $QP$, so is every subspace of $X$). On the other hand, Theorem 4.1 shows that strong nonlunarity is not a hereditary property.

In verifying whether a given point is nonlunar, strongly nonlunar, or $QP$, it is useful to observe that if $v_0 \in S(X)$ and $x \in B(0, 1)$, the following two conditions are equivalent:

1. $v_0 \notin K(v_0, x) \cap S(X)$.
2. There exists an $\epsilon > 0$ such that $B(v_0, \epsilon) \cap K(v_0, x) \subseteq B(0, 1)$.

**THEOREM 2.10.** Let $v_0 \in S(X)$ and consider the following three statements:

1. $v_0$ is $QP$.
2. $v_0$ is strongly nonlunar.
3. $v_0$ is nonlunar.

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3).
In addition, if any one of the following three conditions holds, then (3) ⇒ (1), and so, all three statements above are equivalent.

(a) \( v_0 \in \text{ext } S(X) \).

(b) \( X \) is two-dimensional.

(c) \( X \) is smooth.

Proof. The implications (1) ⇒ (2) ⇒ (3) are trivial (e.g., for (1) ⇒ (2), take \( x = 0 \)). Now suppose \( v_0 \) is nonlunar. Then there is an \( x \in \mathcal{A}(v_0) \) such that \( v_0 \notin K(v_0, x) \cap S(X) \). We shall show that if any one of the conditions (a), (b), or (c) is satisfied, then \( v_0 \) is QP. This will be the case, in particular, when \( K(v_0, x) = K(v_0, 0) \).

Case 1. \( v_0 \in \text{ext } S(X) \).

Since \( v_0 = x + (1 - \| x \|) u \) for some \( u \in S(X) \), it follows that either \( x = 0 \) or \( v_0 = \| x \| x/\| x \| + (1 - \| x \|) u \). If the latter is true, then \( x/\| x \| = u = v_0 \). Hence \( x = \| x \| v_0 \) and, in particular, \( K(v_0, x) = K(v_0, 0) \).

Case 2. \( X \) is two-dimensional.

We may assume \( v_0 \notin \text{ext } S(X) \). Then \( v_0 \) must be interior to some line segment \( L(v_0) \) in \( S(X) \). In particular, \( v_0 \) is a smooth point,

\[ K(v_0, 0) \cap S(X) \subset S(X) \sim L(v_0), \]

and so, \( v_0 \notin K(v_0, 0) \cap S(X) \), i.e., \( v_0 \) is QP.

Case 3. \( X \) is smooth.

The proof in this case, and hence the theorem, will follow immediately from (3) of the following lemma.

**Lemma 2.11.** Let \( v_0 \in S(X) \) and \( x \in \mathcal{A}(v_0) \). Then:

1. \( P(v_0) = P(v_0 - x) \cap P(x) \).
2. \( K(v_0, x) \subset K(v_0, 0) \).
3. If \( X \) is smooth, \( K(v_0, x) = K(v_0, 0) \).

**Proof of the Lemma.** (1) Let \( x^* \in P(v_0) \). Then

\[ \| v_0 - x \| + \| x \| = \| v_0 \| = x^*(v_0) = x^*(v_0 - x) + x^*(x) \]

\[ \leq \| v_0 - x \| + \| x \| \]

and so, \( x^* \in P(v_0 - x) \cap P(x) \). Conversely, suppose \( x^* \in P(v_0 - x) \cap P(x) \). Then

\[ x^*(v_0) = x^*(v_0 - x) + x^*(x) = \| v_0 - x \| + \| x \| = 1, \]

so \( x^* \in P(v_0) \).
(2) From (1) we obtain \(P(v_0) \subseteq P(v_0 - x)\) and so, \(K(v_0, x) \subseteq K(v_0, 0)\).

(3) If \(X\) is smooth, then \(P(y)\) is a singleton for each \(0 \neq y \in X\); so by (1) we obtain \(P(v_0) = P(v_0 - x)\) and hence \(K(v_0, 0) = K(v_0, x)\).

This proves Lemma 2.11 and hence completes the proof of Theorem 2.10. 

Remark 2.12. For a two-dimensional space, we have shown that the concepts "nonlunar," "strongly nonlunar," and "QP" are the same. There exists, however, a three-dimensional space which contains nonlunar points which are not strongly nonlunar \([11]\). Also, we shall see later that there are infinite-dimensional strongly nonlunar spaces which are not QP. However, it is an open question whether there are finite-dimensional spaces with this property.

During the course of the proof of Case 2 in Theorem 2.10, we have actually verified the following result:

**Lemma 2.13.** Let \(X\) be two-dimensional and \(v_0 \in S(X)\). If \(v_0\) is lunar, then \(v_0 \in \text{ext } S(X)\).

It is clear that \(S(X)\)—or, for that matter, any symmetric subset of \(S(X)\)—is never a sun. On the other hand, with the aid of Theorem 2.10, we can give certain conditions which insure that \(S(X)\) is a moon.

**Theorem 2.14.** If \(X\) is strictly convex, then \(S(X)\) is a moon.

**Proof.** Let \(v_0 \in S(X)\) and \(x \in \partial(v_0)\). By the strict convexity, \(x = \|x\|v_0\), and so \(K(v_0, x) = K(v_0, 0)\). Since each \(x^* \in P(v_0)\) attains its norm on \(S(X)\) only at \(v_0\), it follows that \(x^*(v) < 1 = x^*(v_0) \forall v \in S(X) \sim \{v_0\}\). Thus 
\[
K(v_0, x) \cap S(X) = K(v_0, 0) \cap S(X) = S(X) \sim \{v_0\},
\]
and so \(v_0 \in \overline{K(v_0, x)} \cap S(X)\), i.e., \(v_0\) is a lunar point.

By combining Lemma 2.13 and Theorem 2.14, we obtain

**Corollary 2.15.** Let \(X\) be two-dimensional. Then \(S(X)\) is a moon if and only if \(X\) is strictly convex.

A set \(E \subseteq S(X)\) is called an exposed set of \(S(X)\) if \(E\) is the intersection of \(S(X)\) with a supporting hyperplane to \(S(X)\), i.e., if \(E = \{v \in S(X) : x^*(v) = 1\}\) for some \(x^* \in S(X^*)\).

**Theorem 2.16.** Let \(X\) be smooth. Then \(S(X)\) is a moon if and only if each exposed set of \(S(X)\) has an empty interior relative to \(S(X)\).
Proof. If some exposed set $E$ had a relative interior point $v_0$, then (by smoothness)

$$K(v_0, 0) \cap S(X) = S(X) \sim E,$$

and so $v_0 \notin K(v_0, 0) \cap S(X)$. Thus $S(X)$ is not a moon.

