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Constructing quantized enveloping algebras via inverse limits of finite dimensional algebras

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ABSTRACT

It is known that a generalized q-Schur algebra may be constructed as a quotient of a quantized enveloping algebra $\mathbf{U}$ or its modified form $\dot{\mathbf{U}}$. On the other hand, we show here that both $\mathbf{U}$ and $\dot{\mathbf{U}}$ may be constructed within an inverse limit of a certain inverse system of generalized q-Schur algebras. Working within the inverse limit $\widehat{\mathbf{U}}$ clarifies the relation between $\dot{\mathbf{U}}$ and $\mathbf{U}$. This inverse limit is a q-analogue of the linear dual $R[G]^*$ of the coordinate algebra of a corresponding linear algebraic group G.

Introduction

Beilinson, Lusztig, and MacPherson [2] constructed a quantized enveloping algebra $\mathbf{U}$ corresponding to the general linear Lie algebra $\mathfrak{gl}_n$ within the inverse limit of an inverse system constructed from q-Schur algebras. The modified form $\dot{\mathbf{U}}$ of $\mathbf{U}$ was also obtained within the inverse limit. Using a slightly different inverse system, consisting of all the generalized q-Schur algebras connected to a given root datum, we construct both $\mathbf{U}$ and $\dot{\mathbf{U}}$ as subalgebras of the resulting inverse limit. This approach, which is analogous to the inverse limit construction of profinite groups, works uniformly for any root datum of finite type, not just for type A. In particular, this clarifies the relation between $\mathbf{U}$ and $\dot{\mathbf{U}}$. All of this is for the generic case, i.e., working over the field $\mathbb{Q}(v)$ of rational functions in v, v an indeterminate. However, the construction is compatible with the so-called "restricted" integral form of Lusztig, and (in a certain sense made precise in Section 5) is also compatible with specializations defined in terms of the restricted integral form.

Generalized Schur algebras were introduced by Donkin [3], motivated by [7]. In [5] a uniform system of generators and relations was found for them and their q-analogues (this was known earlier [4] in type A) and it was proved that the generalized q-Schur algebras are quasihereditary in any

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specialization to a field. The generators and relations of [5] allow a definition of the generalized q-Schur algebras independent of the theory of quantized enveloping algebras; they also lead directly to the inverse system considered here.

Similar inverse systems appeared in [8] (in a more general context) and in [10] (in the classical case for types A–D). It turns out that the inverse limit we construct is a "procellular" completion of $\dot{\mathbf{U}}$, in the sense of [8]. In particular, its elements may be described as formal, possibly infinite, linear combinations of the canonical basis of $\dot{\mathbf{U}}$.

1. Notation

We fix our notational conventions, which are similar to those of [11].

1.1. Cartan datum. Let a Cartan datum be given. By definition, a Cartan datum consists of a finite set I and a symmetric bilinear form $(\ ,\ )$ on the free abelian group $\mathbb{Z}[I]$ taking values in $\mathbb{Z}$, such that:

(a) $(i,i) \in \{2,4,6,\dots\}$ for any i in I.
(b) $2(i,j)/(i,i) \in \{0,-1,-2,\dots\}$ for any $i \neq j$ in I.

1.2. Root datum. A root datum associated to the given Cartan datum consists of two finitely generated free abelian groups X, Y and a perfect[2] bilinear pairing $\langle\ ,\ \rangle : Y \times X \to \mathbb{Z}$ along with embeddings $I \to Y$ $(i \mapsto h_i)$ and $I \to X$ $(i \mapsto \alpha_i)$ such that

$$\langle h_i, \alpha_j \rangle = 2\frac{(i,j)}{(i,i)}$$

for all i, j in I. The image of the embedding $I \to Y$ is the set $\{h_i\}$ of simple coroots and the image of the embedding $I \to X$ is the set $\{\alpha_i\}$ of simple roots.

1.3. The assumptions on the root datum imply that:

(a) $\langle h_i, \alpha_i \rangle = 2$ for all $i \in I$;
(b) $\langle h_i, \alpha_j \rangle \in \{0,-1,-2,\dots\}$ for all $i \neq j \in I$.

In other words, the matrix $(\langle h_i, \alpha_j \rangle)$ indexed by $I \times I$ is a symmetrizable generalized Cartan matrix.

For each $i \in I$ we set $d_i = (i,i)/2$. Then the matrix $(d_i \langle h_i, \alpha_j \rangle)$ indexed by $I \times I$ is symmetric.

1.4. Let v be an indeterminate. Set $v_i = v^{d_i}$ for each $i \in I$. More generally, given any rational function $P \in \mathbb{Q}(v)$ we let P_i denote the rational function obtained from P by replacing v by v_i.

Set $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$. For $a \in \mathbb{Z}$, $t \in \mathbb{N}$ we set

$$\begin{bmatrix} a \\ t \end{bmatrix} = \prod_{s=1}^{t} \frac{v^{a-s+1} - v^{-a+s-1}}{v^s - v^{-s}}.$$

A priori this is an element of $\mathbb{Q}(v)$, but actually it lies in $\mathcal{A}$ (see [11, §1.3.1(d)]). We set

$$[n] = \begin{bmatrix} n \\ 1 \end{bmatrix} = \frac{v^n - v^{-n}}{v - v^{-1}} \quad (n \in \mathbb{Z})$$

and

$$[n]^! = [1] \cdots [n-1][n] \quad (n \in \mathbb{N}).$$

[2] A 'perfect' pairing is one for which the natural maps $X \to \mathrm{Hom}_{\mathbb{Z}}(Y,\mathbb{Z})$ (given by $x \to \langle -, x \rangle$) and $Y \to \mathrm{Hom}_{\mathbb{Z}}(X,\mathbb{Z})$ (given by $y \to \langle y, - \rangle$) are isomorphisms.

Then it follows that

$$\begin{bmatrix} a \\ t \end{bmatrix} = \frac{[a]^!}{[t]^![a-t]^!} \quad \text{for all } 0 \leqslant t \leqslant a.$$

1.5. The Cartan datum is of *finite type* if the symmetric matrix $((i, j))$ indexed by $I \times I$ is positive definite. This is equivalent to the requirement that the Weyl group associated to the Cartan datum is a finite group.

A root datum is *X-regular* (resp., *Y-regular*) if $\{\alpha_i\}$ (resp., $\{h_i\}$) is linearly independent in X (resp., Y). If the underlying Cartan datum is of finite type then the root datum is automatically both X-regular and Y-regular.

In case a root datum is X-regular, there is a partial order on X given by: $\lambda \leqslant \lambda'$ if and only if $\lambda' - \lambda \in \sum_i \mathbb{N}\alpha_i$. In case a root datum is Y-regular, we define

$$X^+ = \{\lambda \in X \mid \langle h_i, \lambda\rangle \in \mathbb{N}, \text{ for all } i \in I\},$$

the set of dominant weights.

1.6. Corresponding to a given root datum is a quantized enveloping algebra $\mathbf{U}$ over $\mathbb{Q}(v)$. According to [11, Corollary 33.1.5], the algebra $\mathbf{U}$ is the associative algebra with 1 over $\mathbb{Q}(v)$ given by the generators E_i, E_{-i} $(i \in I)$, K_h $(h \in Y)$ subject to the defining relations

(a) $$K_h K_{h'} = K_{h+h'}; \qquad K_0 = 1;$$

(b) $$K_h E_{\pm i} = v^{\pm\langle h, \alpha_i\rangle} E_{\pm i} K_h;$$

(c) $$E_i E_{-j} - E_{-j} E_i = \delta_{ij} \frac{\widetilde{K}_i - \widetilde{K}_{-i}}{v_i - v_i^{-1}} \quad \text{where } \widetilde{K}_{\pm i} := K_{\pm d_i h_i};$$

(d) $$\sum_{s+s'=1-\langle h_i, \alpha_j\rangle} (-1)^{s'} \begin{bmatrix} 1 - \langle h_i, \alpha_j\rangle \\ s \end{bmatrix}_i E_{\pm i}^s E_{\pm j} E_{\pm i}^{s'} = 0 \quad (i \neq j)$$

holding for any $i, j \in I$, and any $h, h' \in Y$.

