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Sharp well-posedness for Kadomtsev–Petviashvili–Burgers (KPBII) equation in $\mathbb{R}^2$

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Abstract

We prove global well-posedness for the Cauchy problem associated with the Kadomtsev–Petviashvili–Burgers equation (KPBII) in $\mathbb{R}^2$ when the initial value belongs to the anisotropic Sobolev space $H^{s_1,s_2}(\mathbb{R}^2)$ for all $s_1 > -\frac{1}{2}$ and $s_2 \geqslant 0$. On the other hand, we prove in some sense that our result is sharp.

1. Introduction

We shall study the initial value problem of the Kadomtsev–Petviashvili–Burgers (KPBII) in $\mathbb{R}^2$:

$$\begin{cases} (\partial_t u + u_{xxx} - u_{xx} + uu_x)_x + u_{yy} = 0, \\ u(0,x,y) = \varphi(x,y). \end{cases} \tag{1.1}$$

This equation is a dissipative version of the Kadomtsev–Petviashvili-II equation (KPII):

$$\begin{cases} (\partial_t u + u_{xxx} + uu_x)_x + u_{yy} = 0, \\ u(0,x,y) = \varphi(x,y). \end{cases} \tag{1.2}$$

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The (KP) equation is a universal model for nearly one directional weakly nonlinear dispersive waves with weak transverse effects. It is a natural two-dimensional extension of the celebrated (KdV) equation:

$$u_t + u_{xxx} + uu_x = 0.$$

In some typical situations, it is not possible to neglect dissipative effects (due to viscosity effects in magneto sonic waves damped by electron–ion collisions for example), and this can lead to the KdV–Burgers equation (cf. [11]):

$$\partial_t u + u_{xxx} + uu_x - u_{xx} = 0.$$

It is then widely accepted that the (KPBII) equation is a natural model for the propagation of the two-dimensional damped waves. Note that as we are interested in nearly one directional propagation, the dissipative term only acts in the main direction of propagation in (1.1).

Bourgain had developed a new method, clarified by Ginibre in [4], for the study of Cauchy problem associated with dispersive nonlinear equations. This method was successfully applied to Schrödinger, (KdV) as well as (KPII) equation (cf. [1–3,6]). It was shown by Molinet and Ribaud [8] that the Bourgain spaces can be used to study the Cauchy problems associated to semi-linear equations with a linear part containing both dispersive and dissipative terms (and consequently this applies to (KPB) equations).

For the Cauchy problem associated to (KPII) equation, the local existence is proved by Bourgain [1] when the initial value is in the space $L^2(\mathbb{R}^2)$ and by Takaoka and Tzvetkov [13] when the initial value $\varphi \in H^{s_1,s_2}(\mathbb{R}^2)$ with $s_1 > -\frac{1}{3}$ and $s_2 \geqslant 0$.

By introducing a Bourgain space associated to the usual (KPII) equation (related only to the dispersive part of the linear symbol of (1.1)), Molinet and Ribaud [8] had proved global existence for the Cauchy problem associated to the (KPBII) equation when the initial value is in $L^2(\mathbb{R}^2)$.

In this paper, we prove local existence for (1.1) with initial value $\varphi \in H^{s_1,s_2}(\mathbb{R}^2)$ when $s_1 > -\frac{1}{2}$ and $s_2 \geqslant 0$. Following [9] (see also [7]), we introduce a Bourgain space associated to the (KPBII) equation. This space is in fact the intersection of the space introduced in [1] and of a Sobolev space. The advantage of this space is that it contains both the dissipative and dispersive parts of the linear symbol of (1.1).

We prove also that our local existence theorem is optimal by constructing a counterexample showing that the application $\varphi \mapsto u$ from H^{s_1,s_2} to $C([0,T]; H^{s_1,s_2})$ cannot be regular for $s_1 < -\frac{1}{2}$ and $s_2 = 0$.

This paper is organized as follows. In Section 2, we introduce our notations and we give an extension of the semi-group of the (KPBII) equation by a linear operator defined on all the real axis. In Section 3 we derive linear estimates and some smoothing properties for the operator L defined by (3.7) in the Bourgain spaces. In Section 4 we state Strichartz type estimates for the (KP) equation which yield bilinear estimates in Section 5. In Section 6, using bilinear estimates, a standard fixed point argument and some smoothing properties, we prove uniqueness and global existence of the solution of (1.1) in anisotropic Sobolev space $H^{s_1,s_2}(\mathbb{R}^2)$ with $s_1 > -\frac{1}{2}$ and $s_2 \geqslant 0$. Finally, we construct in Section 7 a sequence of initial values which ensures that our local existence result is optimal if one requires the smoothness of the flow-map. Note that there is no scaling for (1.1) and that, on the other hand, $H^{-1/2,0}$ is critical for the scaling of (1.2).

2. Notations and main results

We will use C to denote various time independent constants, usually depending only upon s. In case a constant depends upon other quantities, we will try to make it explicit. We use $A \lesssim B$ to denote an estimate of the form $A \leqslant CB$. Similarly, we will write $A \sim B$ to mean $A \lesssim B$ and $B \lesssim A$. We write $\langle \cdot \rangle := (1 + |\cdot|^2)^{1/2} \sim 1 + |\cdot|$. The notation a^+ denotes $a + \epsilon$ for an arbitrarily small ϵ. Similarly $a-$ denotes $a - \epsilon$. For $b \in \mathbb{R}$, we denote respectively by $H^b(\mathbb{R})$ and $\dot{H}^b(\mathbb{R})$ the nonhomogeneous and homogeneous Sobolev spaces which are endowed with the following norms:

$$\|u\|_{H^b}^2 = \int_{\mathbb{R}} \langle \tau \rangle^{2b} \big|\hat{u}(\tau)\big|^2 \, d\tau, \qquad \|u\|_{\dot{H}^b}^2 = \int_{\mathbb{R}} |\tau|^{2b} \big|\hat{u}(\tau)\big|^2 \, d\tau, \tag{2.1}$$

where $\hat{\cdot}$ denotes the Fourier transform from $\mathcal{S}'(\mathbb{R}^2)$ to $\mathcal{S}'(\mathbb{R}^2)$ which is defined by

$$\hat{f}(\xi) := \mathcal{F}(f)(\xi) = \int_{\mathbb{R}^2} e^{i\langle \lambda, \xi \rangle} f(\lambda) \, d\lambda, \quad \forall f \in \mathcal{S}'\big(\mathbb{R}^2\big).$$

Moreover, we introduce the corresponding space (respectively space–time) Sobolev spaces H^{s_1,s_2} (respectively H^{b,s_1,s_2}) which are defined by

$$H^{s_1,s_2}\big(\mathbb{R}^2\big) =: \big\{u \in \mathcal{S}'\big(\mathbb{R}^2\big);\ \|u\|_{H^{s_1,s_2}}\big(\mathbb{R}^2\big) < +\infty\big\}, \tag{2.2}$$

$$H^{b,s_1,s_2}\big(\mathbb{R}^2\big) =: \big\{u \in \mathcal{S}'\big(\mathbb{R}^3\big);\ \|u\|_{H^{b,s_1,s_2}}\big(\mathbb{R}^3\big) < +\infty\big\}, \tag{2.3}$$

where

$$\|u\|_{H^{s_1,s_2}}^2 = \int_{\mathbb{R}^2} \langle \xi \rangle^{2s_1} \langle \eta \rangle^{2s_2} \big|\hat{u}(\nu)\big|^2 \, d\nu, \tag{2.4}$$

$$\|u\|_{H^{b,s_1,s_2}}^2 = \int_{\mathbb{R}^2} \langle \tau \rangle^{b} \langle \xi \rangle^{2s_1} \langle \eta \rangle^{2s_2} \big|\hat{u}(\tau, \nu)\big|^2 \, d\nu \, d\tau, \tag{2.5}$$

and $\nu = (\xi, \eta)$. Let $U(\cdot)$ be the unitary group in H^{s_1,s_2}, $s_1, s_2 \in \mathbb{R}$, defining the free evolution of the (KPII) equation, which is given by

$$U(t) = \exp\big(itP(D_x, D_y)\big), \tag{2.6}$$

where $P(D_x, D_y)$ is the Fourier multiplier with symbol $P(\xi, \eta) = \xi^3 - \eta^2/\xi$. By the Fourier transform, (2.6) can be written like

$$\mathcal{F}_x\big(U(t)\phi\big) = \exp\big(itP(\xi, \eta)\big)\hat{\phi}, \quad \forall \phi \in \mathcal{S}'\big(\mathbb{R}^2\big),\ t \in \mathbb{R}. \tag{2.7}$$

Also, by the Fourier transform, the linear part Eq. (1.1) can be written as

$$i\big(\tau - \xi^3 - \eta^2/\xi\big) + \xi^2 =: i\big(\tau - P(\eta, \xi)\big) + \xi^2. \tag{2.8}$$

Following [8], we introduce a Bourgain space which is in relation with both the dissipative and dispersive parts of (1.1) at the same time, we define this space by

$$X^{b,s_1,s_2} = \{u \in \mathcal{S}'(\mathbb{R}^3),\ \|u\|_{X^{b,s_1,s_2}} < \infty\} \tag{2.9}$$

equipped with the norm

$$\|u\|_{X^{b,s_1,s_2}} = \big\|\langle i\sigma + \xi^2\rangle^b \langle\xi\rangle^{s_1}\langle\eta\rangle^{s_2}\hat{w}(\tau,\nu)\big\|_{L^2(\mathbb{R}^3)}, \tag{2.10}$$

where $\sigma = \tau - P(\nu)$, $\nu = (\xi,\eta) \in \mathbb{R}^2$.

Remark 2.1. It is worth noticing that X^{b,s_1,s_2} is the intersection of the Bourgain space associated with the dispersive part of Eq. (1.1) and Sobolev space. Indeed, by noticing that $\mathcal{F}(U(-t)u)(\tau,\nu) = \mathcal{F}(u)(\tau + P(\nu),\nu)$ and next by performing the change of variable $\tau \to \tau - P(\nu)$, one sees that

$$\begin{aligned}
\|u\|_{X^{b,s_1,s_2}} &= \big\|\langle i\tau + \xi^2\rangle^b \langle\xi\rangle^{s_1}\langle\eta\rangle^{s_2}\hat{w}\big(\tau + P(\nu),\nu\big)\big\|_{L^2_{\tau,\nu}(\mathbb{R}^3)} \\
&= \big\|\langle i\tau + \xi^2\rangle^b \langle\xi\rangle^{s_1}\langle\eta\rangle^{s_2}\mathcal{F}\big(U(-t)u\big)(\tau,\nu)\big\|_{L^2_{\tau,\nu}(\mathbb{R}^3)} \\
&\sim \big\|\langle\tau\rangle^b \langle\xi\rangle^{s_1}\langle\eta\rangle^{s_2}\mathcal{F}\big(U(-t)u\big)(\tau,\nu)\big\|_{L^2_{\tau,\nu}(\mathbb{R}^3)} \\
&\quad + \big\|\langle\xi\rangle^{s_1+2b}\langle\eta\rangle^{s_2}\mathcal{F}\big(U(-t)u\big)(\tau,\nu)\big\|_{L^2_{\tau,\nu}(\mathbb{R}^3)} \\
&= \big\|U(-t)u\big\|_{H^{b,s_1,s_2}} + \|u\|_{L^2_t H^{s_1+2b,s_2}}.
\end{aligned}$$

For $T > 0$, we define the restricted spaces X_T^{b,s_1,s_2} by the norm

$$\|u\|_{X_T^{b,s_1,s_2}} = \inf_{w\in X^{b,s_1,s_2}} \big\{\|w\|_{X^{b,s_1,s_2}};\ w(t) = u(t) \text{ on } [0,T]\big\}. \tag{2.11}$$

We denote by $W(\cdot)$ the semi-group associated with the free evolution of (1.1),

$$\mathcal{F}_x\big(W(t)\phi\big) = \exp\big(itP(\xi,\eta) - |\xi|^2 t\big)\hat{\phi}, \quad \forall\phi \in \mathcal{S}'(\mathbb{R}^2),\ t \geqslant 0. \tag{2.12}$$

Also, we can extend W to a linear operator defined on the whole real axis by setting

$$\mathcal{F}_x\big(W(t)\phi\big) = \exp\big(itP(\xi,\eta) - |\xi|^2|t|\big)\hat{\phi}, \quad \forall\phi \in \mathcal{S}'(\mathbb{R}^2),\ t \in \mathbb{R}. \tag{2.13}$$

By the Duhamel integral formulation, Eq. (1.1) can be written

$$u(t) = W(t)\phi - \frac{1}{2}\int_0^t W(t-t')\partial_x\big(u^2(t')\big)\,dt', \quad t \geqslant 0. \tag{2.14}$$

To prove the local existence result, we will apply a fixed point argument to a truncated version of (2.14) which is defined on all the real axis by

$$u(t)=\psi(t)\left[W(t)\phi-\frac{\chi_{\mathbb{R}_+}(t)}{2}\int_0^t W(t-t')\partial_x\big(\psi_T^2(t')u^2(t')\big)\,dt'\right],\tag{2.15}$$

where $t\in\mathbb{R}$ and ψ indicates a time cutoff function:

$$\psi\in C_0^\infty(\mathbb{R}),\qquad \sup\psi\subset[-2,2],\qquad \psi=1\quad \text{on } [-1,1],\tag{2.16}$$

and $\psi_T(.)=\psi(./T)$.

Remark 2.2. It is clear that if u solves (2.15) then u is a solution of (2.14) on $[0,T]$, $T<1$. Thus it is sufficient to solve (2.15) for a small time ($T<1$ is enough).

Let us now state our results:

Theorem 2.1. *Let $s_1>-1/2$, $s_2\geqslant 0$, $s_c^1\in\,]-1/2,\min(0,s_1)]$ and $\phi\in H^{s_1,s_2}$. Then there exist a time $T=T(\|\phi\|_{H^{s_c^1,0}})>0$ and a unique solution u of* (1.1) *in*

$$Y_T=C\big([0,T];H^{s_1,s_2}\big)\cap X_T^{1/2,s_1,s_2}.\tag{2.17}$$

Moreover, $u\in C(\mathbb{R}_+;H^{s_1,s_2})\cap C(\mathbb{R}_+^*;H^{\infty,s_2})$ and the map $\phi\mapsto u$ is C^∞ from H^{s_1,s_2} to Y_T.