Conversely, suppose each exposed set has an empty relative interior. Let $v_0 \in S(X)$ and $x \in \partial(v_0)$. Since $X$ is smooth, $P(v_0 - x) = \{x^*\}$ is a singleton, so that $E = \{v \in S(X) : x^*(v) = 1\}$ is an exposed set which contains $v_0$. Note that $K(v_0, x) \cap S(X) = S(X) \sim E \neq \emptyset$. Since $E$ has an empty relative interior, it follows that in each neighborhood of $v_0$ there points of $S(X) \sim E$. Thus $v_0 \in K(v_0, x) \cap S(X)$, and so $S(X)$ is a moon.

Remark 2.17. The theorem is not true without the smoothness assumption. A 3-space whose unit ball is a "double ice-cream cone" (i.e., the convex hull of the union of a circle and a line segment through its center, normal to its plane) provides an example. In this case, the vertices (in particular) are nonlunar points, but each exposed set of $S(X)$ has an empty relative interior.

The fundamental result concerning strong nonlunarity is the following.

Theorem 2.18. Let $X$ be strongly nonlunar. A subset of $X$ is a moon if and only if it is a sun.

Proof. Every sun is a moon (Corollary 2.8). Let $V$ be a moon which is not a sun. Then there is a $v_0 \in V$ which is a best approximation to some $x \in X$ with $K(v_0, x) \cap V \neq \emptyset$. Let $v \in K(v_0, x) \cap V$. By the strong nonlunarity of the sphere $S(x, \|v_0 - x\|)(= \|v_0 - x\| S(X) + x)$ at $v_0$, there exists an $x_1 \in B(x, \|v_0 - x\|)$ having $v_0$ as a best approximation in $S(x, \|v_0 - x\|)$ such that $v \in K(v_0, x_1)$ and $v_0 \notin K(v_0, x_1) \cap S(x, \|v_0 - x\|)$, i.e., there is an $\varepsilon > 0$ such that

$$B(v_0, \varepsilon) \cap K(v_0, x_1) \subset B(x, \|v_0 - x\|) \subset X \sim V,$$

and so $v_0 \notin K(v_0, x_1) \cap V$. But this contradicts the fact that $V$ is a moon.

It is an open question whether the converse is true. That is, if every moon in $X$ is a sun, must $X$ be strongly nonlunar?

Before we characterize the finite-dimensional $QP$ spaces, let us observe that $v_0 \in S(X)$ is $QP$ iff there is an $\varepsilon > 0$ such that

$$B(v_0, \varepsilon) \cap K(v_0, 0) = B(v_0, \varepsilon) \cap B(0, 1),$$

which holds iff there is an $\varepsilon > 0$ such that

$$B(v_0, \varepsilon) \cap \text{bd } K(v_0, 0) = B(v_0, \varepsilon) \cap S(X).$$

(Here bd $K(v_0, 0)$ denotes the boundary of $K(v_0, 0)$.)
Theorem 2.19. A finite-dimensional space is QP if and only if its (closed) unit ball is a polytope.

Proof. Let \( B(0, 1) \) be a polytope and let \( v_0 \in S(X) \). Then \( B(0, 1) = \bigcap_{i \in I} E_i \), where \( I \) is finite, \( E_i = \{ x \in X : x_i^*(x) \leq 1 \} \), and \( x_i^* \in S(X^*) \). The hyperplanes \( x_i^*(1) \) which determine the half-spaces \( E_i \) will be denoted by \( H_i \). Let \( I_0 = \{ i \in I : v_0 \in H_i \} \) and set \( \epsilon = \text{dist} (v_0, \bigcup_{i \in I_0} H_i) \). Since \( \bigcup_{i \in I_0} H_i \) is closed, \( \epsilon > 0 \). Now \( \text{dist} (v_0, H_i) = 1 - x_i^*(v_0) \) for every \( i \) (cf., e.g., [14, Lemma 2.1]), so that \( \epsilon = \inf_{i \in I_0} \text{dist} (v_0, H_i) = \inf_{i \in I_0} [1 - x_i^*(v_0)] \). We shall show that

\[
B(v_0, \epsilon) \cap \left( \bigcap_{i \in I} E_i \right) = B(v_0, \epsilon) \cap \left( \bigcap_{i \in I_0} E_i \right).
\]

Indeed, if (1) is false, there is an \( x \in X \) with \( x \in B(v_0, \epsilon) \) and \( x \notin E_i \) for some \( i_0 \in I \sim I_0 \). Then

\[
x_i^*(x) > 1 \geq x_i^*(v_0) + \epsilon,
\]

and hence \( \| x - v_0 \| > \epsilon \). This contradiction establishes (1). From (1) we obtain

\[
B(v_0, \epsilon) \cap B(0, 1) = B(v_0, \epsilon) \cap \left( \bigcap_{i \in I} E_i \right)
\]

\[
= B(v_0, \epsilon) \cap \left( \bigcap_{i \in I_0} E_i \right) = B(v_0, \epsilon) \cap K(v_0, 0).
\]

Thus \( v_0 \) is QP.

Conversely, suppose \( X \) is an \( n \)-dimensional QP space. Consider first the case \( n = 2 \). For each \( v \in S(X) \), there is an \( \epsilon_v > 0 \) such that

\[
B(v, \epsilon_v) \cap \text{bd} K(v, 0) = B(v, \epsilon_v) \cap S(X).
\]

By the compactness of \( S(X) \), there is a finite set of \( v_i \in S(X) \) such that \( \{ B(v_i, \epsilon_{v_i}) \}^m \) covers \( S(X) \). Hence

\[
S(X) = \bigcup_{i=1}^m [B(v_i, \epsilon_{v_i}) \cap \text{bd} K(v_i, 0)].
\]

But since \( \text{bd} K(v, 0) \) consists of at most two lines for each \( v \in S(X) \), it follows that \( S(X) \) consists of a finite number of line segments, i.e., \( X \) is polyhedral. Now suppose \( n > 2 \). Then since the QP property is hereditary, each 2-dimensional subspace of \( X \) is QP. By the above argument, each 2-dimensional subspace of \( X \) is polyhedral. By a well-known result [18a, Theorem 4.7], it follows that \( X \) must be polyhedral. \( \qed \)
Corollary 2.20. Let $X$ be two-dimensional. Then each point of $S(X)$ is nonlunar if and only if $S(X)$ is a polygon.

As an application of this corollary we consider the following unit sphere in the plane which has exactly two lunar points, the remaining being $QP$ points.

Example 2.21. An "infinite polygon" in the plane. Let $t_n = 1 - (1/2^n)$ \((n = 0, 1, 2,...)\) and define a function $f$ on $[0, 1]$ to be linear on each subinterval $[t_n, t_{n+1}]$ and to satisfy $f(t_0) = f(0) = 1$, $f(t_{n+1}) = \frac{1}{2}(t_n + f(t_n))$ \((n = 0, 1,...)\), and $f(1) = 0$. Define $g(t) \equiv t - 1$ \((0 \leq t \leq 1)\),

$$f_1(t) \equiv -f(-t) \ (-1 \leq t \leq 0),$$

and $g_3(t) \equiv -g(-t) \ (-1 \leq t \leq 0)$. Then the union $S(X)$ of the graphs of $f$, $g, f_1$, and $g_3$ is what we call an "infinite polygon" in the plane. Clearly, $S(X)$ is $QP$ at every point, with the exception of the two "infinite" points $(1, 0), (-1, 0)$, and these must be lunar points.