We define, for an element E, the quantized divided power $E^{(m)}$ of E by

$$E^{(m)} := \frac{E^m}{[m]_i^!}$$

for any $m \in \mathbb{N}$. With this convention, one may rewrite relation (d) in the equivalent form

(d′) $$\sum_{s+s'=1-\langle h_i, \alpha_j\rangle} (-1)^{s'} E_{\pm i}^{(s)} E_{\pm j} E_{\pm i}^{(s')} = 0 \quad (i \neq j).$$

As in [11, §3.4], let $\mathcal{C}$ be the category whose objects are $\mathbf{U}$-modules M admitting a weight space decomposition $M = \bigoplus_{\lambda \in X} M_\lambda$ (as $\mathbb{Q}(v)$-vector spaces) where the weight space M_λ is given by

$$M_\lambda = \{m \in M \mid K_h m = v^{\langle h, \lambda\rangle} m, \text{ all } h \in Y\}.$$

The morphisms in $\mathcal{C}$ are $\mathbf{U}$-module homomorphisms.

From now on we assume the root datum is of finite type. Thus it is both X- and Y-regular. We denote by $\Delta(\lambda)$ the simple object (see [11, Corollary 6.2.3, Proposition 3.5.6]) of $\mathcal{C}$ of highest weight $\lambda \in X^+$, for any $\lambda \in X^+$.

2. The algebra $\widehat{\mathbf{U}}$

Fix a root datum $(X, \{\alpha_i\}, Y, \{h_i\})$ of *finite type*. We define the finite dimensional algebras $\mathbf{S}(\pi)$ (the generalized q-Schur algebras) and construct the algebra $\widehat{\mathbf{U}}$ as an inverse limit.

2.1. A nonempty subset π of X^+ is *saturated* if $\lambda \leqslant \mu$ for $\lambda \in X^+$, $\mu \in \pi$ implies $\lambda \in \pi$.

Saturated subsets of X^+ exist in abundance. For instance, given any $\mu \in X^+$, the set $X^+[\leqslant \mu] = \{\lambda \in X^+ \mid \lambda \leqslant \mu\}$ is saturated. In general, a saturated subset of X^+ is a union of such subsets.

2.2. The algebra $\mathbf{S}(\pi)$.

Given a finite saturated set $\pi \subset X^+$ we define an algebra $\mathbf{S}(\pi)$ to be the associative $\mathbb{Q}(v)$-algebra with 1 given by the generators

$$E_i, E_{-i} \quad (i \in I), \qquad 1_\lambda \quad (\lambda \in W\pi)$$

and the relations

(a) $$1_\lambda 1_{\lambda'} = \delta_{\lambda,\lambda'} 1_\lambda, \qquad \sum_{\lambda \in W\pi} 1_\lambda = 1;$$

(b) $$E_{\pm i} 1_\lambda = \begin{cases} 1_{\lambda \pm \alpha_i} E_{\pm i} & \text{if } \lambda \pm \alpha_i \in W\pi, \\ 0 & \text{otherwise;} \end{cases}$$

(b′) $$1_\lambda E_{\pm i} = \begin{cases} E_{\pm i} 1_{\lambda \mp \alpha_i} & \text{if } \lambda \mp \alpha_i \in W\pi, \\ 0 & \text{otherwise;} \end{cases}$$

(c) $$E_i E_{-j} - E_{-j} E_i = \delta_{ij} \sum_{\lambda \in W\pi} \big[\langle h_i, \lambda \rangle\big]_i 1_\lambda;$$

(d) $$\sum_{s+s'=1-\langle h_i, \alpha_j \rangle} (-1)^{s'} E_{\pm i}^{(s)} E_{\pm j} E_{\pm i}^{(s')} = 0 \quad (i \neq j)$$

for all $i, j \in I$ and all $\lambda, \lambda' \in W\pi$. In relation (d), $E_{\pm i}^{(s)}$ is the quantized divided power $E_{\pm i}^{(s)} = E_{\pm i}^s / ([s]_i^!)$.

The algebra $\mathbf{S}(\pi)$ is known as a generalized q-Schur algebra (see [5]). It is a consequence of the defining relations that the generators $E_{\pm i}$ are nilpotent elements of $\mathbf{S}(\pi)$; it follows that $\mathbf{S}(\pi)$ is finite dimensional over $\mathbb{Q}(v)$.

For any π we define elements $K_h \in \mathbf{S}(\pi)$ for each $h \in Y$ by the formula

$$K_h = \sum_{\lambda \in W\pi} v^{\langle h, \lambda \rangle} 1_\lambda.$$

This depends on π as well as h; we rely on the context to make clear in which $\mathbf{S}(\pi)$ a given K_h is to be interpreted. We note that the identities $K_h K_{h'} = K_{h+h'}$, $K_0 = 1$ and $K_{-h} = K_h^{-1}$ hold in $\mathbf{S}(\pi)$ for all $h, h' \in Y$.

2.3. It will be convenient for ease of notation to extend the meaning of the symbols 1_λ to all $\lambda \in X$ by making the convention $1_\lambda = 0$ in $\mathbf{S}(\pi)$ for any $\lambda \notin W\pi$. With this convention $\mathbf{S}(\pi)$ becomes the associative $\mathbb{Q}(v)$-algebra given by generators

$$E_i, E_{-i} \quad (i \in I), \qquad 1_\lambda \quad (\lambda \in W\pi)$$

with the relations

(a) $1_\lambda 1_{\lambda'} = \delta_{\lambda,\lambda'} 1_\lambda$, $\sum_{\lambda \in X} 1_\lambda = 1$;
(b) $E_{\pm i} 1_\lambda = 1_{\lambda \pm \alpha_i} E_{\pm i}$;
(c) $E_i E_{-j} - E_{-j} E_i = \delta_{ij} \sum_{\lambda \in X} [\langle h_i, \lambda \rangle]_i 1_\lambda$;
(d) $\sum_{s+s'=1-\langle h_i, \alpha_j \rangle} (-1)^{s'} E_{\pm i}^{(s)} E_{\pm j} E_{\pm i}^{(s')} = 0$ $(i \neq j)$

for all $i, j \in I$ and all $\lambda, \lambda' \in X$. Note that the sums in (a), (c) are finite since by definition all but finitely many 1_λ are zero in $\mathbf{S}(\pi)$.