Theorem 2.2. *Let $s<-1/2$. Then it does not exist a time $T>0$ such that Eq.* (1.1) *admits a unique solution in $C([0,T[,H^{s,0})$ for any initial data in some ball of $H^{s,0}(\mathbb{R}^2)$ centered at the origin and such that the map*

$$\phi\mapsto u\tag{2.18}$$

is C^2-differentiable at the origin from $H^{s,0}$ to $C([0,T],H^{s,0})$.

3. Linear estimates in X^{b,s_1,s_2}

In this section we study both the free and the forcing terms of the integral equation (2.15) to obtain certain estimates necessary to apply a fixed point argument. The results of this section are essentially contained in [8]. The following lemma will be of constant use in this section:

Lemma 3.1. *Let $b\in\mathbb{R}$ and $\lambda>0$. Then*

$$\big\|f(\lambda t)\big\|_{H^b}=\big(\lambda^{-1/2}+\lambda^{b-1/2}\big)\big\|f(t)\big\|_{H^b},\tag{3.1}$$

$$\big\|f(\lambda t)\big\|_{\dot H^b}=\lambda^{b-1/2}\big\|f(t)\big\|_{\dot H^b}.\tag{3.2}$$

Proposition 3.2. *Let $s_1,s_2\in\mathbb{R}$ and $0\leqslant b\leqslant 1/2$. For all $\phi\in H^{s_1,s_2}$ we have*

$$\big\|\psi(t)W(t)\phi\big\|_{X^{b,s_1,s_2}}\leqslant C\|\phi\|_{H^{s_1+2b-1,s_2}}.\tag{3.3}$$

Proof. By definition of $W(\cdot)$ and X^{b,s_1,s_2}, and by performing the change of variable $\tau \mapsto \sigma := \tau - P(\nu)$ we have

$$\begin{aligned}\left\|\psi(t)W(t)\phi\right\|_{X^{b,s_1,s_2}} &= \left\|\left\langle i\sigma+\xi^2\right\rangle^b\langle\xi\rangle^{s_1}\langle\eta\rangle^{s_2}\mathcal{F}_t\left(\psi(t)e^{-|t|\xi^2}e^{itP(\nu)}\hat{\phi}(\nu)\right)(\tau)\right\|_{L^2}\\ &= \left\|\left\langle i\tau+\xi^2\right\rangle^b\langle\xi\rangle^{s_1}\langle\eta\rangle^{s_2}\mathcal{F}_t\left(\psi(t)e^{-|t|\xi^2}\hat{\phi}(\nu)\right)(\tau)\right\|_{L^2}\\ &\leqslant \left\|\langle\xi\rangle^{s_1+2b}\langle\eta\rangle^{s_2}\hat{\phi}(\nu)\left\|\mathcal{F}_t\left(\psi(t)e^{-|t|\xi^2}\right)(\tau)\right\|_{L^2_\tau}\right\|_{L^2_\nu}\\ &\quad+\left\|\langle\xi\rangle^{s_1}\langle\eta\rangle^{s_2}\hat{\phi}(\nu)\left\|\langle\tau\rangle^b\mathcal{F}_t\left(\psi(t)e^{-|t|\xi^2}\right)(\tau)\right\|_{L^2_\tau}\right\|_{L^2_\nu}.\end{aligned} \tag{3.4}$$

Let $0\leqslant b\leqslant 1/2$. For ξ fixed, we take $G_\xi(\tau)=\langle\tau\rangle^b\mathcal{F}_t(\psi(t)e^{-|t|\xi^2})(\tau)$. Noticing that, as in [8], we have the following estimate:

$$\|G_\xi\|_{L^2_\tau(\mathbb{R})}\leqslant C\langle\xi\rangle^{2b-1},\quad \forall 0\leqslant b\leqslant 1/2. \tag{3.5}$$

By combining these two last inequalities, we obtain the desired result. □

Now, for ξ fixed, we introduce the following time-Sobolev space:

$$Y^b_\xi=\left\{u\in\mathcal{S}'\left(\mathbb{R}^3\right);\ \|u\|_{Y^b_\xi}=:\left\|\left\langle i\tau+\xi^2\right\rangle^b\hat{u}(\tau)\right\|_{L^2_\tau(\mathbb{R})}<\infty\right\}. \tag{3.6}$$

In order to obtain certain estimates in X^{b,s_1,s_2} for the following operator

$$L: f\mapsto \chi_{\mathbb{R}_+}(t)\psi(t)\int_0^t W(t-t')f(t')\,dt' \tag{3.7}$$

we shall study in Y^b_ξ the following linear operator:

$$K: f\mapsto \psi(t)\int_0^t e^{-|t-t'|\xi^2}f(t')\,dt'. \tag{3.8}$$

Proposition 3.3. *Let $\xi\in\mathbb{R}$ fixed and $f\in\mathcal{S}(\mathbb{R}^3)$, $0<\delta\leqslant 1/2$. We consider the operator*

$$t\mapsto K_\xi(t)=\psi(t)\int_0^t e^{-|t-t'|\xi^2}f(t')\,dt'. \tag{3.9}$$

Then the following estimate holds:

$$\left\|K_\xi(t)\right\|_{Y^{1/2}_\xi}\leqslant C\langle\xi\rangle^{-2\delta}\|f\|_{Y^{-1/2+\delta}_\xi}. \tag{3.10}$$

Proof. A simple calculation in [8] gives

$$K_\xi(t) = \psi(t) \int_{\mathbb{R}} \frac{e^{it\tau} - e^{-|t|\xi^2}}{i\tau + \xi^2} \hat{f}(\tau)\, d\tau. \tag{3.11}$$

We can break up K_ξ in $K_\xi = K_{1,0} + K_{1,\infty} + K_{2,0} + K_{2,\infty}$, where

$$K_{1,0} =: \psi(t) \int_{|\tau|\leqslant 1} \frac{e^{it\tau} - 1}{i\tau + \xi^2} \hat{f}(\tau)\, d\tau, \qquad K_{1,\infty} = \psi(t) \int_{|\tau|\geqslant 1} \frac{e^{it\tau}}{i\tau + \xi^2} \hat{f}(\tau)\, d\tau,$$

$$K_{2,0} = \psi(t) \int_{|\tau|\leqslant 1} \frac{1 - e^{-|t|\xi^2}}{i\tau + \xi^2} \hat{f}(\tau)\, d\tau, \qquad K_{2,\infty} = \psi(t) \int_{|\tau|\geqslant 1} \frac{e^{-|t|\xi^2}}{i\tau + \xi^2} \hat{f}(\tau)\, d\tau.$$

Contribution of $K_{1,0}$. In this case, while using the asymptotic expansion, we have

$$K_{1,0} = \psi(t) \sum_{n\geqslant 1} \int_{|\tau|\leqslant 1} \frac{(it\tau)^n}{i\tau + \xi^2} \hat{f}(\tau)\, d\tau, \tag{3.12}$$

it results that

$$\begin{aligned} \big\|\langle i\tau + \xi^2\rangle^{1/2} \mathcal{F}_t(K_{1,0})\big\|_{L^2_\tau(\mathbb{R})} &\leqslant \sum_{n\geqslant 1} \left\| \langle i\tau + \xi^2\rangle^{1/2} \mathcal{F}_t\left(\frac{\psi(t)t^n}{n!}\right)\right\|_{L^2_\tau(\mathbb{R})} \\ &\quad \times \int_{|\tau|\leqslant 1} \frac{|i\tau|^n}{|i\tau + \xi^2|} \big|\hat{f}(\tau)\big|\, d\tau \\ &\leqslant \left(\sum_{n\geqslant 1} \left\| \frac{\psi(t)t^n}{n!} \right\|_{H^{1/2}_\tau} + |\xi| \left\| \frac{t^n\psi(t)}{n!} \right\|_{L^2_\tau} \right) \\ &\quad \times \int_{|\tau|\leqslant 1} \frac{|\tau|^n}{|i\tau + \xi^2|} \big|\hat{f}(\tau)\big|\, d\tau. \end{aligned} \tag{3.13}$$

Using the inequality $\|t^n\psi(t)\|_{H^b} \leqslant Cn$ for $b \in \{0, 1/2\}$, $n \geqslant 1$, together with the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned} \|K_{1,0}\|_{Y^{1/2}_\xi} &\leqslant C\big(1 + |\xi|\big) \left(\int_{|\tau|\leqslant 1} \frac{|\hat{f}(\tau)|^2}{\langle i\tau + \xi^2\rangle} d\tau \right)^{1/2} \left(\int_{|\tau|\leqslant 1} \frac{|\tau|^2 \langle i\tau + \xi^2\rangle}{|i\tau + \xi^2|^2} d\tau \right)^{1/2} \\ &\leqslant C\langle\xi\rangle \left(\int_{|\tau|\leqslant 1} \frac{|\hat{f}(\tau)|^2}{\langle i\tau + \xi^2\rangle} d\tau \right)^{1/2} \langle\xi\rangle^{-1} \\ &\leqslant C \left(\int_{|\tau|\leqslant 1} \frac{|\hat{f}(\tau)|^2}{\langle i\tau + \xi^2\rangle} d\tau \right)^{1/2}. \end{aligned} \tag{3.14}$$

Finally, since for $0 \leqslant \delta < 1/2$ we have $\langle i\tau + \xi^2 \rangle \geqslant \langle i\tau + \xi^2 \rangle^{1-2\delta} \langle \xi \rangle^{4\delta}$, it results that

$$\begin{aligned} \|K_{1,0}\|_{Y_\xi^{1/2}} &\leqslant C\langle \xi \rangle^{-2\delta} \left(\int\limits_{|\tau|\leqslant 1} \frac{|\hat{f}(\tau)|^2}{\langle i\tau + \xi^2 \rangle^{1-2\delta}} \, d\tau \right)^{1/2} \\ &\leqslant C\langle \xi \rangle^{-2\delta} \|f\|_{Y_\xi^{-1/2+\delta}}. \end{aligned} \tag{3.15}$$

Contribution of $K_{2,\infty}$. Note that

$$\begin{aligned} \left\| \langle i\tau + \xi^2 \rangle^{1/2} \mathcal{F}_t(K_{2,\infty}) \right\|_{L^2_\tau(\mathbb{R})} &\leqslant \left\| \langle i\tau + \xi^2 \rangle \mathcal{F}_t\big(\psi(t) e^{-\xi^2|t|}\big) \right\|_{L^2_\tau(\mathbb{R})} \\ &\quad \times \left(\int\limits_{|\tau|\geqslant 1} \frac{|\hat{f}(\tau)|}{\langle i\tau + \xi^2 \rangle} \, d\tau \right). \end{aligned} \tag{3.16}$$

Using the inequality (3.5), we get now that

$$\left\| \langle i\tau + \xi^2 \rangle^{1/2} \mathcal{F}_t\big(\psi(t) e^{-\xi^2|t|}\big) \right\|_{L^2_\tau(\mathbb{R})} \leqslant C, \tag{3.17}$$

therefore, by the Cauchy–Schwarz inequality we obtain

$$\begin{aligned} \|K_{2,\infty}\|_{Y_\xi^{1/2}} &\leqslant C \left(\int\limits_{|\tau|\geqslant 1} \frac{|\hat{f}(\tau)|}{\langle i\tau + \xi^2 \rangle} \, d\tau \right) \\ &\leqslant C \left(\int\limits_{\mathbb{R}} \frac{|\hat{f}(\tau)|^2}{\langle i\tau + \xi^2 \rangle^{1-2\delta}} \, d\tau \right)^{1/2} \left(\int\limits_{|\tau|\geqslant 1} \frac{\langle i\tau + \xi^2 \rangle^{-2\delta}}{\langle i\tau + \xi^2 \rangle} \, d\tau \right)^{1/2}. \end{aligned} \tag{3.18}$$

For $|\xi| \geqslant 1$, the following change of variable $\tau \mapsto r\xi^2$ gives

$$\begin{aligned} \|K_{2,\infty}\|_{Y_\xi^{1/2}} &\leqslant C \left(\int\limits_{\mathbb{R}} \frac{|\hat{f}(\tau)|^2}{\langle i\tau + \xi^2 \rangle^{1-2\delta}} \, d\tau \right)^{1/2} \langle \xi \rangle^{-2\delta} \left(\int\limits_{\mathbb{R}} \frac{1}{\langle r \rangle^{1+2\delta}} \, dr \right)^{1/2} \\ &\leqslant C\langle \xi \rangle^{-2\delta} \|f\|_{Y_\xi^{-1/2+\delta}} \end{aligned} \tag{3.19}$$

since we have $\int_{\mathbb{R}} \frac{1}{\langle r \rangle^{1+2\delta}} \, dr < \infty$.