Maserick [19] has defined a "convex polytope" $P$ as an intersection of a family of half-spaces: \(P = \bigcap_{i \in I} E_i\) (corresponding to the hyperplanes \(\{H_i : i \in I\}\)), such that, for every $x \in X$, there is a finite subcollection $I_0 \subset I$ with $x \in \bigcap_{i \in I_0} E_i$.

Theorem 2.22. If $\overline{B(0, 1)}$ is a convex polytope (in the sense of [19]), then $X$ is a $QP$ space.

Proof. Properties 2.3, 2.4, and 2.5 of [19] assert that, if $\overline{B(0, 1)}$ is a convex polytope and $v_0 \in S(X)$, then $I_0 = \{i \in I : v_0 \in H_i\}$ is a nonempty finite family and $\bigcup_{i \in I_0} H_i$ is a closed set. Setting $\epsilon = \text{dist}(v_0, \bigcup_{i \in I_0} H_i)$, we observe that exactly the same argument used in the proof of Theorem 2.19 shows that $v_0$ is $QP$.

From the results of [19], we quote the following:

1. Convex polytopes in infinite-dimensional spaces have no extreme points.
2. If the unit ball of $X$ is a convex polytope, so is the unit ball of every subspace of $X$.
3. The unit ball of $c_0(T)$ is a convex polytope for every discrete $T$.
4. If the unit ball of $X$ is a convex polytope with a countable number of exposed sets, then $X$ is isometric to a subspace of $c_0$.

We give now an example, which is a simplification of a more general one given in [19], of a $QP$ space whose unit ball is not a convex polytope. Let $X$ be
the $l_1$-product of the real line $R$ and $c_0$, i.e., $X = (R \times c_0)_{l_1}(\omega)$. (See Section 3 for some basic results on product spaces.) Then $X$ is QP since both $R$ and $c_0$ are (Theorem 3.5), but, by property 2.4 of [19], $B(0, 1)$ is not a convex polytope since the vertex $x = (1, 0, 0,...)$ belongs to infinitely many exposed sets.

3. Product Spaces

Let $I$ be an index set and let $Y$ be a normed linear space of real-valued functions on $I$. If, for each $i \in I$, a normed linear space $X_i$ is given, $(\prod_{i \in I} X_i)_Y$ denotes the ($Y$-product) space of all functions $x$ on $I$ such that

1. $x(i) \in X_i$ for every $i \in I$,
2. If $\nu_x$ is the function on $I$ defined by $\nu_x(i) = \| x(i) \|$, then $\nu_x \in Y$.

We define a norm on $(\prod_{i \in I} X_i)_Y$ by \( \| x \| = \| \nu_x \|_Y \).

We shall be mainly interested in the cases where $Y = c_0(I), l_1(I), \text{ or } l_\infty(I)$. It is well known (cf. e.g., [13, p. 311]) that the dual space may be identified with

\[
(\prod_{i \in I} X_i)^*_{c_0(I)} \quad \text{resp.} \quad (\prod_{i \in I} X_i)^*_{l_1(I)}
\]

via the mapping $x^* \rightarrow (x^*(i))_{i \in I}$, with $x^*(i) \in X_i^*$, defined by

\[
x^*(x) = \sum_{i \in I} x^*(i) x(i)
\]

for every $x$ in the product space. If $X = (\prod_{i \in I} X_i)_{l_1(I)}$, then $x \in \text{ext } S(X)$ if and only if $x(i) \in \text{ext } S(X_i)$ for some $i = i_0$, and $x(i) = 0$ if $i \neq i_0$.

If $X = (\prod_{i \in I} X_i)_{l_\infty(I)}$, then $x \in \text{ext } S(X)$ if and only if $x(i) \in \text{ext } S(X_i)$ for every $i \in I$.

We first consider the space $X = (\prod_{i \in I} X_i)_{c_0(I)}$. Let $x \in X$. We define the critical set of $x$ by

\[
\text{crit } x = \{ i \in I : \| x(i) \| = \| x \| \}.
\]

Observe that if $v_0 \in S(X)$ and $x \in \mathcal{C}(v_0)$, then $\text{crit } v_0 \subset \text{crit } (v_0 - x)$ (since if $i \in \text{crit } v_0$, then

\[
\| v_0 \| = \| v_0(i) \| \leq \| v_0(i) - x(i) \| + \| x(i) \| \leq \| v_0 - x \| + \| x \| = \| v_0 \|,
\]

and so $\| v_0(i) - x(i) \| = \| v_0 - x \|$.)
For any \( v_0, x \in (\prod_{i \in I} X_i)_{c_0(i)} \) we have

\[
K(v_0, x) = \{ v : \text{For each } i \in \text{crit}(v_0 - x), \ x^*(i)[v(i) - v_0(i)] < 0 \\
\quad \text{for every } x^*(i) \in (\text{ext}) P[v_0(i) - x(i)] \}
\]

\[
= \{ v : \text{For each } i \in \text{crit}(v_0 - x), \ v(i) \in K_{X_i}(v_0(i), x(i)) \}.
\]

**Lemma 3.1.** Let \( X = (\prod_{i \in I} X_i)_{c_0(i)} \) and \( v_0 \in S(X) \). Then \( v_0 \) is strongly nonlunar (resp. QP) if and only if, for every \( i \in \text{crit} v_0 \), \( v_0(i) \) is strongly nonlunar (resp. QP) in \( S(X_i) \).