2.4. The form of the presentation of $\mathbf{S}(\pi)$ given in 2.3 makes it clear that for any finite saturated subsets π, π' of X^+ with $\pi \subset \pi'$ we have a surjective algebra map

$$f_{\pi,\pi'} : \mathbf{S}(\pi') \to \mathbf{S}(\pi)$$

sending $E_{\pm i} \to E_{\pm i}$, $1_\lambda \to 1_\lambda$ (any $i \in I$, $\lambda \in W\pi'$). Since $f_{\pi,\pi} = 1$ and for any finite saturated subsets π, π', π'' of X^+ with $\pi \subset \pi' \subset \pi''$ we have $f_{\pi,\pi'} f_{\pi',\pi''} = f_{\pi,\pi''}$, the collection

$$\{\mathbf{S}(\pi); f_{\pi,\pi'}\}$$

forms an inverse system of algebras. We denote by $\widehat{\mathbf{U}} = \varprojlim \mathbf{S}(\pi)$ the inverse limit of this inverse system, taken over the collection of all finite saturated subsets of X^+. This is isomorphic with

$$\left\{ (a_\pi)_\pi \in \prod_\pi \mathbf{S}(\pi) \;\middle|\; a_\pi = f_{\pi,\pi'}(a_{\pi'}), \text{ for any } \pi \subset \pi' \right\}$$

with addition and multiplication of such sequences defined componentwise. We set

$$\widehat{1}_\lambda := (1_\lambda)_\pi \in \widehat{\mathbf{U}}$$

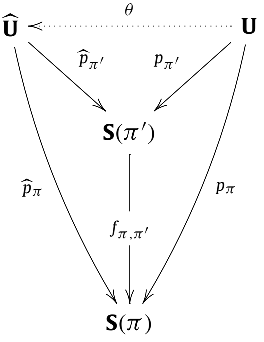
and note that because of the convention introduced in 2.3 a number of the components of this sequence may be zero. However, only finitely many components are zero, so the sequence is eventually constant. We similarly set

$$\widehat{E}_{\pm i} := (E_{\pm i})_\pi \in \widehat{\mathbf{U}}$$

for any $i \in I$. Finally, for any $h \in Y$ we set

$$\widehat{K}_h := (K_h)_\pi \in \widehat{\mathbf{U}}.$$

2.5. Let $\widehat{p}_\pi : \widehat{\mathbf{U}} \to \mathbf{S}(\pi)$ be projection onto the πth component. Let $\mathbf{U}$ be the quantized enveloping algebra determined by the given root datum (see 1.6) and for each $\lambda \in X^+$ let $\Delta(\lambda)$ be the simple $\mathbf{U}$-module of highest weight λ. According to [5, Corollary 3.13], $\mathbf{S}(\pi)$ is the quotient of $\mathbf{U}$ by the ideal consisting of all $u \in \mathbf{U}$ annihilating every simple module $\Delta(\lambda)$ such that $\lambda \in \pi$. Let $p_\pi : \mathbf{U} \to \mathbf{S}(\pi)$ be the corresponding quotient map, which sends $E_{\pm i} \in \mathbf{U}$ to $E_{\pm i} \in \mathbf{S}(\pi)$, $K_h \in \mathbf{U}$ to $K_h \in \mathbf{S}(\pi)$ for all $i \in I$, $h \in Y$. These maps fit into a commutative diagram



for any finite saturated subsets π, π' of X^+ with $\pi \subset \pi'$. The universal property of inverse limits guarantees the existence of a unique algebra map $\theta : \mathbf{U} \to \widehat{\mathbf{U}}$ making the diagram commute.

2.6. Theorem. *The map θ gives an algebra embedding of $\mathbf{U}$ into $\widehat{\mathbf{U}}$ sending $E_{\pm i}$ to $\widehat{E}_{\pm i}$ and K_h to $\widehat{K}_h$ for all $i \in I$, $h \in Y$. Hence, the subalgebra of $\widehat{\mathbf{U}}$ generated by the $\widehat{E}_{\pm i}$ $(i \in I)$, $\widehat{K}_h$ $(h \in Y)$ is isomorphic with $\mathbf{U}$.*

Proof. Suppose $u \in \mathbf{U}$ maps to zero under θ. Then $p_\pi(u) = 0$ for every finite saturated subset π, which means that u annihilates every simple $\mathbf{U}$-module in the category $\mathcal{C}$. By [11, Prop. 3.5.4] it follows that $u = 0$, so the kernel of θ is trivial. The rest of the assertions of the theorem are clear. □

2.7. Proposition. *In $\widehat{\mathbf{U}}$ we have the identity $\widehat{K}_h = \sum_{\lambda \in X} v^{\langle h, \lambda \rangle} \widehat{1}_\lambda$ for any $h \in Y$.*

Proof. We have only to check that this holds when the projection $\widehat{p}_\pi$ is applied to both sides. This is valid by the definition of $K_h \in \mathbf{S}(\pi)$ given in 2.2. □

2.8. From the preceding result it follows by easy calculations that in $\widehat{\mathbf{U}}$ we have the identities

(a) $\widehat{K}_h \widehat{K}_{h'} = \widehat{K}_{h+h'}$,
(b) $\widehat{K}_0 = 1$,
(c) $\widehat{K}_{-h} = \widehat{K}_h^{-1}$

for any $h, h' \in Y$.

2.9. Proposition. *The elements $\widehat{E}_{\pm i}$ $(i \in I)$; $\widehat{1}_\lambda$ $(\lambda \in X)$ of $\widehat{\mathbf{U}}$ satisfy the relations*

(a) $\widehat{1}_\lambda \widehat{1}_{\lambda'} = \delta_{\lambda,\lambda'} \widehat{1}_\lambda$, $\sum_{\lambda \in X} \widehat{1}_\lambda = 1$;
(b) $\widehat{E}_{\pm i} \widehat{1}_\lambda = \widehat{1}_{\lambda \pm \alpha_i} \widehat{E}_{\pm i}$;
(c) $\widehat{E}_i \widehat{E}_{-j} - \widehat{E}_{-j} \widehat{E}_i = \delta_{ij} \sum_{\lambda \in X} [\langle h_i, \lambda \rangle]_i \widehat{1}_\lambda$;
(d) $\sum_{s+s'=1-\langle h_i, \alpha_j \rangle} (-1)^{s'} \widehat{E}_{\pm i}^{(s)} \widehat{E}_{\pm j} \widehat{E}_{\pm i}^{(s')} = 0$ *$(i \neq j)$.*

Proof. The argument is similar to the proof of Proposition 2.7. □

Note that the relations in the preceding result are the same relations as in 2.3 but in this case the sums in (a), (c) are infinite, since $1_\lambda \in \widehat{\mathbf{U}}$ is nonzero for any $\lambda \in X$.

2.10. Remark. It is clear from Theorem 2.6 that the elements $\widehat{E}_{\pm i}$, $\widehat{K}_h$ of $\widehat{\mathbf{U}}$ satisfy the usual defining relations (see 1.6(a)–(d)) for the quantized enveloping algebra $\mathbf{U}$.

On the other hand, if one simply starts with $\mathbf{S}(\pi)$ defined by the presentation given in 2.2 and forms the inverse limit $\widehat{\mathbf{U}}$ without knowledge of $\mathbf{U}$, defining elements $\widehat{E}_{\pm i}$, $\widehat{K}_h$ in $\widehat{\mathbf{U}}$ as we have done above, then the defining relations 1.6(a)–(d) (with $\widehat{E}_{\pm i}$, $\widehat{K}_h$ in place of $E_{\pm i}$, K_h, respectively) may easily be derived from the defining relations for $\mathbf{S}(\pi)$. Then $\mathbf{U}$ could be defined as the subalgebra of $\widehat{\mathbf{U}}$ generated by the $\widehat{E}_{\pm i}$ $(i \in I)$, $\widehat{K}_h$ $(h \in Y)$. In fact, this is clear from Proposition 2.9 in light of Proposition 2.7. For instance, one has

$$\begin{aligned}\widehat{K}_h\widehat{E}_{\pm i} &= \sum_\lambda v^{\langle h,\lambda\rangle}1_\lambda\widehat{E}_{\pm i} = \sum_\lambda v^{\langle h,\lambda\rangle}\widehat{E}_{\pm i}1_{\lambda\mp\alpha_i}\\ &= \widehat{E}_{\pm i}\sum_\lambda v^{\langle h,\lambda\pm\alpha_i\rangle}1_\lambda = v^{\pm\langle h,\alpha_i\rangle}\widehat{E}_{\pm i}\widehat{K}_h\end{aligned}$$

where in the sums λ runs over X. This proves the analogue of relation 1.6(b). The analogue of relation 1.6(c) is proved by a similar calculation, which we leave to the reader. In other words, the defining structure of the quantized enveloping algebra $\mathbf{U}$ is an easy consequence of the defining structure for the $\mathbf{S}(\pi)$.