In the other case when $|\xi| \leqslant 1$ it follows $\langle \xi \rangle^{-2\delta} \sim 1$. Therefore

$$\begin{aligned} \left\| K_\gamma^{2,\infty} \right\|_{Y_\xi^{1/2}} &\lesssim \langle \xi \rangle^{-2\delta} \left(\int\limits_{\mathbb{R}} \frac{|\hat{f}(\tau)|^2}{\langle i\tau + \xi^2 \rangle^{1-2\delta}} \, d\tau \right)^{1/2} \left(\int\limits_{\mathbb{R}} \frac{1}{\langle \tau \rangle^{1+2\delta}} \, d\tau \right)^{1/2} \\ &\lesssim \langle \xi \rangle^{-2\delta} \|f\|_{Y_{\gamma,\xi}^{-1/2+\delta}}. \end{aligned} \tag{3.20}$$

Contribution of $K_{2,0}$. Note that

$$\|K_{2,0}\|_{Y_\xi^{1/2}} \leqslant \big\| \langle i\tau + \xi^2 \rangle^{1/2} \mathcal{F}_t\big(\psi(t)\big(1 - e^{-\xi^2|t|}\big)\big)(\tau)\big\|_{L^2_\tau} \int\limits_{|\tau|\leqslant 1} \frac{|\hat{f}(\tau)|}{|i\tau + \xi^2|}\, d\tau. \tag{3.21}$$

Case 1: $|\xi| \geqslant 1$. Using the inequality (3.5) in the proof of Proposition 3.2, we obtain

$$\begin{aligned} I &=: \big\| \langle i\tau + \xi^2 \rangle^{1/2} \mathcal{F}_t\big(\psi(t)\big(1 - e^{-\xi^2|t|}\big)\big)(\tau)\big\|_{L^2_\tau} \\ &\leqslant \big\| \psi(t)\big(1 - e^{-\xi^2|t|}\big)(\tau)\big\|_{H^{1/2}_\tau} + \langle \xi \rangle \big\| \mathcal{F}_t\big(\psi(t)\big(1 - e^{-\xi^2|t|}\big)\big)(\tau)\big\|_{L^2_\tau} \\ &\leqslant C\big(1 + \langle \xi \rangle\big) \leqslant C\langle \xi \rangle, \end{aligned} \tag{3.22}$$

therefore,

$$\|K_{2,0}\|_{Y_\xi^{1/2}} \leqslant C\langle \xi \rangle \int\limits_{|\tau|\leqslant 1} \frac{|\hat{f}(\tau)|}{|i\tau + \xi^2|}\, d\tau, \tag{3.23}$$

now, we apply the Cauchy–Schwarz inequality to obtain

$$\begin{aligned} \|K_{2,0}\|_{Y_\xi^{1/2}} &\leqslant C\langle \xi \rangle \Bigg(\int\limits_{|\tau|\leqslant 1} \frac{|\hat{f}(\tau)|^2}{\langle i\tau + \xi^2 \rangle}\, d\tau \Bigg)^{1/2} \Bigg(\int\limits_{|\tau|\leqslant 1} \frac{\langle i\tau + \xi^2 \rangle}{|i\tau + \xi^2|^2}\, d\tau \Bigg)^{1/2} \\ &\leqslant C\langle \xi \rangle |\xi|^{-1} \Bigg(\int\limits_{|\tau|\leqslant 1} \frac{|\hat{f}(\tau)|^2}{\langle i\tau + \xi^2 \rangle}\, d\tau \Bigg)^{1/2} \\ &\leqslant C \Bigg(\int\limits_{|\tau|\leqslant 1} \frac{|\hat{f}(\tau)|^2}{\langle i\tau + \xi^2 \rangle}\, d\tau \Bigg)^{1/2} \\ &\leqslant C\langle \xi \rangle^{-2\delta} \Bigg(\int\limits_{|\tau|\leqslant 1} \frac{|\hat{f}(\tau)|^2}{\langle i\tau + \xi^2 \rangle^{1-2\delta}}\, d\tau \Bigg)^{1/2} \\ &\leqslant C\langle \xi \rangle^{-2\delta} \|f\|_{Y_\xi^{-1/2+\delta}}. \end{aligned} \tag{3.24}$$

Case 2: $|\xi| \leqslant 1$. In this case we note that

$$\begin{aligned} I &\leqslant \big\| \psi(t)\big(1 - e^{-\xi^2|t|}\big)(\tau)\big\|_{H^{1/2}_\tau} + \langle \xi \rangle \big\| \mathcal{F}_t\big(\psi(t)\big(1 - e^{-\xi^2|t|}\big)\big)(\tau)\big\|_{L^2_\tau} \\ &\leqslant \big\| \psi(t)\big(1 - e^{-\xi^2|t|}\big)(\tau)\big\|_{H^{1/2}_\tau} \leqslant \sum_{n\geqslant 1} \bigg\| \frac{|t|^n \psi(t) |\xi|^{2n}}{n!} \bigg\|_{H^{1/2}_\tau} \\ &\leqslant C \sum_{n\geqslant 1} \frac{|\xi|^{2n}}{n!} \big\| |t|^n \psi(t) \big\|_{H^{1/2}_\tau} \leqslant C|\xi| \sum_{n\geqslant 1} \frac{1}{(n-1)!} \leqslant C|\xi|. \end{aligned} \tag{3.25}$$

Substituting this inequality in (3.21), then as in Case 1, we obtain using Cauchy–Schwarz inequality,

$$\|K_{2,0}\|_{Y_\xi^{1/2}} \leqslant C|\xi\|\xi|^{-1}\biggl(\int\limits_{|\tau|\leqslant 1} \frac{|\hat{f}(\tau)|^2}{\langle i\tau+\xi^2\rangle}\,d\tau\biggr)^{1/2}$$
$$\leqslant C\langle\xi\rangle^{-2\delta}\|f\|_{Y_\xi^{-1/2+\delta}}. \tag{3.26}$$

Contribution of $K_{1,\infty}$. By the identity $\mathcal{F}(u*v)=\hat{u}\hat{v}$ and the triangle inequality $\langle i\tau+\xi^2\rangle \leqslant \langle\tau_1\rangle + |i(\tau-\tau_1)+\xi^2|$, we see that

$$\|K_{1,\infty}\|_{Y_\xi^{1/2}} = \biggl\|\bigl\langle i\tau+\xi^2\bigr\rangle^{1/2}\biggl|\hat{\psi}(\tau_1) * \biggl(\frac{\hat{f}(\tau_1)}{|i\tau_1+\xi^2|}\chi_{\{|\tau_1|\geqslant 1\}}\biggr)\biggr|(\tau)\biggr\|_{L_\tau^2}$$
$$\leqslant \biggl\|\bigl|\langle\tau_1\rangle^{1/2}\hat{\psi}(\tau_1)\bigr| * \biggl(\frac{|\hat{f}(\tau_1)|}{|i\tau_1+\xi^2|}\chi_{\{|\tau_1|\geqslant 1\}}\biggr)(\tau)\biggr\|_{L_\tau^2}$$
$$+ \biggl\|\bigl|\hat{\psi}(\tau_1)\bigr| * \biggl(\frac{|\hat{f}(\tau_1)|}{|i\tau_1+\xi^2|^{1/2}}\chi_{\{|\tau_1|\geqslant 1\}}\biggr)(\tau)\biggr\|_{L_\tau^2}. \tag{3.27}$$

Due to the convolution inequality $\|u*v\|_{L_\tau^2} \lesssim \|u\|_{L_\tau^1}\|v\|_{L_\tau^2}$, we obtain

$$\|K_{1,\infty}\|_{Y_\xi^{1/2}} \leqslant \bigl\|\langle\tau\rangle\hat{\psi}(t)\bigr\|_{L_\tau^1}\biggl\|\frac{|\hat{f}(\tau)|}{|i\tau+\xi^2|}\chi_{\{|\tau|\geqslant 1\}}\biggr\|_{L_\tau^2}$$
$$+ \bigl\|\psi(t)\bigr\|_{L_\tau^1}\biggl\|\frac{|\hat{f}(\tau)|}{|i\tau+\xi^2|^{1/2}}\chi_{\{|\tau|\geqslant 1\}}\biggr\|_{L_\tau^2}$$
$$\leqslant C\biggl\|\frac{|\hat{f}(\tau)|}{\langle i\tau+\xi^2\rangle^{1/2}}\chi_{\{|\tau|\geqslant 1\}}\biggr\|_{L_\tau^2}$$
$$\leqslant C\langle\xi\rangle^{-2\delta}\bigl\|\bigl\langle i\tau+\xi^2\bigr\rangle^{-1/2+\delta}\hat{f}(\tau)\bigr\|_{L_\tau^2}$$
$$\leqslant C\langle\xi\rangle^{-2\delta}\|f\|_{Y_\xi^{-1/2+\delta}}. \tag{3.28}$$

This completes the proof of the proposition. □

Now, by use of Proposition 3.3, we prove some smoothing properties in the Bourgain spaces for the operator L defined by (3.7).

Proposition 3.4. *Let $0<\delta\leqslant 1/2$ and $s_1,s_2\in\mathbb{R}$, there exists $C=C(\delta)>0$ such that, for all $f\in X^{-1/2+\delta,s_1-2\delta,s_2}$, we have*

$$M =: \biggl\|\chi_{\mathbb{R}_+}(t)\psi(t)\int\limits_0^t W(t-t')f(t')\,dt'\biggr\|_{X^{1/2,s_1,s_2}} \leqslant C_\delta\|f\|_{X^{-1/2+\delta,s_1-2\delta,s_2}}. \tag{3.29}$$

Proof. By definition of $X^{1/2,s_1,s_2}$, we see that M is equal to

$$\left\| \langle\xi\rangle^{s_1}\langle\eta\rangle^{s_2}\big\langle i\big(\tau-P(\nu)\big)+\xi^2\big\rangle^{1/2}\mathcal{F}_{t,x}\left(\chi_{\mathbb{R}_+}(t)\psi(t)\int_0^t W(t-t')f(t')\,dt'\right)(\tau,\nu)\right\|_{L^2_{\tau,\nu}}, \quad (3.30)$$

we note that

$$\begin{aligned}
&\mathcal{F}_{t,x}\left(\chi_{\mathbb{R}_+}(t)\psi(t)\int_0^t W(t-t')f(t')\,dt\right)(\tau)\\
&\quad=\mathcal{F}_t\left(\chi_{\mathbb{R}_+}(t)\psi(t)\int_0^t e^{-|t-t'|\xi^2}e^{it'P(\nu)(t-t')}\mathcal{F}_\nu(f)(t',\nu)\,dt'\right)(\tau)\\
&\quad=\mathcal{F}_t\left(\chi_{\mathbb{R}_+}(t)\psi(t)\int_0^t e^{-|t-t'|\xi^2}e^{-it'P(\nu)t'}\mathcal{F}_\nu\big(U(t)f(t',\nu)\big)\,dt'\right)(\tau)\\
&\quad=\mathcal{F}_t\left(\chi_{\mathbb{R}_+}(t)\psi(t)\int_0^t e^{-|t-t'|\xi^2}e^{-it'P(\nu)t'}\mathcal{F}_\nu(f)(t',\nu)\,dt'\right)\big(\tau-P(\nu)\big),
\end{aligned}$$

then by performing the change of variable $\tau\mapsto\tau-P(\nu)$, we obtain

$$\begin{aligned}
M&=\left\|\langle\xi\rangle^{s_1}\langle\eta\rangle^{s_2}\mathcal{F}_t\left(\chi_{\mathbb{R}_+}(t)\psi(t)\int_0^t e^{-|t-t'|\xi^2}e^{-it'P(\nu)t'}\mathcal{F}_\nu(f)(t',\nu)\,dt'\right)\right\|_{L^2_\nu(Y_\xi^{1/2})}\\
&=\left\|\langle\xi\rangle^{s_1}\langle\eta\rangle^{s_2}\mathcal{F}_t\left(\chi_{\mathbb{R}_+}(t)\psi(t)\int_0^t e^{-|t-t'|\xi^2}\mathcal{F}_\nu\big(U(-t')f\big)(t',\nu)\,dt'\right)\right\|_{L^2_\nu(Y_\xi^{1/2})}. \quad (3.31)
\end{aligned}$$

Now, let us set $w(\tau,\nu)=\mathcal{F}_\nu(U(-t')f)(\tau,\nu)$. To apply Proposition 3.3, we need to assume that $f\in\mathcal{S}(\mathbb{R}^3)$. It is clear that $w\in\mathcal{S}(\mathbb{R}^3)$, and we take

$$K_\xi: f\mapsto\psi(t)\int_0^t e^{-|t-t'|\xi^2}w(t')\,dt, \quad (3.32)$$

therefore,

$$\begin{aligned}
M&=\big\|\langle\xi\rangle^{s_1}\langle\eta\rangle^{s_2}\mathcal{F}_t\big(\chi_{\mathbb{R}_+}(t)K_\xi(t)\big)\big\|_{L^2_\nu(Y_\xi^{1/2})}\\
&\leqslant\big\|\langle\xi\rangle^{s_1}\langle\eta\rangle^{s_2}\big\|\chi_{\mathbb{R}_+}(t)K_\xi(t)\big\|_{H^{1/2}_\tau}\big\|_{L^2_\nu}\\
&\quad+\big\|\langle\xi\rangle^{s_1+1}\langle\eta\rangle^{s_2}\big\|\chi_{\mathbb{R}_+}(t)K_\xi(t)\big\|_{L^2_\tau}\big\|_{L^2_\nu}. \quad (3.33)
\end{aligned}$$

Since $K_\xi(0)=0$, then we have

$$\left\|\chi_{\mathbb{R}_+}(t)K_\xi(t)\right\|_{H^1_\tau} \leqslant \left\|K_\xi(t)\right\|_{H^1_\tau}, \qquad \left\|\chi_{\mathbb{R}_+}(t)K_\xi(t)\right\|_{L^2_\tau} \leqslant \left\|K_\xi(t)\right\|_{L^2_\tau}.$$

Consequently, by noticing that $\|h\|_{L^2}=\|h\|_{H^0}$, it results by interpolation between H^0 and H^1 that

$$\left\|\chi_{\mathbb{R}_+}(t)K_\xi(t)\right\|_{H^{1/2}_\tau} \leqslant \left\|K_\xi(t)\right\|_{H^{1/2}_\tau}. \tag{3.34}$$

Hence

$$\begin{aligned} M &\leqslant \left\|\langle\xi\rangle^{s_1}\langle\eta\rangle^{s_2}\left\|K_\xi(t)\right\|_{H^{1/2}_\tau}\right\|_{L^2_\nu} + \left\|\langle\xi\rangle^{s_1+1}\langle\eta\rangle^{s_2}\left\|K_\xi(t)\right\|_{L^2_\tau}\right\|_{L^2_\nu} \\ &\leqslant C\left\|\langle\xi\rangle^{s_1}\langle\eta\rangle^{s_2}\mathcal{F}_t\big(K_\xi(t)\big)\right\|_{L^2_\nu(Y^{1/2}_\xi)}. \end{aligned} \tag{3.35}$$