**Proof.** Let \( v_0 \) be strongly nonlunar and let \( i_0 \in \text{crit} v_0 \). We show \( v_0(i_0) \) is strongly nonlunar in \( S(X_{i_0}) \). Let \( u(i_0) \in K_{X_{i_0}}(v_0(i_0), 0) \). Define \( u \) by

\[
u(i) = \begin{cases} u(i_0), & \text{if } i = i_0, \\ 0 & \text{if } i \neq i_0. \end{cases}
\]

Then \( u \in K(v_0, 0) \). Thus by strong nonlunarity there exists an \( x \in \mathcal{O}(v_0) \), \( \| x \| < 1 \), such that \( u \in K(v_0, x) \), and there exists an \( \epsilon > 0 \) such that

\[
B(v_0, \epsilon) \cap K(v_0, x) \subset B(0, 1).
\]

In particular, \( x(i) \in B_X(0, 1) \) for every \( i \). Now if \( \| v - v_0 \| < \epsilon \) and if for every \( i \in \text{crit}(v_0 - x) \), \( v(i) \in K_{X_i}(v_0(i), x(i)) \), then \( \| v \| < 1 \). Since \( u \in K(v_0, x) \) and \( i_0 \in \text{crit} v_0 \subset \text{crit}(v_0 - x) \), it follows that \( u(i_0) \in K_{X_{i_0}}(v_0(i_0), x(i_0)) \). Also, if \( \| v(i_0) - v_0(i_0) \| < \epsilon \) and \( v(i_0) \in K_{X_{i_0}}(v_0(i_0), x(i_0)) \), define

\[
v(i) = \begin{cases} v(i_0), & \text{if } i = i_0, \\ \left(1 - \frac{\epsilon}{2}\right)v_0(i) + \frac{\epsilon}{2} x(i), & \text{if } i \neq i_0. \end{cases}
\]

Then \( \| v - v_0 \| < \epsilon \) and \( v \in K(v_0, x) \), so that \( \| v \| < 1 \). In particular, \( \| v(i_0) \| < 1 \). This shows that \( v_0(i_0) \) is strongly nonlunar.

Conversely, suppose that for each \( i \in \text{crit} v_0 \), \( v_0(i) \) is strongly nonlunar. Thus, for every \( i \in \text{crit} v_0 \), if \( u(i) \in K_{X_i}(v_0(i), 0) \), there exist \( y(i) \in B_{X_i}(0, 1) \) and \( \epsilon(i) > 0 \) such that \( u(i) = K_{X_i}(v_0(i), y(i)) \), and if \( \| v(i) - v_0(i) \| < \epsilon(i) \) and \( v(i) \in K_{X_i}(v_0(i), y(i)) \), then \( \| v(i) \| < 1 \). We may assume that \( \| v_0(i) - y(i) \| \) is constant for \( i \in \text{crit} v_0 \). Now

\[
sup_{i \notin \text{crit} v_0} \| v_0(i) \| = 1 - \delta \quad \text{for some } \delta > 0.
\]

Let \( \epsilon = \min\{\delta, \min_{i \in \text{crit} v_0} \epsilon(i)\} \) and define \( x \) by

\[
x(i) = \begin{cases} y(i), & \text{if } i \in \text{crit} v_0, \\ v_0(i), & \text{if } i \notin \text{crit} v_0. \end{cases}
\]
Then \( \| x \| < 1 \) and \( \text{crit}(v_0 - x) = \text{crit} \ v_0 \). Let \( u \in K(v_0, 0) \). Then for every \( i \in \text{crit} v_0 \), we have \( u(i) \in K_{X_i}(v_0(i), 0) \), and so

\[
u(i) \in K_{X_i}(v_0(i), y(i)) = K_{X_i}(v_0(i), x(i)),
\]

i.e., \( u \in K(v_0, x) \). If \( \| v - v_0 \| < \epsilon \) and \( v \in K(v_0, x) \), then \( \| v(i) - v_0(i) \| < \epsilon \) for every \( i \), and for each \( i \in \text{crit} v_0 \), \( v(i) \in K_{X_i}(v_0(i), x(i)) \) and so \( \| v(i) \| < 1 \). If \( i \notin \text{crit} v_0 \), then \( \| v(i) \| < \| v_0(i) \| + \epsilon \leq 1 - \delta + \epsilon \leq 1 \). Thus \( \| v \| < 1 \). We have shown

\[
B(v_0, \epsilon) \cap K(v_0, x) \subset B(0, 1),
\]

and so \( v_0 \) is strongly nonlunar.

The proof of the analogous result with the QP condition is similar, but simpler. □

As an easy consequence of this lemma we obtain

**Theorem 3.2.** Let \( X = (\prod_{i \in I} X_i)^{c_0(I)} \). Then \( X \) is strongly nonlunar (resp. QP) if and only if each \( X_i \) is strongly nonlunar (resp. QP).

**Corollary 3.3.** For any index set \( T \), the space \( c_0(T) \) is QP.

We turn next to the \( l_1 \)-product of a finite number of normed linear spaces. Let \( X = (\prod_{i \in I} X_i)^{c_0(I)} \), and \( v_0, x \in X \). Then

\[
K(v_0, x) = \left\{ v \in X: \sum_{i \in I} x^*(i)[v(i) - v_0(i)] < 0 \quad \text{whenever} \quad x^*(i) \in (\text{ext} \ P[v_0(i) - x(i)] \right\}
\]

**Lemma 3.4.** Let \( X = (X_1 \times X_2)^{l_1(I)} \), where \( I = \{1, 2\} \), and let \( v_0 \in S(X) \). If \( v_0(i) \| v_0(i) \| \) is QP in \( S(X_i) \) whenever \( v_0(i) \neq 0 \), then \( v_0 \) is QP in \( S(X) \).

**Proof.** Assume first that both \( v_0(1) \) and \( v_0(2) \) are \( \neq 0 \). By assumption, we can choose an \( \epsilon \),

\[
0 < \epsilon < \min \left( \frac{\| v_0(1) \|}{2}, \frac{\| v_0(2) \|}{2} \right)
\]

such that for \( i = 1, 2 \),

\[
B_{X_i} \left( \frac{v_0(i)}{\| v_0(i) \|}, \frac{4\epsilon}{\| v_0(i) \|} \right) \cap K_{X_i} \left( \frac{v_0(i)}{\| v_0(i) \|}, 0 \right) \subset B_{X_i}(0, 1).
\]
Now let \( v \in K(v_0, 0) \cap B(v_0, \epsilon) \). Thus \( \| v(1) - v_0(1) \| + \| v(2) - v_0(2) \| < \epsilon \) and

\[
\max_{x^* \in P[v_0(1)]} x^*(1) [v(1) - v_0(1)] + \max_{x^* \in P[v_0(2)]} x^*(2) [v(2) - v_0(2)] < 0.
\]

There is a scalar \( \alpha \) such that

\[
\max_{x^* \in P[v_0(1)]} x^*(1) [v(1) - v_0(1)] < \alpha,
\]

\[
\max_{x^* \in P[v_0(2)]} x^*(2) [v(2) - v_0(2)] < -\alpha.
\]

Since \( v \in B(v_0, \epsilon) \), it follows that \( |\alpha| < \epsilon \). If \( x^*(1) \in P[v_0(1)] \), then

\[
x^*(1) \left[ \frac{v(1)}{\| v_0(1) \| + \alpha} - \frac{v_0(1)}{\| v_0(1) \|} \right] = \frac{\| v_0(1) \|}{\| v_0(1) \|} \left\{ x^*(1) [v(1) - v_0(1)] - \alpha \right\} < 0
\]

so

\[
norm{v_0(1)} + \alpha \in K_{x_1} \left( \frac{v_0(1)}{\| v_0(1) \|} \right), 0 \right).\]