2.11. Remark. The inverse system used here is indexed by the family consisting of all finite saturated subsets of X^+. One could just as well have used the family consisting of all subsets of the form $X^+[\leqslant \lambda] = \{\mu \in X^+ \colon \ \mu \leqslant \lambda\}$, for various $\lambda \in X^+$, or even the family of complements of all the $X^+[\geqslant \lambda] = \{\mu \in X^+ \colon \ \mu \geqslant \lambda\}$. All these families of finite saturated subsets of X^+ lead to the same inverse limit $\widehat{\mathbf{U}}$.

3. Relation with the modified form $\dot{\mathbf{U}}$

In this section we explore the relation between the algebra $\widehat{\mathbf{U}}$ and Lusztig's modified form $\dot{\mathbf{U}}$ of $\mathbf{U}$. We show that $\dot{\mathbf{U}}$ may be identified with a subalgebra of $\widehat{\mathbf{U}}$.

3.1. The modified form $\dot{\mathbf{U}}$ is defined (see [11, Chapter 23]) as follows. For $\lambda, \lambda' \in X$ set

$${}_\lambda\mathbf{U}_{\lambda'} = \mathbf{U}\Big/\Big(\sum_{h\in Y}\big(K_h - v^{\langle h,\lambda\rangle}\big)\mathbf{U} + \sum_{h\in Y}\mathbf{U}\big(K_h - v^{\langle h,\lambda'\rangle}\big)\Big)$$

regarded as a quotient of vector spaces over $\mathbb{Q}(v)$. Then define

$$\dot{\mathbf{U}} := \bigoplus\nolimits_{\lambda,\lambda'\in X}({}_\lambda\mathbf{U}_{\lambda'}).$$

Let $\pi_{\lambda,\lambda'} : \mathbf{U} \to {}_\lambda\mathbf{U}_{\lambda'}$ be the canonical projection. One has a direct sum decomposition $\mathbf{U} = \bigoplus_\nu \mathbf{U}(\nu)$ where ν runs over the root lattice $\sum \mathbb{Z}\alpha_i$, and where $\mathbf{U}(\nu)$ is defined by the requirements $\mathbf{U}(\nu)\mathbf{U}(\nu') \subseteq \mathbf{U}(\nu+\nu')$, $K_h \in \mathbf{U}(0)$, $E_{\pm i} \in \mathbf{U}(\pm\alpha_i)$ for all $i \in I$, $h \in Y$. Then $\dot{\mathbf{U}}$ inherits a natural associative $\mathbb{Q}(v)$-algebra structure from that of $\mathbf{U}$, as follows: for any $\lambda_1, \lambda_2, \lambda_1', \lambda_2' \in X$ and any $t \in \mathbf{U}(\lambda_1 - \lambda_1')$, $s \in \mathbf{U}(\lambda_2 - \lambda_2')$, the product $\pi_{\lambda_1,\lambda_1'}(t)\pi_{\lambda_2,\lambda_2'}(s)$ is equal to $\pi_{\lambda_1,\lambda_2'}(ts)$ if $\lambda_1' = \lambda_2$ and is zero otherwise.

For any $\lambda \in X$, set $1_\lambda = \pi_{\lambda,\lambda}(1)$. Then the elements $1_\lambda \in \dot{\mathbf{U}}$ satisfy the relations

$$1_\lambda 1_{\lambda'} = \delta_{\lambda,\lambda'}1_\lambda$$

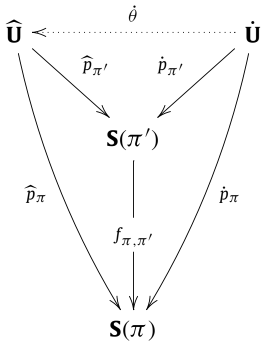
and we have ${}_\lambda\mathbf{U}_{\lambda'} = 1_\lambda\dot{\mathbf{U}}1_{\lambda'}$. The algebra $\dot{\mathbf{U}}$ may be regarded as an $\mathbf{U}$-bimodule by setting, for $t \in \mathbf{U}(\nu)$, $s \in \mathbf{U}$, and $t' \in \mathbf{U}(\nu')$, the product $t\pi_{\lambda,\lambda'}(s)t' = \pi_{\lambda+\nu,\lambda'-\nu'}(tst')$ for any $\lambda, \lambda' \in X$. It follows that the products $1_\lambda E_{\pm i}$ $(i \in I, \lambda \in X)$ are well defined elements of $\dot{\mathbf{U}}$. In fact $\dot{\mathbf{U}}$ is generated by those elements.

3.2. For a given saturated subset π of X^+ we shall write π^c for the set theoretic complement $X^+ - \pi$. In [5] it is shown that $\mathbf{S}(\pi)$ is isomorphic with the quotient algebra $\dot{\mathbf{U}}/\dot{\mathbf{U}}[\pi^c]$ for any finite saturated subset π of X^+. The ideal $\dot{\mathbf{U}}[\pi^c]$ and corresponding quotient both appear in [11, §29.2]; the ideal may be characterized as the set of all elements $u \in \dot{\mathbf{U}}$ such that u annihilates every $\Delta(\lambda)$ with $\lambda \in \pi$. We note for future reference that

$$\bigcap_{\pi} \dot{\mathbf{U}}\big[\pi^c\big] = (0)$$

(see [11, Chapter 29]).

The proof of [5, Theorem 4.2] shows that the quotient map $\dot{p}_\pi : \dot{\mathbf{U}} \to \mathbf{S}(\pi)$ (with kernel $\dot{\mathbf{U}}[\pi^c]$) is defined by sending $E_{\pm i}1_\lambda \in \dot{\mathbf{U}}$ to $E_{\pm i}1_\lambda \in \mathbf{S}(\pi)$. Clearly the quotient maps $\dot{p}_\pi$ fit into a commutative diagram



for any finite saturated subsets π, π' of X^+ with $\pi \subset \pi'$. Again the universal property of inverse limits guarantees the existence of a unique algebra map $\dot{\theta} : \dot{\mathbf{U}} \to \widehat{\mathbf{U}}$ making the diagram commute.

We are now prepared to prove the following result.

3.3. Theorem. *The map $\dot{\theta}$ is an algebra embedding of $\dot{\mathbf{U}}$ into $\widehat{\mathbf{U}}$ sending $E_{\pm i}1_\lambda$ to $\widehat{E}_{\pm i}\widehat{1}_\lambda$ for all $i \in I$, $\lambda \in X$. Hence, the subalgebra of $\widehat{\mathbf{U}}$ generated by the products $\widehat{E}_{\pm i}\widehat{1}_\lambda$ ($i \in I$, $\lambda \in X$) is isomorphic with $\dot{\mathbf{U}}$.*

Proof. Suppose $\dot{\theta}(u) = 0$ for $u \in \dot{\mathbf{U}}$. Then $\dot{p}_\pi(u) = 0$ for each finite saturated subset π of X^+. Hence $u \in \bigcap_\pi \dot{\mathbf{U}}[\pi^c]$; whence $u = 0$. Thus the kernel of $\dot{\theta}$ is trivial. The rest of the claims are clear. □

3.4. Henceforth we identify $\dot{\mathbf{U}}$ with the subalgebra of $\widehat{\mathbf{U}}$ generated by all $\widehat{E}_{\pm i}\widehat{1}_\lambda$. Note that the elements $\widehat{E}_{\pm i}$ of $\widehat{\mathbf{U}}$ are not elements of $\dot{\mathbf{U}}$ since their expression in terms of the generators of $\dot{\mathbf{U}}$ involves infinite sums.