Now, we can apply Proposition 3.3 to obtain

$$\begin{aligned} M &\leqslant C\left\|\langle\xi\rangle^{s_1}\langle\eta\rangle^{s_2}\|w\|_{Y_\xi^{-1/2+\delta}}\langle\xi\rangle^{-2\delta}\right\|_{L^2_\nu} \\ &= C\left\|\langle\xi\rangle^{s_1-2\delta}\langle\eta\rangle^{s_2}\left\|\left\langle i\tau+\xi^2\right\rangle^{-1/2+\delta}\mathcal{F}_t\big(w(t)\big)\right\|_{L^2_\tau}\right\|_{L^2_\nu} \\ &= C\left\|\langle\xi\rangle^{s_1-2\delta}\langle\eta\rangle^{s_2}\left\langle i\tau+\xi^2\right\rangle^{-1/2+\delta}\mathcal{F}_{t,x}\big(U(-t)f(\tau,\nu)\big)\right\|_{L^2_{\tau,\nu}} \\ &= C\left\|\langle\xi\rangle^{s_1-2\delta}\langle\eta\rangle^{s_2}\left\langle i\tau+\xi^2\right\rangle^{-1/2+\delta}\hat f\big(\tau+P(\nu),\nu\big)\right\|_{L^2_{\tau,\nu}}, \end{aligned} \tag{3.36}$$

finally, by performing the change of variable $\tau\mapsto\tau-P(\nu)$ we can deduce that $M\leqslant C\|f\|_{X^{-1/2+\delta,s_1-2\delta,s_2}}$, this for any $f\in\mathcal{S}(\mathbb{R}^3)$. The result for $f\in X^{-1/2+\delta,s_1-2\delta,s_2}$ follows by density. □

Proposition 3.5. *Let s_1, $s_2\in\mathbb{R}$ and $0<\delta\leqslant 1/2$. For all $f\in X^{-1/2+\delta,s_1-2\delta,s2}$ we have*

$$L: t\mapsto\int_0^t W(t-t')f(t')\,dt'\in C\big(\mathbb{R}_+,H^{s_1,s_2}\big), \tag{3.37}$$

moreover, if (f_n) is a sequence with $f_n\to 0$ in $X^{-1/2+\delta,s_1-2\delta,s2}$ as $n\to\infty$, then

$$\left\|\int_0^t W(t-t')f_n(t')\,dt'\right\|_{L^\infty(\mathbb{R}_+,H^{s_1,s_2})}\to 0. \tag{3.38}$$

Proof. By Fubini theorem, and by the definition of $W(.)$ we have

$$L(t) = \int_0^t \mathcal{F}_\nu^{-1}\big(e^{-|t-t'|\xi^2} e^{i(t-t')P(\nu)} \mathcal{F}_\nu\big(f(t')\big)\big)\,dt'$$

$$= \int_{\mathbb{R}^2} e^{i\langle \nu,(x,y)\rangle} e^{itP(\nu)} \int_0^t e^{-|t-t'|\xi^2} \mathcal{F}_{(x,y)}\big(U(-t')f(t')\big)(\nu)\,dt'$$

$$= U(t)\mathcal{F}_{(x,y)}^{-1}\left[\int_0^t e^{-|t-t'|\xi^2} \mathcal{F}_{(x,y)}\big(g(t',.)\big)(\nu)\,dt'\right], \tag{3.39}$$

where $g(t,\nu) = (U(-t)f(t))(\nu)$. As noticed in [5] since $U(\cdot)$ is a strongly continuous unitary group in $L^2(\mathbb{R}^2)$, it is enough to prove that $t \mapsto U(-t)L(t) \in C(\mathbb{R}_+, H^{s_1,s_2})$, then it is equivalent to show that

$$F : t \mapsto \langle\xi\rangle^{s_1}\langle\eta\rangle^{s_2} \int_0^t e^{-|t-t'|\xi^2} \mathcal{F}_{(x,y)}\big(g(t',.)\big)(\nu)\,dt' \tag{3.40}$$

is continuous from $\mathbb{R}_+$ in $L^2(\mathbb{R}^2)$, for $f \in X^{-1/2+\delta,s_1-2\delta,s_2}$, $0<\delta \leqslant 1/2$. Note that by the Fubini theorem we have

$$F(t) = \langle\xi\rangle^{s_1}\langle\eta\rangle^{s_2} e^{-t|\xi|^2} \int_0^t e^{t'|\xi|^2} \mathcal{F}_{(x,y)}\big(g(t',.)\big)(\nu)\,dt'$$

$$= \langle\xi\rangle^{s_1}\langle\eta\rangle^{s_2} e^{-t|\xi|^2} \int_{\mathbb{R}} \hat{g}(\tau,\nu) \int_0^t e^{(i\tau+|\xi|^2)t'}\,dt'\,d\tau$$

$$= \langle\xi\rangle^{s_1}\langle\eta\rangle^{s_2} \int_{\mathbb{R}} \hat{g}(\tau,\nu) \frac{e^{it\tau} - e^{-|\xi|^2 t}}{i\tau+\xi^2}\,d\tau. \tag{3.41}$$

One fixes $t_1, t_2 \in \mathbb{R}_+$, then

$$F(t_1) - F(t_2) = \langle\xi\rangle^{s_1}\langle\eta\rangle^{s_2} \int_{\mathbb{R}} \frac{\hat{g}(\tau,\nu)}{i\tau+\xi^2}\big[\big(e^{it_1\tau} - e^{it_2\tau}\big) - \big(e^{-|\xi|^2 t_1} - e^{-|\xi|^2 t_2}\big)\big]\,d\tau$$

$$=: \langle\xi\rangle^{s_1}\langle\eta\rangle^{s_2} \int_{\mathbb{R}} J_{t_1,t_2}(\tau)\,d\tau. \tag{3.42}$$

We deal first with the case $|\xi| \geqslant 1$. Using Cauchy–Schwarz inequality we obtain

$$\begin{aligned}\big|F(t_1)-F(t_2)\big| &\leqslant 4\langle\xi\rangle^{s_1}\langle\eta\rangle^{s_2}\int_{\mathbb{R}}\frac{|\hat{g}(\tau,\nu)|}{|i\tau+\xi^2|}\,d\tau\\ &\leqslant 4\langle\xi\rangle^{s_1}\langle\eta\rangle^{s_2}\bigg(\int_{\mathbb{R}}\frac{|\hat{g}(\tau,\nu)|^2}{\langle i\tau+\xi^2\rangle^{1-2\delta}}\,d\tau\bigg)^{1/2}\bigg(\int_{\mathbb{R}}\frac{\langle i\tau+\xi^2\rangle^{1-2\delta}}{|i\tau+\xi^2|^2}\,d\tau\bigg)^{1/2}. \quad (3.43)\end{aligned}$$

Since in this case we have $|i\tau+\xi^2|\sim\langle i\tau+\xi^2\rangle$, then the change of variable $\tau\mapsto r\xi^2$ leads to

$$\begin{aligned}\big|F(t_1)-F(t_2)\big| &\leqslant C\langle\xi\rangle^{s_1}\langle\eta\rangle^{s_2}\bigg(\int_{\mathbb{R}}\frac{|\hat{g}(\tau,\nu)|^2}{\langle i\tau+\xi^2\rangle^{1-2\delta}}\,d\tau\bigg)^{1/2}|\xi|^{-2\delta}\bigg(\int_{\mathbb{R}}\frac{dr}{\langle r\rangle^{1+2\delta}}\,d\tau\bigg)^{1/2}\\ &\leqslant C\langle\xi\rangle^{s_1-2\delta}\langle\eta\rangle^{s_2}\big\|\big\langle i\tau+\xi^2\big\rangle^{-1/2+\delta}\hat{g}(.,\nu)\big\|_{L^2_\tau}. \quad (3.44)\end{aligned}$$

In the other case when $|\xi|\leqslant 1$, we assume that $|t_1-t_2|$ is small enough. We can write $|F(t_1)-F(t_2)|\leqslant I_1+I_2$, where

$$I_1=:\langle\xi\rangle^{s_1}\langle\eta\rangle^{s_2}\bigg|\int_{\mathbb{R}}\frac{\hat{g}(\tau,\nu)}{i\tau+\xi^2}\big[e^{it_1\tau}-e^{it_2\tau}\big]\,d\tau\bigg|, \quad (3.45)$$

$$I_2=:\langle\xi\rangle^{s_1}\langle\eta\rangle^{s_2}\bigg|\int_{\mathbb{R}}\frac{\hat{g}(\tau,\nu)}{i\tau+\xi^2}\big[e^{-\xi^2t_1}-e^{-\xi^2t_2}\big]\,d\tau\bigg|. \quad (3.46)$$

We first estimate I_1. By Cauchy–Schwarz inequality, we see that

$$\begin{aligned}I_1 &=: \langle\xi\rangle^{s_1}\langle\eta\rangle^{s_2}\bigg[|t_1-t_2|\int_{|\tau|\leqslant 1}|\tau|\bigg|\frac{\hat{g}(\tau,\nu)}{i\tau+\xi^2}\bigg|\,d\tau+2\int_{|\tau|\geqslant 1}\bigg|\frac{\hat{g}(\tau,\nu)}{i\tau+\xi^2}\bigg|\,d\tau\bigg]\\ &\leqslant C\langle\xi\rangle^{s_1}\langle\eta\rangle^{s_2}\big\|\big\langle i\tau+\xi^2\big\rangle^{-1/2+\delta}\hat{g}(.,\nu)\big\|_{L^2_\tau}\bigg[\bigg(\int_{|\tau|\leqslant 1}\frac{|\tau|^2\langle i\tau+\xi^2\rangle^{1-2\delta}}{|i\tau+\xi^2|^2}\,d\tau\bigg)^{1/2}\\ &\quad+\bigg(\int_{|\tau|\geqslant 1}\frac{|i\tau+\xi^2|^{1-2\delta}}{|i\tau+\xi^2|}\,d\tau\bigg)^{1/2}\bigg]. \quad (3.47)\end{aligned}$$

Since for $|\xi|\leqslant 1$, we have $\langle i\tau+\xi^2\rangle^{1-2\delta}\sim\langle\tau\rangle^{1-2\delta}$. Using this approximation in the first integral of (3.47) together with the change of variable $\tau\mapsto r\xi^2$ in the second one, we obtain

$$\begin{aligned}I_1 &\leqslant C\langle\xi\rangle^{s_1}\langle\eta\rangle^{s_2}\big\|\big\langle i\tau+\xi^2\big\rangle^{-1/2+\delta}\hat{g}(.,\nu)\big\|_{L^2_\tau}\\ &\quad\times\bigg(\bigg(\int_{|\tau|\leqslant 1}\langle\tau\rangle^{1-2\delta}\,d\tau\bigg)^{1/2}+\bigg(\int_{\mathbb{R}}\frac{1}{\langle r\rangle^{1+2\delta}}\,dr\bigg)^{1/2}\bigg)\\ &\leqslant C\langle\xi\rangle^{s_1}\langle\eta\rangle^{s_2}\big\|\big\langle i\tau+\xi^2\big\rangle^{-1/2+\delta}\hat{g}(.,\nu)\big\|_{L^2_\tau}. \quad (3.48)\end{aligned}$$

Note that in this case $\langle\xi\rangle^{-2\delta} \sim 1$, therefore

$$I_1 \leqslant C\langle\xi\rangle^{s_1-2\delta}\langle\eta\rangle^{s_2}\left\|\left\langle i\tau+\xi^2\right\rangle^{-1/2+\delta}\hat{g}(.,\nu)\right\|_{L^2_\tau}. \tag{3.49}$$

Now, we pass to estimate I_2. By Cauchy–Schwarz inequality it results that

$$I_2 \leqslant C\langle\xi\rangle^{s_1}\langle\eta\rangle^{s_2}|\xi|^2\left\|\left\langle i\tau+\xi^2\right\rangle^{-1/2+\delta}\hat{g}(.,\nu)\right\|_{L^2_\tau}\left(\int_{\mathbb{R}}\frac{\langle i\tau+\xi^2\rangle^{1-2\delta}}{|i\tau+\xi^2|^2}\,d\tau\right)^{1/2}. \tag{3.50}$$

Since we have

$$\begin{aligned}\int_{\mathbb{R}}\frac{\langle i\tau+\xi^2\rangle^{1-2\delta}}{|i\tau+\xi^2|^2}\,d\tau &= \int_{|\tau|\leqslant 1}\frac{\langle i\tau+\xi^2\rangle^{1-2\delta}}{|i\tau+\xi^2|^2}\,d\tau + \int_{|\tau|\geqslant 1}\frac{\langle i\tau+\xi^2\rangle^{1-2\delta}}{|i\tau+\xi^2|^2}\,d\tau\\ &\leqslant C\left(|\xi|^{-4}+|\xi|^{-4\delta}\right),\end{aligned} \tag{3.51}$$

then

$$\begin{aligned}I_2 &\leqslant C\langle\xi\rangle^{s_1}\langle\eta\rangle^{s_2}|\xi|^2\left(|\xi|^{-2}+|\xi|^{-2\delta}\right)\left\|\left\langle i\tau+\xi^2\right\rangle^{-1/2+\delta}\hat{g}(.,\nu)\right\|_{L^2_\tau}\\ &\leqslant C\langle\xi\rangle^{s_1-2\delta}\langle\eta\rangle^{s_2}\left\|\left\langle i\tau+\xi^2\right\rangle^{-1/2+\delta}\hat{g}(.,\nu)\right\|_{L^2_\tau}.\end{aligned} \tag{3.52}$$

Finally, gathering (3.44), (3.49), (3.52), one infers that

$$\begin{aligned}\left\|F(t_1)-F(t_2)\right\|_{L^2_\tau} &\leqslant C\left\|\langle\xi\rangle^{s_1-2\delta}\langle\eta\rangle^{s_2}\left\langle i\tau+\xi^2\right\rangle^{-1/2+\delta}\hat{g}(.,\nu)\right\|_{L^2_{\tau,\nu}}\\ &= C\|f\|_{X^{-1/2+\delta,s_1-2\delta,s_2}}.\end{aligned} \tag{3.53}$$

It is clear that the integrand in (3.42) tends to 0 pointwise in (τ,ν) as soon as $|t_1-t_2| \to 0$ and is bounded uniformly in $|t_1-t_2|$ by the right member of (3.53). The result follows then from Lebesgue dominated convergence theorem.

To show (3.38) it suffices to notice that one has

$$\sup_{t\in\mathbb{R}_+}\left\|F_n(t)\right\|_{L^2(\mathbb{R}^2)} \leqslant C\|f_n\|_{X^{-1/2+\delta,s_1-2\delta,s_2}},$$

where F_n is defined as F with $g_n(.) = \mathcal{F}_\nu(U(-t)f_n(t))$ instead of g. This completes the proof. □

4. Strichartz and multilinear estimates for the KP-equation

The goal in this section is to prepare certain Strichartz and multilinear estimates by using result derived by Molinet and Ribaud in [8] and Saut in [12]. This type of estimates is necessary to treat in the next section the nonlinear term $\partial(u^2)$ in X^{b,s_1,s_2}. The following lemma is prepared by Molinet and Ribaud in [8].