Moreover,

\[
\frac{\| v(1) \|}{\| v_0(1) \| + \alpha} - \frac{v_0(1)}{\| v_0(1) \|} \leq \frac{\| v(1) - v_0(1) \|}{\| v_0(1) \| + \alpha} + \frac{|\alpha|}{\| v_0(1) \| + \alpha} \leq \frac{\epsilon}{\| v_0(1) \| + \alpha} + \frac{\epsilon}{\| v_0(1) \| + \alpha} < \frac{4\epsilon}{\| v_0(1) \|},
\]

so that, by (1), we have

\[
\frac{\| v(1) \|}{\| v_0(1) \| + \alpha} < 1 \quad \text{or} \quad \| v(1) \| < \| v_0(1) \| + \alpha.
\]

Similarly, we get \( \| v(2) \| < \| v_0(2) \| - \alpha \). Hence

\[
\| v \| = \| v(1) \| + \| v(2) \| < \| v_0(1) \| + \| v_0(2) \| = 1.
\]

Thus \( B(v_0, \epsilon) \cap K(v_0, 0) \subset B(0, 1) \) and so \( v_0 \) is \( QP \). In the case when \( v_0(1) = 0 \) or \( v_0(2) = 0 \), the proof is similar but simpler.

By induction, we obtain

**Lemma 3.5.** Let \( X = (\prod_{i \in I} X_i)_{i \in I} \), where \( I \) is finite and \( v_0 \in S(X) \). If \( v_0(i)/v_0(i) \) is \( QP \) in \( S(X_i) \) whenever \( v_0(i) \neq 0 \), then \( v_0 \) is \( QP \) in \( S(X) \).
As an immediate consequence of this lemma, we have

**Theorem 3.6.** Let $X = \prod_{i \in I} X_i$, where $I$ is finite. If each of the spaces $X_i$ is QP, then $X$ is QP.

**Remark 3.7.** An analogous result for $I$ infinite is not valid since (by Theorem 5.6 in the sequel) $I_1$ is not a QP-space.

4. **The Space $C_0(T)$**

Throughout this section $T$ will denote a locally compact Hausdorff space and $X = C_0(T)$—the space of real-valued continuous functions on $T$, vanishing at infinity, endowed with the uniform norm [18; p. 86]. Thus $x \in X$ iff $x$ is continuous and, for each $\epsilon > 0$, the set $\{t \in T : |x(t)| > \epsilon\}$ is compact. Since the extreme points of $S(X^*)$ are just (plus or minus) the "point evaluations", we may identify $\text{ext } P(x)$ with

$$\text{crit } x = \text{crit } x^+ \cup \text{crit } x^-,$$

where

$$\text{crit } x^\pm = \{t \in T : x(t) = \pm \|x\|\}.$$

Hence, for any $v_0, x \in X$, we have

$$K(v_0, x) = \{v \in X : v(t) < v_0(t) \quad \text{if} \quad t \in \text{crit } (v_0 - x)^+, \quad v(t) > v_0(t) \quad \text{if} \quad t \in \text{crit } (v_0 - x)^-\}.$$

**Theorem 4.1.** $C_0(T)$ is strongly nonlunar.

**Proof.** Let $v_0 \in S(X)$ and $v_1 \in K(v_0, 0)$. Choose $0 < \delta < 1$ such that

$$\delta < \min\{|v_0(t) - v_1(t)| : t \in \text{crit } v_0\},$$

and set

$$K^+ = \left\{t : v_0(t) \geq 1 - \frac{\delta}{3} > 1 - \frac{2\delta}{3} \geq v_1(t)\right\},$$

$$K^- = \left\{t : v_0(t) \leq -1 + \frac{\delta}{3} < -1 + \frac{2\delta}{3} \leq v_1(t)\right\}.$$

Let $V^+, V^-$ be, respectively, disjoint neighborhoods of $K^+, K^-$. Note that $K^+, K^-$ are compact $G_\delta$'s, $K^+ \cap \text{crit } v_0^+$, and $K^- \cap \text{crit } v_0^-$. By Urysohn's lemma, we can choose a function $f \in C_0(T)$ such that

$$f = \begin{cases} 
1/2 & \text{on } K^+ \\
-1/2 & \text{on } K^- \\
0 & \text{off } V^+ \cup V^-.
\end{cases}$$
and $|f| < \frac{1}{2}$ off $K^+ \cup K^-$. Set $x = v_0 - f$. Then $\|x - v_0\| = \frac{1}{2}$,

$$\text{crit}(v_0 - x)^+ = K^+,$$

and $\text{crit}(v_0 - x)^- = K^-$. Since $v_1 < v_0$ on $K^+$, and $v_1 > v_0$ on $K^-$, $v_1 \in K(v_0, x)$.

Let $J = \{t : |v_0(t)| \geq 1/2\}$. Since $\text{crit} v_0 \subset \text{int}(K^+ \cup K^-)$ and $J \sim \text{int}(K^+ \cup K^-)$ is compact ("int" means "interior of"), it follows that

$$\sup\{|v_0(t)| : t \in J \sim \text{int}(K^+ \cup K^-)\} = 1 - \delta_1$$

for some $\delta_1 > 0$. Set $\epsilon = \min\{\delta/6, \delta_1/2\}$. Let $v \in B(v_0, \epsilon) \cap K(v_0, x)$, i.e.,

$$|v_0 - v| < \epsilon, v < v_0 \text{ on } K^+, \text{ and } v > v_0 \text{ on } K^-.$$ In particular, $|v| < 1$ on $K^+ \cup K^-$. If $t \in J \sim K^+ \cup K^-$, then

$$|v(t)| < |v_0(t)| + \epsilon < 1 - \delta_1 + \epsilon < 1.$$ If $t \notin J$, then

$$|v(t)| < |v_0(t)| + \epsilon < 1/2 + \epsilon < 1.$$ Thus $\|v\| < 1$ and so, $B(v_0, \epsilon) \cap K(v_0, x) \subset B(0, 1)$, i.e., $v_0$ is strongly non-

lunar. $lacksquare$

From Theorems 2.18 and 4.1, we immediately obtain

**Corollary 4.2.** In $C_0(T)$, a set is a sun if and only if it is a moon.

**Lemma 4.3.** Let $v_0 \in S(C_0(T))$. Then $v_0$ is QP if and only if $\text{crit} v_0$ is clopen (i.e., both open and closed).