3.5. For convenience, choose a total ordering $\pi_1, \pi_2, \ldots$ on the finite saturated sets π which is compatible with the partial order given by set inclusion, in the following sense: $i \leqslant j$ implies that $\pi_i \subseteq \pi_j$. Then the completion of $\dot{\mathbf{U}}$ with respect to the descending sequence of ideals

$$\dot{\mathbf{U}} \supseteq \dot{\mathbf{U}}\big[\pi_1^c\big] \supseteq \dot{\mathbf{U}}\big[\pi_2^c\big] \supseteq \cdots$$

is isomorphic with $\widehat{\mathbf{U}}$.

As in [8], we put a topology on the ring $\dot{\mathbf{U}}$ by letting the collection $\{\dot{\mathbf{U}}[\pi^c]\}$, as π varies over the finite saturated subsets of X^+, define a neighborhood base of 0. Then $\widehat{\mathbf{U}}$ may be regarded as the set of equivalence classes of Cauchy sequences $(x_n)_{n=1}^\infty$ of elements of $\dot{\mathbf{U}}$ under usual Cauchy equivalence.

Here a sequence (x_n) is Cauchy if for each neighborhood $\dot{\mathbf{U}}[\pi^c]$ there exists some positive integer $N(\pi)$ such that

$$x_m - x_n \in \dot{\mathbf{U}}\big[\pi^c\big] \quad \text{for all } m, n \geqslant N(\pi),$$

and given sequences (x_n), (y_n) are Cauchy equivalent if $x_n - y_n \to 0$ as $n \to \infty$. The proof is standard (see e.g. [1, Chapter 10]). Given a Cauchy sequence (x_n) its image in $\mathbf{S}(\pi)$ is eventually constant, say a_π. The resulting sequence $(a_\pi) \in \prod_\pi \mathbf{S}(\pi)$ satisfies $a_\pi = f_{\pi,\pi'}(a_{\pi'})$ for any $\pi \subset \pi'$, so $(a_\pi) \in \widehat{\mathbf{U}}$. On the other hand, given any $(a_\pi) \in \widehat{\mathbf{U}}$ we can define a corresponding Cauchy sequence by setting x_n equal to any element of the coset $\dot{p}_{\pi_n}(a_{\pi_n}) \in \dot{\mathbf{U}}/\dot{\mathbf{U}}[\pi_n^c]$, where $\pi = \pi_n$.

Thus, $\widehat{\mathbf{U}}$ is a complete topological algebra. It is Hausdorff, thanks to the triviality of the intersection of the elements of the neighborhood base of 0.

3.6. We say that a basis B of $\dot{\mathbf{U}}$ is *coherent* if the set of nonzero elements of $\dot{p}_\pi(B)$ is a basis of $\mathbf{S}(\pi)$, for each finite saturated $\pi \subset X^+$.

Assume that B is any such basis. Write $B[\pi]$ for the set of nonzero elements of $\dot{p}_\pi(B)$. The following result is a consequence of [8, Corollary 2.2.5].

3.7. Proposition. *Given any coherent basis B of $\dot{\mathbf{U}}$, the completion $\widehat{\mathbf{U}}$ may be identified with the algebra of all formal infinite linear combinations of elements of B.*

Proof. Any formal sum of the form $a = \sum_{b\in B} a_b b$ (for $a_b \in \mathbb{Q}(v)$) determines an element $a_\pi = \sum_{b\in B[\pi]} a_b b$ of $\mathbf{S}(\pi)$. Clearly, the sequence (a_π) is an element of $\widehat{\mathbf{U}}$.

We must show that every element of $\widehat{\mathbf{U}}$ is expressible in such a form. Let $a = (a_\pi)$ be an element of $\widehat{\mathbf{U}}$. Each $a_\pi \in \mathbf{S}(\pi)$ may be written in the form $a_\pi = \sum_{b\in B[\pi]} a_b b$ where $a_b \in \mathbb{Q}(v)$. Moreover, the coefficient a_b of any $b \in B$ will always be the same value, for any π' such that $b \in B[\pi']$. To see this, let π'' be any finite saturated subset of X^+ containing both π and π' (such must exist) and consider the projections $f_{\pi,\pi'}$ and $f_{\pi,\pi''}$. Since $\cup_\pi B[\pi] = B$ this shows that a determines a well-defined infinite sum $\sum_{b\in B} a_b b$. □

3.8. In [11, Chapter 25] it is proved that the canonical basis can be lifted from the positive part of $\mathbf{U}$ to a canonical basis $\dot{\mathbf{B}}$ of $\dot{\mathbf{U}}$. (This was a primary motivation for the introduction of $\dot{\mathbf{U}}$.) Moreover, $\dot{\mathbf{B}}$ is coherent with respect to the inverse system $\{\mathbf{S}(\pi)\}$; see [11, §29.2.3]. Thus it follows from the preceding proposition that elements of $\widehat{\mathbf{U}}$ may be regarded as formal infinite linear combinations of $\dot{\mathbf{B}}$.

3.9. Remark. It is easy to see that $\widehat{\mathbf{U}}$ is a procellular algebra in the sense of R.M. Green [8]. This is a consequence of Lusztig's refined Peter–Weyl theorem [11, Theorem 29.3.3], which implies that $\dot{\mathbf{B}}$ is a *cellular basis* of $\dot{\mathbf{U}}$. (See [6] for the definition of cellular basis.)

3.10. Lemma. *The algebra $\widehat{\mathbf{U}}$ is topologically generated by the elements $\widehat{E}_{\pm i}$ $(i \in I)$, $\widehat{1}_\lambda$ $(\lambda \in X)$ in the sense that every element of $\widehat{\mathbf{U}}$ is expressible as a formal* (*possibly infinite*) *linear combination of finite products of those elements.*

Proof. By 3.8 every element of $\widehat{\mathbf{U}}$ is a formal linear combination of elements of $\dot{\mathbf{B}}$. But elements of $\dot{\mathbf{B}}$ are themselves expressible as finite linear combinations of finite products of the elements $\widehat{E}_{\pm i}$ $(i \in I)$, $\widehat{1}_\lambda$ $(\lambda \in X)$, since $\dot{\mathbf{U}}$ is generated by elements of the form $\widehat{E}_{\pm i}\widehat{1}_\lambda$ for various $i \in I$, $\lambda \in X$ (see [11, Chapter 23]). □

3.11. Theorem. *The algebra $\widehat{\mathbf{U}}$ is the associative algebra with* 1 *given by the generators $\widehat{E}_{\pm i}$ $(i \in I)$, $\widehat{1}_\lambda$ $(\lambda \in X)$ with the relations* (a)–(d) *of Proposition* 2.9, *in the following sense*:

$$\widehat{\mathbf{U}} \simeq \mathbb{Q}(v)\langle\!\langle \widehat{E}_{\pm i}, \widehat{1}_\lambda \rangle\!\rangle / J$$

where $\mathbb{Q}(v)\langle\!\langle \widehat{E}_{\pm i}, \widehat{1}_\lambda \rangle\!\rangle$ *is the free complete algebra on the generators* $\widehat{E}_{\pm i}, \widehat{1}_\lambda$ *(consisting of all formal linear combinations of finite products of generators) and* J *is the ideal generated by relations* 2.9(a)–(d).