Lemma 4.1. *Let $v \in L^2(\mathbb{R}^2)$ with* $\operatorname{supp} v \subset \{(t,x,y)\colon |t| \leqslant T\}$ *and let* $\epsilon > 0$, $\delta(r) = 1 - 2/r$. *Then for all* (r, β, θ) *with*

$$2 \leqslant r < \infty, \quad 0 \leqslant \beta \leqslant 1/2, \quad 0 \leqslant \delta(r) \leqslant \frac{\theta}{1-\beta/3} \tag{4.1}$$

there exists $\mu = \mu(\epsilon) > 0$ *such that*

$$\left\| \mathcal{F}_{t,x}^{-1}\left(|\xi|^{-\frac{\beta\delta(r)}{2}} \langle \tau - P(\nu)\rangle^{\frac{-\theta}{2}(1+\epsilon)} \left|\hat{v}(\tau,\nu)\right|\right)\right\|_{L^{q,r}_{t,x}} \leqslant CT^{\mu}\|v\|_{L^2(\mathbb{R}^3)}, \tag{4.2}$$

where q is defined by

$$2/q = (1-\beta/3)\delta(r) + (1-\theta). \tag{4.3}$$

Now, we will use Lemma 4.1 to derive a first multilinear estimate.

Lemma 4.2. *Let u, v with compact support in* $\{(x,y,t)\colon |t| \leqslant T\}$. *For* $b > 0$ *small enough, there exists* $\mu > 0$ *such that*

$$\begin{aligned} I =: \int_{\mathbb{R}^6} \frac{|\hat{u}(\tau_1,\nu_1)\|\hat{v}(\tau-\tau_1,\nu-\nu_1)\|\hat{w}(\tau,\nu)|}{\langle\sigma\rangle^{1/2-b}|\xi_1|^{b/4}\langle\sigma_1\rangle^{b}\langle\sigma_2\rangle^{1/2-b/24}}\,d\tau\,d\tau_1\,d\nu\,d\nu_1 \\ \leqslant CT^{\mu}\|u\|_{L^2_{t,x}}\|v\|_{L^2_{t,x}}\|w\|_{L^2_{t,x}}, \end{aligned} \tag{4.4}$$

where σ, σ_1 *and* σ_2 *are defined by*

$$\sigma = \tau - P(\nu), \qquad \sigma_1 = \tau_1 - P(\nu_1), \qquad \sigma_2 = \tau - \tau_1 - P(\nu - \nu_1). \tag{4.5}$$

Proof. By the Plancherel theorem we see that

$$I = \int_{\mathbb{R}^3} \mathcal{F}_{t,x}^{-1}\left(\frac{|\hat{w}(\tau,\nu)|}{\langle\sigma\rangle^{1/2-b}}\right)\mathcal{F}_{t,x}^{-1}\left(\frac{|\hat{u}(\tau,\nu)|}{\langle\sigma\rangle^{b}|\xi|^{b/4}} * \frac{|\hat{v}(\tau,\nu)|}{\langle\sigma\rangle^{1/2-b/24}}\right)(\tau,\nu)\,d\tau\,d\nu, \tag{4.6}$$

by using the fact that $\mathcal{F}_{t,x}^{-1}(h * f) = \mathcal{F}_{t,x}^{-1}(h) * \mathcal{F}_{t,x}^{-1}(f)$ then by applying Hölder inequality in space and next in time we obtain that I is bounded by the product of the three terms

$$\left\|\mathcal{F}_{t,x}^{-1}\left(\frac{|\hat{w}(\tau,\nu)|}{\langle\sigma\rangle^{1/2-b}}\right)\right\|_{L^{q_1,r_1}_{t,x}} \left\|\mathcal{F}_{t,x}^{-1}\left(\frac{|\hat{u}(\tau,\nu)|}{\langle\sigma\rangle^{b}|\xi|^{b/4}}\right)\right\|_{L^{q_2,r_2}_{t,x}} \left\|\mathcal{F}_{t,x}^{-1}\left(\frac{|\hat{v}(\tau,\nu)|}{\langle\sigma\rangle^{1/2-b/24}}\right)\right\|_{L^{q_3,r_3}_{t,x}}, \tag{4.7}$$

where

$$\sum_{i=1}^{3} 1/q_i = 1, \qquad \sum_{i=1}^{3} 1/r_i = 1. \tag{4.8}$$

Our goal now is to estimate the three terms of (4.7) by using Lemma 4.1. Let b be small enough. We take first $\epsilon_1 = \epsilon_2 = \epsilon_3 = \epsilon$, where $\epsilon = \epsilon(b)$ will be a small parameter. Also we choose

$$\theta_1 = \frac{1-2b}{1+\epsilon}, \qquad \theta_2 = \frac{2b}{1+\epsilon}, \qquad \theta_3 = \frac{1-b/12}{1+\epsilon}, \tag{4.9}$$

we choose $\beta_1 = \beta_3 = 0$. From (4.3), it remains to find β_2, q_i and r_i with

$$\frac{2}{q_1} = \delta(r_1) + (1-\theta_1), \qquad \frac{2}{q_2} = \delta(r_2) + (1-\theta_2) - \frac{\beta_2\delta(r_2)}{3},$$

$$\frac{2}{q_3} = \delta(r_3) + (1-\theta_3), \tag{4.10}$$

$$\beta_2\delta(r_2) = \frac{b}{2} \tag{4.11}$$

such that (4.1) remains valid for $i = 1, 2, 3$. It is simple to check that $\sum_{i=1}^{3} 2/q_i = 2$, $\sum_{i=1}^{3} \delta(r_i) = 3 - 2\sum_{i=1}^{3} 1/r_i = 1$, $\sum_{i=1}^{3} \theta_i = \frac{2-b/12}{1+\epsilon}$. Hence, adding the three equations in (4.10), we see necessarily that $\sum 2/q_i = \sum \delta(r_i) + 3 - \sum \theta_i - \frac{\beta_1\delta(r_1)}{3}$. Thanks to (4.11) this relation is equivalent to $2 = 4 - \frac{2-b/12}{1+\epsilon} - \frac{b}{6}$, i.e.

$$\frac{\epsilon + b/24}{1+\epsilon} = \frac{b}{12}. \tag{4.12}$$

Therefore, for b small enough, it is clear that $\epsilon = \epsilon(b) = \frac{b}{24-2b} = 0^+$. Now, we choose $(r_1, r_2, r_3) = (\frac{4}{1+b}, \frac{2}{1-b}, \frac{4}{1+b})$. It is simple to see that $\sum 1/r_i = 1$ and $(\delta(r_1), \delta(r_2), \delta(r_3)) = (1/2 - b/2, b, 1/2 - b/2)$ and the relations of (4.10) can be written

$$2/q_1 = 1/2 - b/2 + \frac{\epsilon + 2b}{1+\epsilon}, \qquad 2/q_2 = b + 1 - \frac{2b}{1+\epsilon} - \frac{b}{6},$$

$$2/q_3 = 1/2 - b/2 + \frac{\epsilon + b/12}{1+\epsilon}. \tag{4.13}$$

Now, due to the fact that $\epsilon = \frac{b}{24-2b}$, we infer that

$$2/q_1 = 1/2 - b/2 + \frac{49b - 4b^2}{24-b} = 1/2 + \frac{74b - 7b^2}{2(24-b)}, \tag{4.14}$$

$$2/q_2 = 1 + 5b/6 - \frac{2b(24-2b)}{24-b} = 1 + \frac{-168b + 19b^2}{6(24-b)}, \tag{4.15}$$

$$2/q_3 = 1/2 - b/2 + \frac{18b - b^2}{6(24-2b)} = 1/2 + \frac{-27b + b^2}{3(24-b)}. \tag{4.16}$$

Therefore by the relations (4.14)–(4.16), it is clear for b small enough that $(2/q_1, 2/q_2, 2/q_3) = (\frac{1}{2}^+, 1^-, \frac{1}{2}^-)$, i.e. $(q_1, q_2, q_3) = (4^-, 2^+, 4^+)$ and by construction we have $\sum 1/q_i = 1$. It remains to check that (4.1) is valid for our parameter. Indeed, by definition of θ_i in (4.9) it is clear that $0<\theta_i<1$ for $i=1,2,3$. Since $(\delta(r_2),\theta_2)=(b,\frac{2b}{1+\epsilon})$ we see that

$$0\leqslant \beta_2=\frac{b}{2\delta(r_2)}=\frac{1}{2},\quad 0\leqslant \delta(r_2)=b\leqslant 2b\left(\frac{24-2b}{24-b}\right)=\frac{2b}{1+\epsilon}=\theta_2\leqslant\frac{\theta_2}{1-\beta/3}. \tag{4.17}$$

Moreover, since we have the equality $(\delta(r_1),\delta(r_3))=(\frac{1}{2}^-,\frac{1}{2}^-)$ and $(\theta_1,\theta_3)=(\frac{1-2b}{1+\epsilon},\frac{1-b/12}{1+\epsilon})$, it is simple to see for $i=1,3$ that

$$0\leqslant\beta_i\leqslant 1/2,\quad 0\leqslant\delta(r_i)\leqslant\theta_i\leqslant\frac{\theta_i}{1-\beta_i/3}. \tag{4.18}$$

By combining Eqs. (4.17) and (4.18) we have (4.1), and now we can apply Lemma 4.1 to obtain

$$\left\|F_{t,x}^{-1}\left(\frac{|\hat{w}(\tau,\nu)|}{\langle\sigma\rangle^{1/2-b}}\right)\right\|_{L_{t,x}^{q_1,r_1}}\leqslant CT^{\mu}\|w\|_{L_{\tau,\nu}^2}, \tag{4.19}$$

$$\left\|F_{t,x}^{-1}\left(\frac{|\hat{u}(\tau,\nu)|}{\langle\sigma\rangle^{b}|\xi|^{b/4}}\right)\right\|_{L_{t,x}^{q_2,r_2}}\leqslant CT^{\mu}\|u\|_{L_{\tau,\nu}^2}, \tag{4.20}$$

$$\left\|F_{t,x}^{-1}\left(\frac{|\hat{v}(\tau,\nu)|}{\langle\sigma\rangle^{1/2-b/24}}\right)\right\|_{L_{t,x}^{q_3,r_3}}\leqslant C\|v\|_{L_{\tau,\nu}^2}. \tag{4.21}$$

This completes the proof. □

Lemma 4.3. *Let $u,v\in L^2(\mathbb{R}^3)$ with compact support in $\{(x,y,t)\colon |t|\leqslant T\}$. For $b>0$ and $c>0$ small enough there exists $\mu>0$ such that*

$$\int_{\mathbb{R}^6}\frac{|\hat{u}(\tau_1,\nu_1)||\hat{v}(\tau-\tau_1,\nu-\nu_1)||\hat{w}(\tau,\nu)|}{\langle\sigma_1\rangle^{1/2}|\xi_1|^{3b+c}\langle\sigma_2\rangle^{1/2-b}}\,d\tau\,d\tau_1\,d\nu\,d\nu_1$$
$$\leqslant CT^{\mu}\|u\|_{L_{t,x}^2}\|v\|_{L_{t,x}^2}\|w\|_{L_{t,x}^2}, \tag{4.22}$$

where σ, σ_1 and σ_2 are defined by

$$\sigma=\tau-P(\nu),\qquad \sigma_1=\tau_1-P(\nu_1),\qquad \sigma_2=\tau-\tau_1-P(\nu-\nu_1). \tag{4.23}$$

Proof. By Plancherel theorem and by Hölder inequality in space and time we see that the right-hand side of (4.22) is bounded by

$$\left\|\mathcal{F}_{t,x}^{-1}\left(\frac{|\hat{u}(\tau,\nu)|}{\langle\sigma\rangle^{1/2}|\xi|^{3b+c}}\right)\right\|_{L_{t,\nu}^{q_1,r_1}}\left\|\mathcal{F}_{t,x}^{-1}\left(\frac{|\hat{v}(\tau,\nu)|}{\langle\sigma\rangle^{1/2-b}}\right)\right\|_{L_{t,\nu}^{q_2,r_2}}\|w\|_{L_{t,\nu}^2} \tag{4.24}$$

provided

$$1/r_1+1/r_2=1/2,\qquad 1/q_1+1/q_2=1/2. \tag{4.25}$$

To apply Lemma 4.1 to each of the first two terms in (4.24), for b and c small enough we take $\epsilon_1 = \epsilon_2 = \epsilon$, where $\epsilon = \epsilon(b, c)$, we set

$$\theta_1 = \frac{1}{1+\epsilon}, \qquad \theta_2 = \frac{1-2b}{1+\epsilon}, \tag{4.26}$$

and we choose $\beta_2 = 0$ and β_1 such that $\frac{\beta_1 \delta(r_1)}{2} = 3b + c$. From (4.3), it remains to find β_1, q_i and r_i with

$$\frac{2}{q_1} = \delta(r_1) + (1-\theta_1) - \frac{\beta_1 \delta(r_1)}{3}, \qquad \frac{2}{q_2} = \delta(r_2) + (1-\theta_2) \tag{4.27}$$

such that (4.1) remains valid for $i = 1, 2$. It is simple to see that $\sum_{i=1}^{2} 2/q_i = 1$, $\sum_{i=1}^{2} \delta(r_i) = 1$ and $\sum_{i=1}^{3} \theta_i = \frac{2-2b}{1+\epsilon}$. Hence, adding the two equations of (4.27), we see necessarily that, $1 = 1 + (2 - \frac{2-2b}{1+\epsilon}) - \frac{6b+2c}{3}$. This relation is equivalent to $\frac{3-3b-c}{3} = \frac{1-b}{1+\epsilon}$, i.e.

$$\epsilon = \frac{c}{3-3b-c}. \tag{4.28}$$

Therefore, for b and c small enough, it is clear that $\epsilon = \epsilon(b, c) = 0^+$. Now we choose $(r_1, r_2) = (4, 4)$. It follows that $\sum 1/r_i = 1$ and $(\delta(r_1), \delta(r_2)) = (1/2, 1/2)$ and the relations of (4.10) can be written

$$2/q_1 = 1/2 + \frac{\epsilon}{1+\epsilon} - \frac{6b+2c}{3}, \qquad 2/q_2 = 1/2 + \frac{2b+\epsilon}{1+\epsilon}. \tag{4.29}$$