**Proof.** Let $v_0$ be QP. Choose an $\epsilon > 0$ such that

$$B(v_0, \epsilon) \cap K(v_0, 0) \subset B(0, 1).$$

Suppose $\sup\{|v_0(t)| : t \notin \text{crit} v_0\} = 1$. Without loss of generality, we may assume $\sup\{|v_0(t)| : t \notin \text{crit} v_0\} = 1$. Choose $t_0 \in T \sim \text{crit} v_0$ such that

$$v_0(t_0) > 1 - \epsilon/2.$$ Using Urysohn's lemma, choose an $x \in C_0(T)$ such that

$$x = \begin{cases} -\epsilon/2 & \text{on } \text{crit} v_0^+, \\ \epsilon/2 & \text{on } \{t_0\} \cup \text{crit} v_0^-, \end{cases}$$

and $|x| \leq \epsilon/2$ everywhere. Setting $v = v_0 + x$, we see that $\|v - v_0\| < \epsilon$

and $|v(t)| < 1$ on $\text{crit} v_0$, i.e., $v \in B(v_0, \epsilon) \cap K(v_0, 0)$. But

$$v(t_0) = v_0(t_0) + \epsilon/2 > 1, \text{ so } \|v\| > 1.$$
This contradiction shows that
\[ \sup \{ |v_0(t)| : t \notin \text{crit } v_0 \} < 1, \]
i.e., there exists \( \delta > 0 \) such that
\[ \text{crit } v_0 = \{ t \in T : |v_0(t)| > 1 - \delta \}. \]

Hence \( \text{crit } v_0 \) is open. Also, \( \text{crit } v_0 \) is always closed.

Conversely, suppose \( \text{crit } v_0 \) is open. Then there is \( \delta > 0 \) such that
\[ \text{crit } v_0 = \{ t : |v_0(t)| > 1 - \delta \}. \]

Let \( \epsilon = \delta/2 \). If \( v \in X, \| v - v_0 \| < \epsilon, \) and \( |v(t)| < 1 \) on \( \text{crit } v_0 \), then for any \( t \in T \sim \text{crit } v_0 \), we have
\[ |v(t)| < |v_0(t)| + \epsilon \leq 1 - \delta + \epsilon < 1 \]
and so, \( \| v \| < 1 \). We have shown that \( B(v_0, \epsilon) \cap K(v_0, 0) \subset B(0, 1) \) and so, \( v_0 \) is \( QP \).

**THEOREM 4.4.** The following are equivalent:

1. \( C_0(T) \) is a \( QP \)-space.
2. \( \text{crit } v_0 \) is clopen for every \( v_0 \in C_0(T) \).
3. \( T \) is discrete.

**Proof.** The equivalence of (1) and (2) is an immediate consequence of Lemma 4.3.

(3) \( \Rightarrow \) (2). If \( T \) is discrete, then every subset of \( T \) is clopen.

(2) \( \Rightarrow \) (3). Suppose \( \text{crit } v_0 \) is open for every \( v_0 \in X \). If \( T_0 \subset T \) is compact, then every continuous function on \( T_0 \) must have a finite range. Using the regularity of \( T_0 \), it would then follow that \( T_0 \) is finite. Hence compact sets are finite; so \( T \) is discrete.

5. THE SPACE \( L_1(T, \Sigma, \mu) \)

In this section, unless otherwise specified, \( (T, \Sigma, \mu) \) will denote a \( \sigma \)-finite measure space and \( X = L_1 = L_1(T, \Sigma, \mu) \) the space of all real-valued integrable functions \( x \) on \( T \), endowed with the norm
\[ \| x \| = \int_T |x(t)| \, d\mu. \]
We shall abbreviate "μ-almost everywhere" to "a.e." The zero set of a given measurable function \( x \) is defined, modulo a set of measure zero, by
\[
Z(x) = \{ t \in T : x(t) = 0 \}.
\]
The support of \( x \) is defined by
\[
supp x = T \sim Z(x) = \{ t : x(t) \neq 0 \}.
\]
A set \( A \in \Sigma \) is called an atom if \( 0 < \mu(A) < \infty \) and each measurable subset \( B \subset A \) satisfies either \( \mu(B) = 0 \) or \( \mu(B) = \mu(A) \). It is well known (and easy to prove) that \( (T, \Sigma, \mu) \) can have at most countably many atoms. A subset of \( T \) is called purely atomic if it is the union of atoms. Each measurable function \( x \) must be constant a.e. on an atom \( A \). We denote this value by \( x(A) \).

**Lemma 5.1.** Let \( v_0 \in S(L_1) \). Then
\[
\mathcal{A}(v_0) = \{ x \in X : | x | \leq | v_0 | \text{ a.e.}, \text{and } sgn x = sgn v_0 \text{ a.e. on } supp x \}.
\]

*Proof.* We have \( x \in \mathcal{A}(v_0) \) iff \( | v_0 - x | + | x | = | v_0 | \). By the condition for equality in the triangle inequality [18, p. 192], this is equivalent to the existence of a positive measurable function \( \rho \) such that
\[
v_0 = (1 + \rho) x \text{ a.e., on } supp[v_0 - x] x.
\]
But (*) is clearly equivalent to \( | x | \leq | v_0 | \text{ a.e. and } sgn x = sgn v_0 \text{ a.e. on } supp x \).

The following result is the main tool of this section.

**Lemma 5.2.** Let \( v_0 \in S(L_1) \). Consider the statements:

1. supp \( v_0 \) is purely atomic,
2. \( v_0 \) is strongly nonlunar,
3. \( v_0 \) is nonlunar,
4. supp \( v_0 \) contains an atom.

Then (1) \( \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \).

*Proof.* (1) \( \Rightarrow (2) \). Let supp \( v_0 = \bigcup_{i \in I} A_i \), where the \( A_i \) are atoms and \( I \) is some (countable) index set. Let \( v_1 \in K(v_0, 0) \), i.e.
\[
\int_{v_0 > 0} (v_1 - v_0) \, d\mu - \int_{v_0 < 0} (v_1 - v_0) \, d\mu + \int_{Z(v_0)} | v_1 - v_0 | \, d\mu < 0.
\]
By a limit argument (using e.g., the dominated convergence theorem) one can readily show that there exists $\delta > 0$ such that

$$
\sum_{I_5^+} [v_1(A_i) - v_0(A_i)] \mu(A_i) - \sum_{I_5^-} [v_1(A_i) - v_0(A_i)] \mu(A_i)
+ \int_{Z(v_0)} |v_1| \, d\mu + \sum_{I = I_5^+ \cup I_5^-} |v_1(A_i) - v_0(A_i)| \mu(A_i) < 0,
$$

where $I_5^+ = \{i \in I : v_0(A_i) \mu(A_i) > \delta\}$, and $I_5^- = \{i \in I : v_0(A_i) \mu(A_i) < -\delta\}$.