Proof. Let $\mathbf{P}$ be the algebra $\mathbb{Q}(v)\langle\!\langle \widehat{E}_{\pm i}, \widehat{1}_\lambda \rangle\!\rangle / J$. Notice (see 2.3) that by definition $\mathbf{S}(\pi)$ is the quotient of $\mathbf{P}$ by the ideal generated by all $\widehat{1}_\lambda$ with $\lambda \notin W\pi$. Thus we have surjective quotient maps

$$q_\pi : \mathbf{P} \to \mathbf{S}(\pi) \quad (\pi \text{ finite saturated})$$

such that $f_{\pi,\pi'} q_{\pi'} = q_\pi$ whenever $\pi \subset \pi'$. These maps fit into a commutative diagram similar to the one appearing in 2.5, and by the universal property of inverse limits there is an algebra map $\Psi : \mathbf{P} \to \widehat{\mathbf{U}}$ sending $\widehat{E}_{\pm i}$ to $\widehat{E}_{\pm i}$, and $\widehat{1}_\lambda$ to $\widehat{1}_\lambda$.

The map Ψ is injective since the intersection of the kernels of the various q_π is trivial. On the other hand, by the preceding lemma combined with Proposition 2.9 Ψ must also be surjective, since the generators of $\widehat{\mathbf{U}}$ satisfy the defining relations of $\mathbf{P}$. □

3.12. Remark. The topology on $\widehat{\mathbf{U}}$ is induced from the topology on $\dot{\mathbf{U}}$. The basic neighborhoods of 0 are of the form $\widehat{\mathbf{U}}[\pi^c]$ for the various finite saturated subsets π of X^+, where $\widehat{\mathbf{U}}[\pi^c]$ is the set of all formal $\mathbb{Q}(v)$-linear combinations of elements of $\dot{\mathbf{B}}[\pi^c]$, where the notation $\dot{\mathbf{B}}[\pi^c]$ is as defined in [11, §29.2.3].

4. Integral forms

We will now extend the results obtained thus far to integral forms (over the ring $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$ of Laurent polynomials in v).

4.1. One has an integral form ${}_{\mathcal{A}}\mathbf{S}(\pi)$ in $\mathbf{S}(\pi)$. It is by definition the $\mathcal{A}$-subalgebra of $\mathbf{S}(\pi)$ generated by all $E_{\pm i}^{(m)}$ ($i \in I$, $m \in \mathbb{N}$) and 1_λ ($\lambda \in W\pi$). There is an algebra isomorphism

$$\mathbf{S}(\pi) \simeq \mathbb{Q}(v) \otimes_{\mathcal{A}} \big({}_{\mathcal{A}}\mathbf{S}(\pi)\big)$$

which carries $E_{\pm i}^{(m)}$ to $1 \otimes E_{\pm i}^{(m)}$ and 1_λ to $1 \otimes 1_\lambda$. Note that the elements K_h ($h \in Y$) in $\mathbf{S}(\pi)$ in fact belong to the subalgebra ${}_{\mathcal{A}}\mathbf{S}(\pi)$.

It is easy to see (see [5, §5.1]) that ${}_{\mathcal{A}}\mathbf{S}(\pi)$ is isomorphic with a quotient of the Lusztig $\mathcal{A}$-form ${}_{\mathcal{A}}\mathbf{U}$ of $\mathbf{U}$, which is by definition [11, §3.1.13] the $\mathcal{A}$-subalgebra of $\mathbf{U}$ generated by all $E_{\pm i}^{(m)}$ ($i \in I$, $m \geqslant 0$) and K_h ($h \in Y$). The quotient map ${}_{\mathcal{A}}\mathbf{U} \to {}_{\mathcal{A}}\mathbf{S}(\pi)$ sends $E_{\pm i}^{(m)}$ to $E_{\pm i}^{(m)}$ and K_h to K_h ($i \in I$, $m \geqslant 0$, $h \in Y$). Hence it is just the restriction of p_π to ${}_{\mathcal{A}}\mathbf{U}$; we denote it also by p_π.

Clearly the integral form on $\mathbf{S}(\pi)$ is compatible with the maps $f_{\pi,\pi'}$ in the sense that the restriction of $f_{\pi,\pi'}$ to ${}_{\mathcal{A}}\mathbf{S}(\pi')$ is a surjective map of $\mathcal{A}$-algebras from ${}_{\mathcal{A}}\mathbf{S}(\pi')$ onto ${}_{\mathcal{A}}\mathbf{S}(\pi)$. Recall the identification

$$\widehat{\mathbf{U}} = \left\{ (a_\pi)_\pi \in \prod_\pi \mathbf{S}(\pi) \;\middle|\; f_{\pi,\pi'}(a_{\pi'}) = a_\pi \text{ whenever } \pi \subset \pi' \right\}.$$

Inside this algebra we have an $\mathcal{A}$-subalgebra

$${}_{\mathcal{A}}\widehat{\mathbf{U}} = \left\{ (a_\pi)_\pi \in \prod_\pi \big({}_{\mathcal{A}}\mathbf{S}(\pi)\big) \;\middle|\; f_{\pi,\pi'}(a_{\pi'}) = a_\pi \text{ whenever } \pi \subset \pi' \right\}.$$

It is clear that ${}_{\mathcal{A}}\widehat{\mathbf{U}}$ is isomorphic with $\varprojlim({}_{\mathcal{A}}\mathbf{S}(\pi))$ (an isomorphism of $\mathcal{A}$-algebras).

4.2. Theorem. *The map θ (see 2.5) restricts to an algebra embedding (also denoted θ) of ${}_{\mathcal{A}}\mathbf{U}$ into ${}_{\mathcal{A}}\widehat{\mathbf{U}}$ sending $E_{\pm i}^{(m)} \to \widehat{E}_{\pm i}^{(m)}$ and K_h to $\widehat{K}_h$ for all $i \in I$, $m \geqslant 0$, $h \in Y$. Hence, ${}_{\mathcal{A}}\mathbf{U}$ is isomorphic with the $\mathcal{A}$-subalgebra of ${}_{\mathcal{A}}\widehat{\mathbf{U}}$ generated by all $\widehat{E}_{\pm i}^{(m)}$, $\widehat{K}_h$ ($i \in I$, $m \geqslant 0$, $h \in Y$).*

Proof. If $\widehat{p}_\pi$ denotes projection to the πth component as before, we have a commutative diagram of $\mathcal{A}$-algebras similar to the commutative diagram considered in 2.5, where the algebras are replaced by their integral forms and each map is just the restriction to the integral form of the corresponding map in the diagram given in 2.5. The existence of the map θ is guaranteed by the universal property of inverse limits, and by considering its effect on generators we see that it must in fact be the restriction to ${}_{\mathcal{A}}\mathbf{U}$ of the map θ given already in 2.5. Since θ is a restriction of an injective map, it is itself injective. □

4.3. Now consider the $\mathcal{A}$-subalgebra ${}_{\mathcal{A}}\dot{\mathbf{U}}$ of $\dot{\mathbf{U}}$ generated by all products of the form $E_{\pm i}^{(m)} 1_\lambda$ ($i \in I$, $m \geqslant 0$, $\lambda \in X$). This integral form of $\dot{\mathbf{U}}$ was studied in [11, §23.2].