From (4.28), we get that $(2/q_1, 2/q_2) = (\frac{1}{2}^-, \frac{1}{2}^+)$, i.e. $(q_1, q_2) = (4^+, 4^-)$ and by construction we have $\sum 1/q_i = 1/2$. Moreover (4.1) is valid for our parameter. Indeed, for b and c small we have that

$$(\theta_1, \theta_2) = (1^-, 1^-), \qquad \beta_2 = \frac{6b+2c}{\delta(r_2)} = 12b + 4c = 0^+ \leqslant 1/2$$

and

$$\delta(r_i) \sim 1/2 < 1^- = \theta_i \leqslant \frac{\theta_i}{1-\beta_i/3}.$$

Now we apply Lemma 4.1 to give a suitable bound for each of the first two terms in (4.24). This ends the proof of Lemma 4.3. □

Lemma 4.4. *Let $2 \leqslant q \leqslant 4$ and $u \in L^2(\mathbb{R}^2)$ with compact support in $\{(x, y, t)\colon |t| \leqslant T\}$. For $\epsilon > 0$ and $b = (1-2/q)(\frac{1+\epsilon}{2})$, there exists $\mu = \mu(\epsilon) > 0$ such that*

$$\left\| \mathcal{F}^{-1}\big(\langle\sigma\rangle^{-b} \big|\hat{u}(\tau, \nu)\big|\big) \right\|_{L^q_{t,\nu}} \leqslant C T^{\mu} \|u\|_{L^2_{t,\nu}}. \tag{4.30}$$

Proof. For any $\phi \in L^2(\mathbb{R}^2)$ the Strichartz inequality in [12] (see Proposition 2.3) yields

$$\left\| U(t)\phi \right\|_{L^4_{t,\nu}} \leqslant \|\phi\|_{L^2_\nu}, \tag{4.31}$$

where

$$\mathcal{F}_{t,\nu}\big(U(t)\phi\big) = \exp\left(it\left(\xi^3 + n\frac{\eta^2}{\xi}\right)\right)\hat{\phi}(\xi,\eta), \quad n = \pm 1.$$

Using (4.31) together with Lemma 3.3 of [4], we see for all $\epsilon > 0$ that

$$\left\| u(\tau,\nu) \right\|_{L^4_{t,\nu}} \leqslant C \left\| \langle\sigma\rangle^{1/2+\epsilon/4} \hat{u}(\tau,\nu) \right\|_{L^2_{t,\nu}}. \tag{4.32}$$

Since we have $\|u\|_{L^2_{t,\nu}} = \|u\|_{X^{0,0,0}}$, therefore by interpolation between $(L^4_{t,\nu}, X^{1/2+\epsilon/4,0,0})$ and $(L^2_{t,\nu}, X^{0,0,0})$, for $0 \leqslant \theta \leqslant 1$, we obtain

$$\left\| u(\tau,\nu) \right\|_{L^q_{t,\nu}} \leqslant C \left\| \langle\sigma\rangle^{\theta(1/2+\epsilon/4)} \hat{u}(\tau,\nu) \right\|_{L^2_{t,\nu}}, \tag{4.33}$$

where

$$2/q = \theta/4 + \frac{1-\theta}{2}. \tag{4.34}$$

Next, using the assumption on the support of u and the results in [5], we get that there exists $\mu = \mu(\epsilon)$ such that

$$\left\| u(\tau,\nu) \right\|_{L^q_{t,\nu}} \leqslant CT^\mu \left\| \langle\sigma\rangle^{\theta(1/2+\epsilon/2)} \hat{u}(\tau,\nu) \right\|_{L^2_{t,\nu}} \tag{4.35}$$

from (4.34), the desired result is deduced. □

Using Lemma 4.4 together with the proof of Lemma 2.2 of [13], we obtain the following lemma.

Lemma 4.5. *Let $u, v, w \in L^2(\mathbb{R}^3)$ with compact support in $\{(x,y,t)\colon |t| \leqslant T\}$ and $\alpha, \beta, \gamma \in [0, 1/2+\epsilon]$. For any $\epsilon > 0$ there exists $\mu = \mu(\epsilon) > 0$ such that*

$$\int_{\mathbb{R}^6} \frac{|\hat{u}(\tau_1,\nu_1)||\hat{v}(\tau-\tau_1,\nu-\nu_1)||\hat{w}(\tau,\nu)|}{\langle\sigma\rangle^\alpha \langle\sigma_1\rangle^\beta \langle\sigma_2\rangle^\gamma}\, d\tau\, d\tau_1\, d\nu\, d\nu_1$$
$$\leqslant CT^\mu \|u\|_{L^2_{t,x}} \|v\|_{L^2_{t,x}} \|w\|_{L^2_{t,x}}, \tag{4.36}$$

proven for $\alpha + \beta + \gamma \geqslant 1 + 2\epsilon$.

5. Bilinear estimates

In this section we will prepare certain bilinear estimates on $\partial_x(uv)$ in the Bourgain space X^{b,s_1,s_2}. These bilinear estimates will be the main tools in the next section to apply a fixed point argument which will give the local existence result.

Proposition 5.1. *Let $\delta > 0$ small enough, $s_2 \geqslant 0$ and $s_1 \in [\frac{-1}{2} + 8\delta, 0]$. For all $u, v \in X^{1/2,s_1,s_2}$ with compact support in time and included in the subset $\{(t,x,y)\colon t \in [-T,T]\}$, there exists $\mu > 0$ such that the following bilinear estimate holds*

$$\left\|\partial_x(uv)\right\|_{X^{-1/2+\delta,s_1-2\delta+\epsilon,s_2}} \leqslant CT^{\mu}\|u\|_{X^{1/2,s_1,s_2}}\|v\|_{X^{1/2,s_1,s_2}} \tag{5.1}$$

for some $\epsilon > 0$ such that $\epsilon \ll \delta$.

Proof. We proceed by duality. It is equivalent to show that for $\delta > 0$ small enough and $\epsilon \ll \delta$ for all $w \in X^{1/2-\delta,-s_1+2\delta-\epsilon,-s_2}$,

$$\left|\left\langle\partial_x(uv), w\right\rangle\right| \leqslant CT^{\mu}\left[\|u\|_{X^{1/2,s_1,s_2}}\|v\|_{X^{1/2,s_1,s_2}}\right]\|w\|_{X^{1/2-\delta,-s_1+2\delta-\epsilon,-s_2}}. \tag{5.2}$$

Let f, g and h respectively defined by

$$\hat{f}(\tau,\nu) = \left\langle i\left(\tau - P(\nu)\right) + \xi^2\right\rangle^{1/2}\langle\xi\rangle^{s_1}\langle\eta\rangle^{s_2}\hat{u}(\tau,\nu), \tag{5.3}$$

$$\hat{g}(\tau,\nu) = \left\langle i\left(\tau - P(\nu)\right) + \xi^2\right\rangle^{1/2}\langle\xi\rangle^{s_1}\langle\eta\rangle^{s_2}\hat{v}(\tau,\nu), \tag{5.4}$$

$$\hat{h}(\tau,\nu) = \left\langle i\left(\tau - P(\nu)\right) + \xi^2\right\rangle^{-1/2+\delta}\langle\xi\rangle^{-s_1+2\delta-\epsilon}\langle\eta\rangle^{-s_2}\hat{w}(\tau,\nu). \tag{5.5}$$

It is clear that

$$\|u\|_{X^{1/2,s_1,s_2}} = \|f\|_{L^2_{t,\nu}}, \qquad \|v\|_{X^{1/2,s_1,s_2}} = \|g\|_{L^2_{t,\nu}}, \qquad \|w\|_{X^{-1/2+\delta,-s_1+2\delta-\epsilon,-s_2}} = \|h\|_{L^2_{t,\nu}}.$$

Thus by Plancherel theorem, (5.2) is equivalent

$$\begin{aligned}
&\int_{\mathbb{R}^6} \frac{|\xi||\hat{h}(\tau_1,\nu_1)||\hat{g}(\tau-\tau_1,\nu-\nu_1)||\hat{h}(\tau,\nu)|}{\langle i\sigma+\xi^2\rangle^{1/2-\delta}\langle i\sigma_1+\xi_1^2\rangle^{1/2}\langle i\sigma_2+(\xi-\xi_1)^2\rangle^{1/2}}\frac{\langle\xi\rangle^{s_1-2\delta+\epsilon}}{\langle\xi-\xi_1\rangle^{s_1}\langle\xi_1\rangle^{s_1}} \\
&\quad\times\frac{\langle\eta\rangle^{s_2}}{\langle\eta-\eta_1\rangle^{s_2}\langle\eta_1\rangle^{s_2}}\,d\tau\,d\tau_1\,d\nu\,d\nu_1 \\
&\quad\leqslant CT^{\mu}\|u\|_{L^2_{t,x}}\|v\|_{L^2_{t,x}}\|w\|_{L^2_{t,x}}.
\end{aligned} \tag{5.6}$$

Moreover, we can assume that $s_2 = 0$ since in the case $s_2 \geqslant 0$ we have

$$\frac{\langle\eta\rangle^{s_2}}{\langle\eta-\eta_1\rangle^{s_2}\langle\eta_1\rangle^{s_2}} \leqslant C, \quad \forall\eta,\eta_1\in\mathbb{R}. \tag{5.7}$$

Therefore, setting $s = -s_1 \in [0, 1/2 - 8\delta]$, it is enough to estimate,

$$I =: \int_{\mathbb{R}^6} \frac{|\hat{h}(\tau_1,\nu_1)||\hat{g}(\tau-\tau_1,\nu-\nu_1)||\hat{h}(\tau,\nu)|}{\langle i\sigma+\xi^2\rangle^{1/2-\delta}\langle i\sigma_1+\xi_1^2\rangle^{1/2}\langle i\sigma_2+(\xi-\xi_1)^2\rangle^{1/2}} \times \frac{|\xi|\langle\xi-\xi_1\rangle^s\langle\xi_1\rangle^s}{\langle\xi\rangle^{s+2\delta-\epsilon}}\,d\tau\,d\tau_1\,d\nu\,d\nu_1. \tag{5.8}$$

To estimate I we will use an algebraic relation between σ, σ_1 and σ_2:

$$\sigma_1+\sigma_2-\sigma = 3\xi\xi_1(\xi-\xi_1)+\frac{(\xi_1\eta-\xi\eta_1)^2}{\xi\xi_1(\xi-\xi_1)} \tag{5.9}$$

(see [1]), which ensures that

$$\max\bigl(|\sigma|,|\sigma_1|,|\sigma_2|\bigr) \geqslant \frac{|\sigma_1+\sigma_2-\sigma|}{3} \geqslant \bigl|\xi\xi_1(\xi-\xi_1)\bigr|. \tag{5.10}$$

A symmetry argument shows that it is enough to estimate the contribution to I of the subset of $\mathbb{R}^6$, $\Omega=\{(\tau,\tau_1,\nu,\nu_1)\in\mathbb{R}^6\colon |\sigma_1|\geqslant|\sigma_2|\}$. To do this we split Ω in $\Omega=\Omega_1\cup\Omega_2$ where

$$\Omega_1=\Omega\cap\bigl\{(\tau,\tau_1,\nu,\nu_1)\in\mathbb{R}^6\colon |\xi|\leqslant C_0,\ C_0\gg 1\bigr\},$$
$$\Omega_2=\Omega\cap\bigl\{(\tau,\tau_1,\nu,\nu_1)\in\mathbb{R}^6\colon |\xi|\geqslant C_0,\ C_0\gg 1\bigr\}.$$

Case 1. Contribution of Ω_1 to I. We divide Ω_1 in three regions:

$$\Omega_1^1=\Omega_1\cap\bigl\{(\tau,\tau_1,\nu,\nu_1)\in\mathbb{R}^6\colon |\xi_1|\leqslant 2C_0\bigr\},$$
$$\Omega_1^2=\Omega_1\cap\bigl\{(\tau,\tau_1,\nu,\nu_1)\in\mathbb{R}^6\colon |\sigma|\geqslant|\sigma_1|,\ |\xi_1|\geqslant 2C_0\bigr\},$$
$$\Omega_1^3=\Omega_1\cap\bigl\{(\tau,\tau_1,\nu,\nu_1)\in\mathbb{R}^6\colon |\sigma_1|\geqslant|\sigma|,\ |\xi_1|\geqslant 2C_0\bigr\}.$$

It is clear that $\Omega_1=\Omega_1^1\cup\Omega_2^2\cup\Omega_1^3$.