Define a function $x$ by

$$
x(t) = \begin{cases} 
v_0(A_i), & \text{if } t \in A_i \text{ and } i \in I \sim I_5^+ \cup I_5^-, \\ 0, & \text{otherwise.}
\end{cases}
$$

Then $x \in C(v_0)$ and

$$
K(v_0, x) = \left\{ v \in X : \int_{v_0 > x} (v - v_0) \, d\mu - \int_{v_0 < x} (v - v_0) \, d\mu
+ \int_{v_0 = x} |v - v_0| \, d\mu < 0 \right\}
$$

$$
= \left\{ v \in X : \sum_{I_5^+} [v(A_i) - v_0(A_i)] \mu(A_i) - \sum_{I_5^-} [v(A_i) - v_0(A_i)] \mu(A_i)
+ \int_{Z(v_0)} |v| \, d\mu + \sum_{I = I_5^+ \cup I_5^-} |v(A_i) - v_0(A_i)| \mu(A_i) < 0 \right\}.
$$

In particular, $v_1 \in K(v_0, x)$. Choose any $0 < \epsilon < \delta$. Let

$$
v \in B(v_0, \epsilon) \cap K(v_0, x).
$$

Then

$$
|v_0(A_i) - v(A_i)| \mu(A_i) < \epsilon \leq \delta,
$$

so that $\text{sgn} \, v(A_i) = \text{sgn} \, v_0(A_i)$ if $i \in I_5^+ \cup I_5^-$. Thus

$$
\|v\| - 1 = \|v\| - \|v_0\|
\leq \sum_{I_5^+} [v(A_i) - v_0(A_i)] \mu(A_i) - \sum_{I_5^-} [v(A_i) - v_0(A_i)] \mu(A_i)
+ \sum_{I = I_5^+ \cup I_5^-} |v(A_i) - v_0(A_i)| \mu(A_i) + \int_{Z(v_0)} |v| \, d\mu
\leq \sum_{I_5^+} [v(A_i) - v_0(A_i)] \mu(A_i) - \sum_{I_5^-} [v(A_i) - v_0(A_i)] \mu(A_i)
+ \sum_{I = I_5^+ \cup I_5^-} |v(A_i) - v_0(A_i)| \mu(A_i) + \int_{Z(v_0)} |v| \, d\mu < 0,
$$

640/6/2-6
since \( v \in K(v_0, x) \). Hence
\[
B(v_0, \varepsilon) \cap K(v_0, x) \subseteq B(0, 1)
\]
and so, \( v_0 \) is strongly nonlunar.

The implication (2) \( \Rightarrow \) (3) is obvious.

(3) \( \Rightarrow \) (4). If \( v_0 \) is nonlunar, then there is \( x \in \mathcal{C}(v_0) \) and \( \varepsilon > 0 \) such that
\[
B(v_0, \varepsilon) \cap K(v_0, x) \subseteq B(0, 1).
\]

By Lemma 5.1, for almost all \( t \in T \), either \( 0 \leq x(t) \leq v_0(t) \) or \( v_0(t) \leq x(t) \leq 0 \). Now
\[
K(v_0, x) = \left\{ v \in L_1 : \int_{v \geq x} (v - v_0) \, d\mu - \int_{v_0 < x} (v - v_0) \, d\mu + \int_{v_0 = x} |v - v_0| \, d\mu < 0 \right\}.
\]

Let \( T^+ = \{ t \in \text{supp} v_0 : v_0(t) > x(t) \} \) and \( T^- = \{ t \in \text{supp} v_0 : v_0(t) < x(t) \} \). It follows that either \( \mu(T^+) > 0 \) or \( \mu(T^-) > 0 \). We may assume \( \mu(T^+) > 0 \); the case \( \mu(T^-) > 0 \) can be treated similarly. If \( \text{supp} \ v_0 \) contained no atom, then neither would \( T^+ \). Hence we can choose a sequence \( (E_n) \) of disjoint subsets of \( T^+ \) with \( 0 < \mu(E_n) < \infty \). Since
\[
\sum_{n=1}^{\infty} \int_{E_n} |v_0| \, d\mu \leq \int_{T} |v_0| \, d\mu = 1,
\]
we have \( \int_{E_n} |v_0| \, d\mu \to 0 \). Choose \( N \) such that \( \int_{E_N} v_0 \, d\mu < \varepsilon/4 \), and let \( E = \bigcup_{1}^{\infty} E_n \). Define a function \( v \) by
\[
v = \begin{cases} v_0 & \text{on } T \sim E, \\ (1 + \delta) v_0 & \text{on } E \sim E_N, \\ -2v_0 & \text{on } E_N, \end{cases}
\]
where
\[
\delta = \left[ \int_{E_N} v_0 \, d\mu \right]^{-1} \int_{E_N} v_0 \, d\mu.
\]
Then
\[
\begin{align*}
\int_{v \geq x} (v - v_0) \, d\mu &- \int_{v_0 < x} (v - v_0) \, d\mu + \int_{v_0 = x} |v - v_0| \, d\mu \\
&= \delta \int_{E_N} v_0 \, d\mu - 3 \int_{E_N} v_0 \, d\mu = -2 \int_{E_N} v_0 \, d\mu < 0,
\end{align*}
\]
i.e., \( v \in K(v_0, x) \),
\[
\| v - v_0 \| = \delta \int_{E \sim E_N} |v_0| \, d\mu + 3 \int_{E_N} |v_0| \, d\mu = 4 \int_{E_N} |v_0| \, d\mu < \epsilon,
\]
i.e., \( v \in B(v_0, \epsilon) \), but
\[
\| v \| = \int_{T \sim E} |v_0| \, d\mu + (1 + \delta) \int_{E \sim E_N} |v_0| \, d\mu + 2 \int_{E_N} |v_0| \, d\mu
\]
\[= 1 + \delta \int_{E \sim E_N} |v_0| \, d\mu + \int_{E_N} |v_0| \, d\mu > 1.
\]
However, this contradicts
\[B(v_0, \epsilon) \cap K(v_0, x) \subset B(0, 1)\]
and completes the proof. \( \blacksquare \)

From this result we immediately obtain

**Corollary 5.3.** If \( \text{supp} \ v_0 \) contains no atom, then \( v_0 \) is a lunar point of \( S(L_1) \). In particular, if \( T \) contains no atoms, \( S(L_1) \) is a moon.

Another easy consequence of Lemma 5.2 is

**Theorem 5.4.** The following are equivalent:

1. \( L_1(T, \Sigma, \mu) \) is strongly nonlunar,
2. each point of \( S(L_1) \) is nonlunar,
3. \( T \) is purely atomic,
4. \( L_1(T, \Sigma, \mu) \) is (isometrically isomorphic to) a space of type \( l_1 \) or \( l_1^n \), for some \( n \).

**Proof.** The implication (1) \( \Rightarrow \) (2) is obvious.

(2) \( \Rightarrow \) (3). If \( T \) were not purely atomic, there would exist a set \( E \in \Sigma \), with \( 0 < \mu(E) < \infty \), containing no atoms. Then the support of the element \( v_0 = [\mu(E)]^{-1} \chi_E \) would contain no atom. By Lemma 5.2, \( v_0 \) would be lunar.

The equivalence (3) \( \Leftrightarrow \) (4) is well known.