The restriction of the quotient map $\dot{p}_\pi$ (see 3.2) to ${}_{\mathcal{A}}\dot{\mathbf{U}}$ gives a surjective map (also denoted by $\dot{p}_\pi$) from ${}_{\mathcal{A}}\dot{\mathbf{U}}$ to ${}_{\mathcal{A}}\mathbf{S}(\pi)$. This is clear from the definition of ${}_{\mathcal{A}}\mathbf{S}(\pi)$ given in 4.1. There is a commutative diagram similar to the diagram considered in 3.2, in which all the algebras are replaced by their integral forms, and the maps are just the restrictions of the maps considered in the diagram 3.2. As before, the universal property of inverse limits guarantees the existence of a unique algebra map $\dot{\theta} : {}_{\mathcal{A}}\dot{\mathbf{U}} \to {}_{\mathcal{A}}\widehat{\mathbf{U}}$ making the diagram commute.

4.4. Theorem. *The map $\dot{\theta}$ is an algebra embedding of ${}_{\mathcal{A}}\dot{\mathbf{U}}$ into ${}_{\mathcal{A}}\widehat{\mathbf{U}}$ sending $E_{\pm i}^{(m)} 1_\lambda$ to $\widehat{E}_{\pm i}^{(m)} \widehat{1}_\lambda$ for all $i \in I$, $m \geqslant 0$, $\lambda \in X$. Hence, the $\mathcal{A}$-subalgebra of ${}_{\mathcal{A}}\widehat{\mathbf{U}}$ generated by the $\widehat{E}_{\pm i}^{(m)} \widehat{1}_\lambda$ ($i \in I$, $m \geqslant 0$, $\lambda \in X$) is isomorphic with ${}_{\mathcal{A}}\dot{\mathbf{U}}$.*

Proof. The map $\dot{\theta}$ is the restriction to ${}_{\mathcal{A}}\dot{\mathbf{U}}$ of the injective map $\dot{\theta}$ considered in the proof of 3.3, thus injective. □

4.5. As before, fix a total ordering $\pi_1, \pi_2, \ldots$ on the finite saturated sets π compatible with the partial order given by set inclusion. Then the completion of ${}_{\mathcal{A}}\dot{\mathbf{U}}$ with respect to the descending sequence of ideals

$$ {}_{\mathcal{A}}\dot{\mathbf{U}} \supseteq {}_{\mathcal{A}}\dot{\mathbf{U}}\big[\pi_1^c\big] \supseteq {}_{\mathcal{A}}\dot{\mathbf{U}}\big[\pi_2^c\big] \supseteq \cdots $$

is isomorphic with ${}_{\mathcal{A}}\widehat{\mathbf{U}}$.

One may put a topology on ${}_{\mathcal{A}}\dot{\mathbf{U}}$ exactly as in 3.5, by letting the collection $\{{}_{\mathcal{A}}\dot{\mathbf{U}}[\pi^c]\}$, as π varies over the finite saturated subsets of X^+, define a neighborhood base of 0. Here ${}_{\mathcal{A}}\dot{\mathbf{U}}[\pi^c] = {}_{\mathcal{A}}\dot{\mathbf{U}} \cap \dot{\mathbf{U}}[\pi^c]$ is the kernel of the surjection $\dot{p}_\pi : {}_{\mathcal{A}}\dot{\mathbf{U}} \to {}_{\mathcal{A}}\mathbf{S}(\pi)$.

Since the canonical basis $\dot{\mathbf{B}}$ is an $\mathcal{A}$-basis of ${}_{\mathcal{A}}\dot{\mathbf{U}}$, it follows that elements of ${}_{\mathcal{A}}\widehat{\mathbf{U}}$ may be regarded as formal (possibly infinite) $\mathcal{A}$-linear combinations of $\dot{\mathbf{B}}$. Then the subalgebra ${}_{\mathcal{A}}\dot{\mathbf{U}}$ may be regarded as the set of all finite $\mathcal{A}$-linear combinations of $\dot{\mathbf{B}}$.

5. Specialization

By specializing to a commutative ring R (via the ring homomorphism $\mathcal{A} \to R$ determined by $v \to \xi$ for an invertible $\xi \in R$) one obtains generalized q-Schur algebras ${}_R\mathbf{S}(\pi)$ over R for each saturated π. These algebras form an inverse system; we study the corresponding inverse limit ${}_R\widehat{\mathbf{U}}$.

5.1. Let R be a given commutative ring with 1, and $\xi \in R$ a given invertible element. Regard R as an $\mathcal{A}$-algebra via the ring homomorphism $\mathcal{A} \to R$ such that $v^n \to \xi^n$ for all $n \in \mathbb{Z}$. Consider the R-algebras

(a) $$_R\mathbf{U} = R \otimes_{\mathcal{A}} ({}_{\mathcal{A}}\mathbf{U}), \qquad {}_R\dot{\mathbf{U}} = R \otimes_{\mathcal{A}} ({}_{\mathcal{A}}\dot{\mathbf{U}}).$$

In the literature, these algebras are sometimes denoted by alternative notations such as $\mathbf{U}_\xi$, $\dot{\mathbf{U}}_\xi$. We note that one has isomorphisms ${}_{\mathbb{Q}(v)}\mathbf{U} \simeq \mathbf{U}$, ${}_{\mathbb{Q}(v)}\dot{\mathbf{U}} \simeq \dot{\mathbf{U}}$ given by the obvious maps.

For a finite saturated subset π of X^+ set ${}_R\mathbf{S}(\pi) = R \otimes_{\mathcal{A}} ({}_{\mathcal{A}}\mathbf{S}(\pi))$, a generalized q-Schur algebra specialized at $v \to \xi$. Note that ${}_{\mathbb{Q}(v)}\mathbf{S}(\pi) \simeq \mathbf{S}(\pi)$. The elements $E_{\pm i}^{(m)}$, 1_λ, $K_h \in {}_{\mathcal{A}}\mathbf{S}(\pi)$ give rise to corresponding elements

$$1 \otimes E_{\pm i}^{(m)}, \qquad 1 \otimes 1_\lambda, \qquad 1 \otimes K_h \in {}_R\mathbf{S}(\pi)$$

for $i \in I$, $m \geqslant 0$, $\lambda \in X$, $h \in Y$. These elements of ${}_R\mathbf{S}(\pi)$ will be respectively denoted again by $E_{\pm i}^{(m)}$, 1_λ, K_h, since the intended meaning will be clear from the context.

Since tensoring is right exact, we have a surjective quotient map

(b) $$1 \otimes \dot{p}_\pi : {}_R\dot{\mathbf{U}} \to {}_R\mathbf{S}(\pi),$$

arising (by tensoring with the identity map on R) from the corresponding quotient map $\dot{p}_\pi : {}_{\mathcal{A}}\dot{\mathbf{U}} \to {}_{\mathcal{A}}\mathbf{S}(\pi)$ over $\mathcal{A}$.

5.2. Lemma. *The kernel of the map* $1 \otimes \dot{p}_\pi$ *is* ${}_R\dot{\mathbf{U}}[\pi^c] = R \otimes_{\mathcal{A}} ({}_{\mathcal{A}}\dot{\mathbf{U}}[\pi^c])$.