Case 1.1. Contribution of Ω_1^1 to I. Denote by I_1^1 the contribution of this region to I. In this case we have $|\xi-\xi_1|\leqslant|\xi|+|\xi_1|\leqslant C$ and we see that

$$\frac{|\xi|\langle\xi-\xi_1\rangle^s\langle\xi_1\rangle^s}{\langle\xi\rangle^{s+2\delta-\epsilon}}\leqslant C$$

and hence,

$$I_1^1 \leqslant C\int_{\mathbb{R}^6} \frac{|\hat{h}(\tau_1,\nu_1)||\hat{g}(\tau-\tau_1,\nu-\nu_1)||\hat{h}(\tau,\nu)|}{\langle i\sigma+\xi^2\rangle^{1/2-\delta}\langle i\sigma_1+\xi_1^2\rangle^{1/2}\langle i\sigma_2+(\xi-\xi_1)^2\rangle^{1/2}}\,d\tau\,d\tau_1\,d\nu\,d\nu_1$$
$$\leqslant C\int_{\mathbb{R}^6} \frac{|\hat{h}(\tau_1,\nu_1)||\hat{g}(\tau-\tau_1,\nu-\nu_1)||\hat{h}(\tau,\nu)|}{\langle\sigma\rangle^{1/2-\delta}\langle\sigma_1\rangle^{1/2}\langle\sigma_2\rangle^{1/2}}\,d\tau\,d\tau_1\,d\nu\,d\nu_1.$$

Now, we can apply Lemma 4.5 to deduce

$$I_1^1 \leqslant CT^{\mu}\|u\|_{L^2_{t,x}}\|v\|_{L^2_{t,x}}\|w\|_{L^2_{t,x}}.$$

Case 1.2. Contribution of Ω_1^2 to I. Denote by I_1^2 the contribution of this region to I. Since we have, in this case, $|\xi| \leqslant 1/2|\xi_1|$, it follows that $|\xi_1| \sim |\xi - \xi_1|$. Therefore

$$\frac{|\xi|\langle\xi-\xi_1\rangle^s\langle\xi_1\rangle^s}{\langle\xi\rangle^{s+2\delta-\epsilon}} \leqslant C|\xi-\xi_1|^{2s}|\xi|^s.$$

Moreover, since $|\sigma| = \max(|\sigma|, |\sigma_1|, |\sigma_2|)$, by the relation between σ, σ_1 and σ_2 in (5.9) it results that $|\sigma| \geqslant |\xi||\xi_1||\xi-\xi_1|$. Therefore

$$|\sigma|^s \geqslant |\xi|^s|\xi_1|^s|\xi-\xi_1|^s \geqslant |\xi-\xi_1|^{2s}|\xi|^s$$

and hence,

$$\begin{aligned} I_1^2 &\leqslant C\int\limits_{\mathbb{R}^6} \frac{|\xi-\xi_1|^{2s}|\xi|^s|\hat{h}(\tau_1,\nu_1)||\hat{g}(\tau-\tau_1,\nu-\nu_1)||\hat{h}(\tau,\nu)|}{\langle i\sigma+\xi^2\rangle^{1/2-\delta}\langle i\sigma_1+\xi_1^2\rangle^{1/2}\langle i\sigma_2+(\xi-\xi_1)^2\rangle^{1/2}}\,d\tau\,d\tau_1\,d\nu\,d\nu_1 \\ &\leqslant C\int\limits_{\mathbb{R}^6} \frac{|\hat{h}(\tau_1,\nu_1)||\hat{g}(\tau-\tau_1,\nu-\nu_1)||\hat{h}(\tau,\nu)|}{\langle\sigma\rangle^{1/2-s-\delta}\langle\sigma_1\rangle^{1/2}\langle\sigma_2\rangle^{1/2}}\,d\tau\,d\tau_1\,d\nu\,d\nu_1, \end{aligned}$$

since $s \in [0, 1/2 - 8\delta]$, we see that $(1/2 - s - \delta) + 1/2 + 1/2 > 1 + 7\delta$, therefore a use of Lemma 4.5 provides a good bound for I_1^1.

Case 1.3. Contribution of Ω_1^3 to I. We denote by I_1^3 the contribution of this region to I. In this case $|\sigma_1|$ dominates and $|\xi_1| \sim |\xi - \xi_1|$. Because of (5.9), we obtain

$$\frac{|\xi|\langle\xi-\xi_1\rangle^s\langle\xi_1\rangle^s}{\langle\xi\rangle^{s+2\delta-\epsilon}} \leqslant C|\xi-\xi_1|^{2s}|\xi|^s \leqslant \langle\sigma_1\rangle^s.$$

As in Case 1.2, we obtain by Lemma 4.5 that

$$\begin{aligned} I_1^3 &\leqslant C\int\limits_{\mathbb{R}^6} \frac{|\hat{h}(\tau_1,\nu_1)||\hat{g}(\tau-\tau_1,\nu-\nu_1)||\hat{h}(\tau,\nu)|}{\langle\sigma\rangle^{1/2-\delta}\langle\sigma_1\rangle^{8\delta}\langle\sigma_2\rangle^{1/2}}\,d\tau\,d\tau_1\,d\nu\,d\nu_1 \\ &\leqslant CT^{\mu}\|f\|_{L^2_{t,x}}\|g\|_{L^2_{t,x}}\|h\|_{L^2_{t,x}}. \end{aligned}$$

Case 2. Contribution of Ω_2 to I. We divide Ω_2 into three subdomains Ω_2^i, $i = 1, 2, 3$, such that $\Omega_2 = \Omega_2^1 \cup \Omega_2^2 \cup \Omega_2^3$, where

$$\begin{aligned} \Omega_2^1 &= \Omega_2 \cap \big\{(\tau,\tau_1,\nu,\nu_1) \in \mathbb{R}^6\colon\ \min\big(|\xi_1|,|\xi_2|\big) \leqslant 1\big\}, \\ \Omega_2^2 &= \Omega_2 \cap \big\{(\tau,\tau_1,\nu,\nu_1) \in \mathbb{R}^6\colon\ |\sigma| \geqslant |\sigma_1|,\ \min\big(|\xi_1|,|\xi_2|\big) \geqslant 1\big\}, \\ \Omega_2^3 &= \Omega_2 \cap \big\{(\tau,\tau_1,\nu,\nu_1) \in \mathbb{R}^6\colon\ |\sigma_1| \geqslant |\sigma|,\ \min\big(|\xi_1|,|\xi_2|\big) \geqslant 1\big\}. \end{aligned}$$

Case 2.1. Contribution of I in Ω_2^1. We denote by I_2^1 the contribution of this region to I. We first assume that $\min(|\xi_1|, |\xi - \xi_1|) = |\xi_1|$ and thus $|\xi - \xi_1| \leqslant 1 + |\xi| \leqslant (1 + C_0)|\xi|$, therefore $|\xi| \sim |\xi - \xi_1|$. It follows

$$\frac{|\xi|\langle \xi - \xi_1\rangle^s \langle \xi_1\rangle^s}{\langle \xi\rangle^{s+2\delta-\epsilon}} \leqslant C|\xi|^{1-2\delta+\epsilon}.$$

Since $\langle i\sigma + \xi^2\rangle^{1/2-\delta} \geqslant |\xi|^{1-2\delta}$, $\langle i\sigma_2 + (\xi - \xi_2)^2\rangle^{1/2} \geqslant \langle \sigma_2\rangle^{1/2-\delta}|\xi - \xi_1|^{\epsilon}$ and $|\xi_1| \leqslant 1$, it results that

$$I_2^1 \leqslant C \int_{\mathbb{R}^6} \frac{|\hat{h}(\tau_1, \nu_1)||\hat{g}(\tau - \tau_1, \nu - \nu_1)||\hat{h}(\tau, \nu)|}{\langle \sigma_1\rangle^{1/2}\langle \sigma_2\rangle^{1/2-\delta}} \, d\tau \, d\tau_1 \, d\nu \, d\nu_1$$
$$\leqslant \int_{\mathbb{R}^6} \frac{|\hat{h}(\tau_1, \nu_1)||\hat{g}(\tau - \tau_1, \nu - \nu_1)||\hat{h}(\tau, \nu)|}{\langle \sigma_1\rangle^{1/2}|\xi_1|^{4\delta}\langle \sigma_2\rangle^{1/2-\delta}} \, d\tau \, d\tau_1 \, d\nu \, d\nu_1.$$

Now a use of Lemma 4.3 provides a bound for I_2^1. The other case where $\min(|\xi_1|, |\xi - \xi_1|) = |\xi - \xi_1|$, follows exactly in the same manner since in this case we did not use the supposition: $|\sigma_1| \geqslant |\sigma_2|$, that was allowed by symmetry.

Case 2.2. Contribution of Ω_2^2 to I. In this case we need to divide Ω_2^2 in two regions defined by

$$\Omega_2^{21} = \Omega_2^2 \cap \left\{(\tau, \tau_1, \nu, \nu_1) \in \mathbb{R}^6\colon \min\bigl(|\xi_1|, |\xi_2|\bigr) \leqslant \frac{1}{C_0}|\xi|\right\},$$
$$\Omega_2^{22} = \Omega_2^2 \cap \left\{(\tau, \tau_1, \nu, \nu_1) \in \mathbb{R}^6\colon \min\bigl(|\xi_1|, |\xi_2|\bigr) \geqslant \frac{1}{C_0}|\xi|\right\}.$$

Case 2.21. Contribution of Ω_2^{21} to I. We denote by I_2^{21} the contribution of this region to I. By symmetry argument we can assume that $|\xi_1| \leqslant |\xi - \xi_1|$. It follows $|\xi| \leqslant \frac{1}{C_0}|\xi|$ ($C_0 \gg 1$). Therefore $|\xi - \xi_1| \leqslant |\xi| + |\xi_1| \leqslant C|\xi|$ and $|\xi| \leqslant |\xi_1| + |\xi - \xi_1| \leqslant \frac{1}{C_0}|\xi| + |\xi - \xi_1|$, i.e. $|\xi| \leqslant \frac{1}{1-1/C_0}|\xi - \xi_1|$ and thus $|\xi - \xi_1| \sim |\xi|$. It results that

$$\frac{|\xi|\langle \xi - \xi_1\rangle^s \langle \xi_1\rangle^s}{\langle \xi\rangle^{s+2\delta-\epsilon}} \leqslant C|\xi|^{1-2\delta+\epsilon}|\xi_1|^s,$$

and hence

$$I_2^{21} \leqslant C \int_{\mathbb{R}^6} \frac{|\xi|^{1-2\delta+\epsilon}|\xi_1|^s|\hat{h}(\tau_1, \nu_1)||\hat{g}(\tau - \tau_1, \nu - \nu_1)||\hat{h}(\tau, \nu)|}{\langle i\sigma + \xi^2\rangle^{1/2-\delta}\langle i\sigma_1 + \xi_1^2\rangle^{1/2}\langle i\sigma_2 + (\xi - \xi_1)^2\rangle^{1/2-\delta}|\xi - \xi_1|^{\epsilon}} | \, d\tau \, d\tau_1 \, d\nu \, d\nu_1.$$

Since in this case $|\sigma|$ dominates, we obtain

$$\bigl\langle i\sigma + \xi^2\bigr\rangle^{1/2-\delta} \geqslant \langle \sigma\rangle^{1/2-\delta} \geqslant |\xi|^{1/2-\delta}|\xi_1|^{1/2-\delta}|\xi - \xi_1|^{1/2-\delta} \geqslant |\xi|^{1-2\delta}|\xi_1|^{1/2-\delta}$$

therefore,

$$I_2^{21} \leqslant C \int_{\mathbb{R}^6} \frac{|\hat{h}(\tau_1, \nu_1)||\hat{g}(\tau - \tau_1, \nu - \nu_1)||\hat{h}(\tau, \nu)|}{|\xi_1|^{1/2-\delta-s} \langle \sigma_1 \rangle^{1/2} \langle \sigma_2 \rangle^{1/2-\delta}} \, d\tau \, d\tau_1 \, d\nu \, d\nu_1.$$

It is clear that $1/2 - \delta - s \geqslant 4\delta$ for $s \in [0, 1/2 - 8\delta]$. Now we can apply Lemma 4.3 to estimate I_2^{21}.

Case 2.22. Contribution of Ω_2^{22} to I. We denote by I_2^{22} the contribution of this region to I. In this case, we notice that $|\xi| \lesssim |\xi_1|$ and $|\xi| \lesssim |\xi - \xi_1|$, it results that

$$\begin{aligned} \frac{|\xi| \langle \xi - \xi_1 \rangle^s \langle \xi_1 \rangle^s}{\langle \xi \rangle^{s+2\delta-\epsilon}} &\leqslant C|\xi|^{1-(s+2\delta-\epsilon)} |\xi_1|^s |\xi - \xi_1|^s \\ &\leqslant C|\xi|^{\frac{1+s+2\delta-\epsilon}{3}} |\xi|^{\frac{2}{3}(1-2(s+\delta-\epsilon))} |\xi_1|^s |\xi - \xi_1|^s \\ &\leqslant C|\xi|^{\frac{1+s+2\delta-\epsilon}{3}} |\xi_1|^{\frac{1+s+2\delta-\epsilon}{3}} |\xi - \xi_1|^{\frac{1+s+2\delta-\epsilon}{3}} \\ &\leqslant \langle \sigma \rangle^{\frac{1+s+2\delta-\epsilon}{3}}, \end{aligned}$$

and hence

$$I_2^{22} \leqslant C \int_{\mathbb{R}^6} \frac{|\hat{h}(\tau_1, \nu_1)||\hat{g}(\tau - \tau_1, \nu - \nu_1)||\hat{h}(\tau, \nu)|}{\langle \sigma \rangle^{1/2 - \frac{1+s+2\delta-\epsilon}{3} - \delta} \langle \sigma_1 \rangle^{1/2} \langle \sigma_2 \rangle^{1/2}} \, d\tau \, d\tau_1 \, d\nu \, d\nu_1,$$

since $s \in [0, 1/2 - 8\delta]$ we see, for $\epsilon \ll \delta$, that $1/2 - \frac{1+s+2\delta-\epsilon}{3} - \delta = \frac{1-2s-10\delta+2\epsilon}{3} \geqslant \delta$, therefore we can apply Lemma 4.5 to estimate I_2^{22}.