(3) \( \Rightarrow \) (1). Since \( T \) is purely atomic, so is \( \text{supp} \ v_0 \) for every \( v_0 \in S(L_1) \). By Lemma 5.2, it follows that \( L_1(T, \Sigma, \mu) \) is strongly nonlunar. \( \blacksquare \)

**Lemma 5.5.** Let \( v_0 \in S(L_1) \). Then \( v_0 \) is a QP point if and only if \( \text{supp} \ v_0 \) is a finite union of atoms.
Proof. Let $v_0$ be $QP$. Then there is an $\epsilon > 0$ such that

$$B(v_0, \epsilon) \cap K(v_0, 0) \subset B(0, 1).$$

Let

$$T^+ = \{t \in T : v_0(t) > 0\}, \quad T^- = \{t \in T : v_0(t) < 0\}.$$

If $\text{supp } v_0$ were not a finite union of atoms, then $\text{supp } v_0$ would contain either an infinite number of atoms or a set of positive measure which has no atoms. In either case, one of the sets $T^+$ or $T^-$ would contain a sequence $(E_n)$ of disjoint sets with $0 < \mu(E_n) < \infty$. We may assume it is $T^+$ as the other possibility can be treated similarly. The proof now proceeds exactly as that of the implication $(3) \Rightarrow (4)$ in Lemma 5.2 (taking $x = 0$). Thus we can construct a function $v \in B(v_0, \epsilon) \cap K(v_0, 0)$ with $\|v\| > 1$ and get a contradiction.

Conversely, suppose $\text{supp } v_0 = \bigcup_{i=1}^n A_i$, where each $A_i$ is an atom; we can assume $\mu(A_i \cap A_j) = 0$ if $i \neq j$. Choose $\epsilon > 0$ such that

$$\epsilon < \frac{1}{\min_{1 \leq i \leq n} \|v_0(A_i)\| \mu(A_i)}.$$

Let $v \in B(v_0, \epsilon)$. Then $\text{sgn } v(A_i) = \text{sgn } v_0(A_i)$ for $i = 1, \ldots, n$. If $v$ is also in $K(v_0, 0)$, then

$$\|v\| - 1 = \|v\| - \|v_0\| = \int_{v_0 > 0} (v - v_0) \, d\mu - \int_{v_0 < 0} (v - v_0) \, d\mu + \int_{Z(v_0)} |v| \, d\mu < 0,$$

i.e., $\|v\| < 1$. Hence $B(v_0, \epsilon) \cap K(v_0, 0) \subset B(0, 1)$ and so, $v_0$ is $QP$. 

From this lemma we immediately obtain

**Theorem 5.6.** The following are equivalent:

1. $L_1(T, \Sigma, \mu)$ is a $QP$-space.
2. $T$ is a finite union of atoms.
3. $L_1(T, \Sigma, \mu)$ is (isometrically isomorphic to) a space of type $l_1^n$ for some $n$.

6. Related Matters and Some Open Questions

Let $T$ be a compact Hausdorff space, $X$ a real normed linear space, and let $C(T, X)$ be the normed linear space of all $X$-valued continuous functions $f$ on $T$, with the max-norm: $\|f\| = \max_{t \in T} \|f(t)\|_X$. If $T$ is a singleton, $C(T, X) = X$; while if $X = R$, $C(T, X) = C(T)$. It is natural to ask questions
like "if $X$ has a certain property, does $C(T, X)$ too have this property?" In particular, the following problem is unsettled:

**Problem 6.1.** If $X$ is strongly nonlunar (or even $QP$), is $C(T, X)$ strongly nonlunar?

We do have the following partial answer to this question: *If $T$ is finite, then $C(T, X)$ is strongly nonlunar (resp. $QP$), if $X$ is strongly nonlunar (resp. $QP$).* This fact is a consequence of Theorem 3.2, since we may regard $C(T, X)$ as $(\prod_{t \in T} X_t)_{\mathcal{Q}}(T)$, where $X_t = X$ for every $t \in T$.

Another open question is whether the converse of Theorem 2.18 holds. Thus

**Problem 6.2.** If each moon in $X$ is a sun, must $X$ be strongly nonlunar?

We have seen that there are strongly nonlunar spaces which are not $QP$. However, we know of no finite-dimensional example.

In [12], Brown introduced the concept of a normed linear space having property $(P)$. ($X$ has property $(P)$ if for each pair of points $x, z$ in $X$, with $\| x + z \| \leq \| x \|$, there are positive constants $\lambda, \epsilon$ such that $\| y + \lambda z \| \leq \| y \|$ whenever $\| x - y \| < \epsilon$.) Brown observed that every strictly convex space has $(P)$, and so does every finite-dimensional space whose unit ball is a convex polytope. He also showed that a space $X$ has $(P)$ if and only if the metric projection onto any finite-dimensional subspace of $X$ is lower semicontinuous (cf. also [2]). Blatter, Morris and Wulbert [2] have shown that $C_0(T)$ has property $(P)$ if and only if $T$ is discrete. Also, they verified that $L_1(T, \Sigma, \mu)$ has property $(P)$ if and only if $T$ is a finite union of atoms. In [1] Blatter proved, among other things, that $(\prod_{t \in T} X_t)_{\mathcal{Q}}(T)$ has property $(P)$ if and only if each of the spaces $X_t$ has $(P)$. Thus, in the spaces $C_0(T), L_1(T, \Sigma, \mu)$ and $(\prod_{t \in T} X_t)_{\mathcal{Q}}(T)$, property $(P)$ is equivalent to $QP$.

Deutsch and Lindahl [15] have studied the minimal extremal subsets of the unit sphere. Let $v_0 \in S(X)$ and let $E(v_0)$ denote the minimal extremal subset of $S(X)$ which contains $v_0$. Then $X$ is said to have property $Q$ if, for each $v_0 \in S(X)$, the set $E(v_0)$ is the intersection of all the exposed sets in $S(X)$ which contain $v_0$. It was shown in [15] that $C_0(T)$ has property $Q$ if and only if $T$ is discrete; $L_1(T, \Sigma, \mu)$ has property $Q$ if and only if $T$ is a finite union of atoms; every finite-dimensional space whose unit ball is a polytope has property $Q$.

Thus, from the preceding two paragraphs, we have

**Theorem 6.3.** Let $X = C_0(T)$ or $X = L_1(T, \Sigma, \mu)$. Then the following are equivalent:

1. $X$ is $QP$.
2. $X$ has property $(P)$ (of [12]).
3. $X$ has property $Q$ (of [15]).
If \( X = C_{0}(T) \), each of these conditions is equivalent to \( T \) being discrete. If \( X = L_{1}(T, \Sigma, \mu) \), each of these conditions is equivalent to \( T \) being a finite union of atoms.

ACKNOWLEDGMENTS

The authors are indebted to Professor B. Brosowski for many stimulating conversations on various topics of this study. We also thank Mr. Wolfgang Warth for pointing out an error in the first draft of this article.

REFERENCES