Proof. This is a consequence of the fact that the canonical basis of ${}_{\mathcal{A}}\dot{\mathbf{U}}$ is a “cellular” basis (in the sense of [6]) and the kernel ${}_{\mathcal{A}}\dot{\mathbf{U}}[\pi^c]$ of the map $\dot{p}_\pi : {}_{\mathcal{A}}\dot{\mathbf{U}} \to {}_{\mathcal{A}}\mathbf{S}(\pi)$ is the cell ideal spanned by the canonical basis elements in $\dot{\mathbf{B}}[\pi^c]$. See [5, §5] for details. □

5.3. Whenever $\pi \subset \pi'$ (for finite saturated subsets of X^+) there is a surjective algebra map $1 \otimes f_{\pi,\pi'} : {}_R\mathbf{S}(\pi') \to {}_R\mathbf{S}(\pi)$ obtained from the map $f_{\pi,\pi'} : {}_{\mathcal{A}}\mathbf{S}(\pi') \to {}_{\mathcal{A}}\mathbf{S}(\pi)$ by tensoring with the identity map on R. Thus we have an inverse system

(a) $$\{{}_R\mathbf{S}(\pi);\ 1 \otimes f_{\pi,\pi'}\}.$$

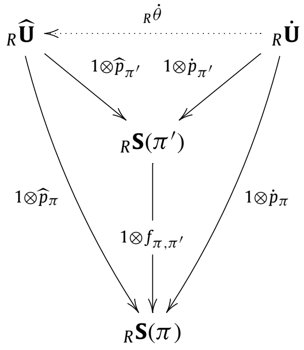
We define ${}_R\widehat{\mathbf{U}}$ to be the inverse limit $\varprojlim {}_R\mathbf{S}(\pi)$ of this inverse system. In ${}_R\widehat{\mathbf{U}}$ we have elements

(b) $$\widehat{E}_{\pm i}^{(m)} := \big(E_{\pm i}^{(m)}\big)_\pi, \qquad \widehat{1}_\lambda := (1_\lambda)_\pi, \qquad \widehat{K}_h := (K_h)_\pi$$

of ${}_R\widehat{\mathbf{U}}$ defined by the corresponding constant sequences, for $i \in I$, $m \geqslant 0$, $\lambda \in X$, $h \in Y$. Actually, the sequence defining 1_λ is not necessarily constant, but it is eventually constant.

5.4. Theorem. *There is an embedding* ${}_R\dot{\mathbf{U}} \to {}_R\widehat{\mathbf{U}}$ *of R-algebras sending* $E_{\pm i}^{(m)}1_\lambda$ *to* $\widehat{E}_{\pm i}^{(m)}\widehat{1}_\lambda \in {}_R\widehat{\mathbf{U}}$ *for all* $i \in I$, $m \geqslant 0$, $\lambda \in X$. *Thus* ${}_R\dot{\mathbf{U}}$ *may be identified with the R-subalgebra of* ${}_R\widehat{\mathbf{U}}$ *generated by all* $\widehat{E}_{\pm i}^{(m)}\widehat{1}_\lambda$ ($i \in I$, $m \geqslant 0$, $\lambda \in X$).

Proof. Consider the commutative diagram of R-algebra maps



for any finite saturated $\pi \subset \pi'$. The universal property of inverse limits guarantees the existence of a unique algebra map $_R\dot{\theta} : {_R\dot{\mathbf{U}}} \to {_R\widehat{\mathbf{U}}}$ making the diagram commute. This map has the desired properties.

By Lemma 5.2, the kernel of $_R\dot{\theta}$ is the intersection over π of all $_R\dot{\mathbf{U}}[\pi^c]$. From Lusztig's results [11, Chapter 29] it follows that this intersection is the zero ideal (0). Indeed, the intersection is contained in the intersection of all $_R\dot{\mathbf{U}}[\geqslant \lambda]$ as λ runs through all dominant weights, and by known properties of the canonical basis the latter intersection is (0). This proves the injectivity of $_R\dot{\theta}$, as desired. □

5.5. As before, fix a total ordering $\pi_1, \pi_2, \ldots$ on the finite saturated sets π which is compatible with the partial order given by set inclusion. Then the completion of $_R\dot{\mathbf{U}}$ with respect to the descending sequence of ideals

$$_R\dot{\mathbf{U}} \supseteq {_R\dot{\mathbf{U}}}\big[\pi_1^c\big] \supseteq {_R\dot{\mathbf{U}}}\big[\pi_2^c\big] \supseteq \cdots$$

is isomorphic with $_R\widehat{\mathbf{U}}$.

One may put a topology on $_R\dot{\mathbf{U}}$, by letting the collection $\{{_R\dot{\mathbf{U}}}[\pi^c]\}$, as π varies over the finite saturated subsets of X^+, define a neighborhood base of 0. Elements of $_R\widehat{\mathbf{U}}$ may be regarded as formal (possibly infinite) R-linear combinations of $\dot{\mathbf{B}}$. Then the subalgebra $_R\dot{\mathbf{U}}$ may be regarded as the set of all finite R-linear combinations of $\dot{\mathbf{B}}$. The topology on $_R\widehat{\mathbf{U}}$ is induced from the topology on $_R\dot{\mathbf{U}}$; i.e., the basic neighborhoods of 0 are of the form $_R\widehat{\mathbf{U}}[\pi^c]$ for the various finite saturated sets π, where $_R\widehat{\mathbf{U}}[\pi^c]$ is the set of all formal R-linear combinations of elements of $\dot{\mathbf{B}}[\pi^c]$.

5.6. Remark. It is not immediately clear how the completion $_R\widehat{\mathbf{U}}$ is related to $R \otimes_{\mathcal{A}} ({_{\mathcal{A}}\widehat{\mathbf{U}}})$. We note that there is a well-defined homomorphism of R-algebras

$$R \otimes_{\mathcal{A}} ({_{\mathcal{A}}\widehat{\mathbf{U}}}) \to {_R\widehat{\mathbf{U}}} \tag{a}$$

sending $1 \otimes (a_\pi)$ to $(1 \otimes a_\pi)$, where $(a_\pi) \in \prod_\pi ({_{\mathcal{A}}\mathbf{S}(\pi)})$ satisfies the condition $f_{\pi',\pi}(a_{\pi'}) = a_\pi$ whenever $\pi \subset \pi'$. It seems unlikely that this map is an isomorphism.

5.7. Remark. Does the algebra $_R\mathbf{U}$ embed in $_R\widehat{\mathbf{U}}$? There is an R-algebra homomorphism

$$_R\mathbf{U} \to {_R\widehat{\mathbf{U}}} \tag{a}$$

defined on generators by sending $E_{\pm i}^{(m)}$ to $\widehat{E}_{\pm i}^{(m)}$ (for $i \in I$, $m \geqslant 0$), and sending K_h to $\widehat{K}_h$ (for $h \in Y$). But it is not clear that this map is injective.

One would like to prove the injectivity of this map, since then one may identify $_R\mathbf{U}$ with a subalgebra of the completion $_R\widehat{\mathbf{U}}$.

We sketch a possible approach to this question. To prove the injectivity, one needs to show that the intersection of the kernels of the quotient maps ${}_R\mathbf{U} \to {}_R\mathbf{S}(\pi)$ is (0). The quotient map ${}_R\mathbf{U} \to {}_R\mathbf{S}(\pi)$ is the map $1 \otimes p_\pi$ obtained from the quotient map $p_\pi : {}_\mathcal{A}\mathbf{U} \to {}_\mathcal{A}\mathbf{S}(\pi)$ defined in 4.1, by tensoring with the identity map on R.

Suppose some $u \in {}_R\mathbf{U}$ belongs to the intersection of the kernels of the quotient maps ${}_R\mathbf{U} \to {}_R\mathbf{S}(\pi)$, as π varies over the finite saturated subsets of X^+. Then one can show that u acts as zero on any finite dimensional ${}_R\mathbf{U}$-module, since such a module will be a well-defined module for ${}_R\mathbf{S}(\pi)$ for some (large enough) saturated set π. Now if [9, Proposition 5.11] can be generalized to our setting, we would be able to conclude that $u = 0$, and the desired injectivity statement would be established. However, such a generalization is not available in the published literature on quantum groups, to the author's knowledge.

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