Case 2.3. Contribution of Ω_2^3 to I. We divide Ω_2^3 in two parts:

$$\Omega_2^{31} = \Omega_2^3 \cap \left\{ (\tau, \tau_1, \nu, \nu_1) \in \mathbb{R}^6 \colon \min\bigl(|\xi_1|, |\xi_2|\bigr) \leqslant \frac{1}{C_0} |\xi| \right\},$$
$$\Omega_2^{32} = \Omega_2^3 \cap \left\{ (\tau, \tau_1, \nu, \nu_1) \in \mathbb{R}^6 \colon \min\bigl(|\xi_1|, |\xi_2|\bigr) \geqslant \frac{1}{C_0} |\xi| \right\}.$$

Case 2.31. Contribution of Ω_2^{31} to I. Because there is no symmetry between $|\xi_1|$ and $|\xi - \xi_1|$ we distinguish between two regions of Ω_2^{31}:

$$\Omega_2^{311} = \Omega_2^{31} \cap \bigl\{ (\tau, \tau_1, \nu, \nu_1) \in \mathbb{R}^6 \colon \min\bigl(|\xi_1|, |\xi_2|\bigr) = |\xi_1| \bigr\},$$
$$\Omega_2^{312} = \Omega_2^{31} \cap \bigl\{ (\tau, \tau_1, \nu, \nu_1) \in \mathbb{R}^6 \colon \min\bigl(|\xi_1|, |\xi_2|\bigr) = |\xi - \xi_1| \bigr\}.$$

Case 2.311. Contribution of Ω_2^{311} to I. We denote by I_2^{311} the contribution of this region to I. In this case we have $|\xi_1| \leqslant \frac{1}{C_0} |\xi|$ and thus $|\xi - \xi_1| \sim |\xi|$. Therefore

$$\frac{|\xi|\langle\xi-\xi_1\rangle^s\langle\xi_1\rangle^s}{\langle\xi\rangle^{s+2\delta-\epsilon}} \leqslant C|\xi|^{1-2\delta+\epsilon}|\xi_1|^s$$
$$\leqslant C|\xi-\xi_1|^{1-2\delta+\epsilon}|\xi_1|^s$$

and it results that

$$I_2^{311} \leqslant C\int\limits_{\mathbb{R}^6} \frac{|\xi-\xi_1|^{1-2\delta+\epsilon}|\xi_1|^s|\hat{h}(\tau_1,\nu_1)||\hat{g}(\tau-\tau_1,\nu-\nu_1)||\hat{h}(\tau,\nu)|}{\langle i\sigma+\xi^2\rangle^{1/2-\delta}\langle i\sigma_1+\xi_1^2\rangle^{1/2}\langle i\sigma_2+(\xi-\xi_1)^2\rangle^{1/2}}\,d\tau\,d\tau_1\,d\nu\,d\nu_1.$$

Note that

$$\left\langle i\sigma_2+(\xi-\xi_1)^2\right\rangle^{1/2} \geqslant \langle\sigma_2\rangle^{1/2-\epsilon/2}|\xi-\xi_1|^\epsilon.$$

On the other hand, since $|\sigma_1|$ dominates, we see that

$$\begin{aligned}\left\langle i\sigma_1+\xi_1^2\right\rangle^{1/2} &\geqslant \langle\sigma_1\rangle^\delta\langle\sigma_1\rangle^{1/2-\delta}\\ &\geqslant \langle\sigma_1\rangle^\delta|\xi|^{1/2-\delta}|\xi_1|^{1/2-\delta}|\xi-\xi_1|^{1/2-\delta}\\ &\geqslant \langle\sigma_1\rangle^\delta|\xi-\xi_1|^{1-2\delta}|\xi_1|^{1/2-\delta}.\end{aligned}$$

By combining the two last inequalities, we infer that

$$I_2^{311} \leqslant C\int\limits_{\mathbb{R}^6} \frac{|\hat{h}(\tau_1,\nu_1)||\hat{g}(\tau-\tau_1,\nu-\nu_1)||\hat{h}(\tau,\nu)|}{\langle\sigma\rangle^{1/2-\delta}\langle\sigma_1\rangle^\delta|\xi_1|^{1/2-\delta-s}\langle\sigma_2\rangle^{1/2-\epsilon/2}}\,d\tau\,d\tau_1\,d\nu\,d\nu_1.$$

Since $1/2-\delta-s > \frac{\delta}{4}$, it follows by virtue of Lemma 4.2, for $\epsilon \ll \delta$ ($\epsilon \leqslant \delta/12$ is enough), that

$$\begin{aligned}I_2^{311} &\leqslant C\int\limits_{\mathbb{R}^6} \frac{|\hat{h}(\tau_1,\nu_1)||\hat{g}(\tau-\tau_1,\nu-\nu_1)||\hat{h}(\tau,\nu)|}{\langle\sigma\rangle^{1/2-\delta}\langle\sigma_1\rangle^\delta|\xi_1|^{\frac{\delta}{4}}\langle\sigma_2\rangle^{1/2-\delta/24}}\,d\tau\,d\tau_1\,d\nu\,d\nu_1\\ &\leqslant CT^\mu\|f\|_{L^2_{t,x}}\|g\|_{L^2_{t,x}}\|h\|_{L^2_{t,x}}.\end{aligned}$$

Case 2.312. Contribution of Ω_2^{312} to I. We denote by I_2^{312} the contribution of this region to I. In this case we have $|\xi-\xi_1| \leqslant \frac{1}{C_0}|\xi|$ and thus $|\xi_1| \sim |\xi|$. It results that

$$\frac{|\xi|\langle\xi-\xi_1\rangle^s\langle\xi_1\rangle^s}{\langle\xi\rangle^{s+2\delta-\epsilon}} \leqslant C|\xi|^{1-2\delta+\epsilon}|\xi-\xi_1|^s. \tag{5.11}$$

Since $|\sigma_1|$ dominates in this case, for $\epsilon < \delta$ we obtain

$$\begin{aligned}\left\langle i\sigma_1+\xi_1^2\right\rangle^{1/2} &\geqslant \left\langle i\sigma_1+\xi_1^2\right\rangle^\delta\langle\sigma_1\rangle^{1/2-\delta}\\ &\geqslant |\xi|^\epsilon\langle\sigma_1\rangle^{\delta-\epsilon/2}|\xi|^{1/2-\delta}|\xi_1|^{1/2-\delta}|\xi-\xi_1|^{1/2-\delta}\\ &\geqslant \langle\sigma_1\rangle^{\delta-\epsilon/2}|\xi-\xi_1|^{1/2-\delta}|\xi|^{1-2\delta+\epsilon}\end{aligned}$$

and since $|\sigma_1| \geqslant |\sigma|$, it results, for $s \in [0, 1/2 - 8\delta]$ and $\epsilon < \delta$, that

$$\left\langle i\sigma_1 + \xi_1^2 \right\rangle^{1/2} \geqslant \langle \sigma \rangle^{\delta - \epsilon/2} |\xi - \xi_1|^{1/2-\delta} |\xi|^{1-2\delta+\epsilon}. \tag{5.12}$$

By combining (5.11) and (5.12), we can deduce that

$$I_2^{312} \leqslant C \int_{\mathbb{R}^6} \frac{|\hat{h}(\tau_1, \nu_1)||\hat{g}(\tau - \tau_1, \nu - \nu_1)||\hat{h}(\tau, \nu)|}{\langle \sigma \rangle^{1/2-\epsilon} |\xi - \xi_1|^{1/2-s-\delta} \langle \sigma_2 \rangle^{1/2}} \, d\tau \, d\tau_1 \, d\nu \, d\nu_1$$

$$\leqslant C \int_{\mathbb{R}^6} \frac{|\hat{h}(\tau_1, \nu_1)||\hat{g}(\tau - \tau_1, \nu - \nu_1)||\hat{h}(\tau, \nu)|}{\langle \sigma \rangle^{1/2-\epsilon} |\xi - \xi_1|^{4\epsilon} \langle \sigma_2 \rangle^{1/2}} \, d\tau \, d\tau_1 \, d\nu \, d\nu_1.$$

Now Lemma 4.3 provides a bound for I_2^{312}.

Case 2.33. Contribution of Ω_2^{33} to I. We indicate by I_2^{33} the contribution of this region to I. In this case $|\sigma_1|$ dominates and we have $|\xi - \xi_1| \geqslant \frac{1}{C_0}|\xi|$, $|\xi_1| \geqslant \frac{1}{C_0}|\xi|$. Hence

$$\begin{aligned}
\frac{|\xi| \langle \xi - \xi_1 \rangle^s \langle \xi_1 \rangle^s}{\langle \xi \rangle^{s+2\delta-\epsilon}} &\leqslant C|\xi|^{1-(s+2\delta-\epsilon)} |\xi_1|^s |\xi - \xi_1|^s \\
&\leqslant C|\xi|^{\frac{1+s+2\delta-\epsilon}{3}} |\xi_1|^{\frac{1+s+2\delta-\epsilon}{3}} |\xi - \xi_1|^{\frac{1+s+2\delta-\epsilon}{3}} \\
&\leqslant \langle \sigma_1 \rangle^{\frac{1+s+2\delta-\epsilon}{3}},
\end{aligned}$$

it follows

$$I_2^{33} \leqslant C \int_{\mathbb{R}^6} \frac{|\hat{h}(\tau_1, \nu_1)||\hat{g}(\tau - \tau_1, \nu - \nu_1)||\hat{h}(\tau, \nu)|}{\langle \sigma \rangle^{1/2-\delta} \langle \sigma_1 \rangle^{1/2 - \frac{1+s+2\delta-\epsilon}{3}} \langle \sigma_2 \rangle^{1/2}} \, d\tau \, d\tau_1 \, d\nu \, d\nu_1.$$

It is clear that, for $\epsilon \ll \delta$, $(\frac{1}{2} - \delta) + (\frac{1}{2} - \frac{1+s+2\delta-\epsilon}{3}) = \frac{1}{2} + \frac{1-2s-10\delta+2\epsilon}{6} \geqslant \frac{1}{2}^+$. Therefore we can apply Lemma 4.5 to estimate I_2^{33}.

This completes the proof of Proposition 5.1. □

Actually, we will mainly use the following version, which is a direct consequence of Proposition 5.1, together with the two triangle inequality

$$\forall s_1 \geqslant s_c^1, \quad \langle \xi \rangle^{s_1} \leqslant \langle \xi \rangle^{s_c^1} \langle \xi_1 \rangle^{s_1 - s_c^1} + \langle \xi \rangle^{s_c^1} \langle \xi - \xi_1 \rangle^{s_1 - s_c^1}, \tag{5.13}$$

$$\langle \eta \rangle^{s_2} \leqslant \langle \eta_1 \rangle^{s_2} + \langle \eta - \eta_1 \rangle^{s_2}. \tag{5.14}$$

Proposition 5.2. *Let $s_c^1 \in \,]-1/2, 0]$, $s_2 \geqslant 0$. For all $s_1 \geqslant s_c^1$ and $u, v \in X^{1/2, s_1, s_2}$ with compact support in time and included in the subset $\{(t, x, y)\colon t \in [-T, T]\}$, there exists $\mu > 0$ such that the following bilinear estimate holds*:

$$\begin{aligned}&\left\|\partial_x(uv)\right\|_{X^{-1/2+\delta,s_1-2\delta+\epsilon,s_2}}\\&\quad\leqslant CT^{\mu}\big(\|u\|_{X^{1/2,s_c^1,0}}\|v\|_{X^{1/2,s_1,s_2}}+\|u\|_{X^{1/2,s_1,s_2}}\|v\|_{X^{1/2,s_c^1,0}}+\|u\|_{X^{1/2,s_1,0}}\|v\|_{X^{1/2,s_c^1,s_2}}\\&\qquad+\|u\|_{X^{1/2,s_c^1,s_2}}\|v\|_{X^{1/2,s_1,0}}\big)\end{aligned}\tag{5.15}$$

this for some $\delta>0$ small enough and $\epsilon>0$ such that $\epsilon\ll\delta$.

6. Proof of Theorem 2.1

6.1. Existence result

Let $\phi\in H^{s_1,s_2}$ with $s_1>-1/2$, $s_2\geqslant 0$ and $s_c^1\in\,]-1/2,\min(0,s_1)]$. We suppose that $T\leqslant 1$, if u is a solution of the integral equation $u=L(u)$ with

$$L(u)=\psi(t)\left[W(t)\phi-\frac{\chi_{\mathbb{R}_+}(t)}{2}\int_0^t W(t-t')\partial_x\big(\psi_T^2(t')u^2(t')\big)\,dt'\right],\tag{6.1}$$

then u solve (KPBII) equation on $[0,T/2]$.We introduce the Bourgain spaces defined by

$$Z_1=\big\{u\in X^{1/2,s_1,s_2};\ \|u\|_{Z_1}=\|u\|_{X^{1/2,s_1,0}}+\gamma_1\|u\|_{X^{1/2,s_1,s_2}}\big\},\tag{6.2}$$

$$Z_2=\big\{u\in X^{1/2,s_1,0};\ \|u\|_{Z_2}=\|u\|_{X^{1/2,s_c^1,0}}+\gamma_2\|u\|_{X^{1/2,s_1,0}}\big\},\tag{6.3}$$

where

$$\gamma_1=\frac{\|\phi\|_{H^{s_1,0}}}{\|\phi\|_{H^{s_1,s_2}}},\qquad \gamma_2=\frac{\|\phi\|_{H^{s_c^1,0}}}{\|\phi\|_{H^{s_1,0}}}.\tag{6.4}$$

The goal to introduce two Bourgain spaces is to show in a first time that there exists $T_1=T(\|\phi\|_{H^{s_1,0}})$ and a solution u of Eq. (6.1) in a ball of Z_1, and then to solve (6.1) in Z_2 in order to check that the time of existence $T=T(\|\phi\|_{H^{s_c^1,0}})$ with $s_c^1\in\,]-1/2,0]$.

Step 1. Resolution of (6.1) in Z_1. By Propositions 3.2 and 3.4, it results that

$$\left\|L(u)\right\|_{X^{1/2,s_1,0}}\leqslant C\|\phi\|_{H^{s_1,0}}+C\left\|\partial_x\big(\psi_T^2(t)u^2\big)\right\|_{X^{-1/2+\delta,s_1-2\delta+\epsilon,0}},\tag{6.5}$$

$$\left\|L(u)\right\|_{X^{1/2,s_1,s_2}}\leqslant C\|\phi\|_{H^{s_1,s_2}}+C\left\|\partial_x\big(\psi_T^2(t)u^2\big)\right\|_{X^{-1/2+\delta,s_1-2\delta+\epsilon,s_2}}.\tag{6.6}$$

By Propositions 5.1 and 5.2, we can deduce

$$\left\|L(u)\right\|_{X^{1/2,s_1,0}}\leqslant C\|\phi\|_{H^{s_1,0}}+CT^{\mu}\left\|\psi_T(t)u\right\|^2_{X^{1/2,s_1,0}},\tag{6.7}$$

$$\left\|L(u)\right\|_{X^{1/2,s_1,s_2}}\leqslant C\|\phi\|_{H^{s_1,s_2}}+CT^{\mu}\left\|\psi_T(t)u\right\|_{X^{1/2,s_1,0}}\left\|\psi_T(t)u\right\|_{X^{1/2,s_1,s_2}}.\tag{6.8}$$

By Leibniz rule for fractional derivative and Sobolev inequalities in time we have, for all $\epsilon>0$ and $0<T\leqslant 1$, that

$$\left\|\psi_T(t)u\right\|_{X^{1/2,s_1,s_2}}\leqslant C_\epsilon T^{-\epsilon}\|u\|_{X^{1/2,s_1,s_2}}.$$

Taking $\epsilon = \mu/4$ we obtain

$$\|L(u)\|_{X^{1/2,s_1,0}} \leqslant C\|\phi\|_{H^{s_1,0}} + CT^{\mu/2}\|u\|^2_{X^{1/2,s_1,0}}, \tag{6.9}$$

$$\|L(u)\|_{X^{1/2,s_1,s_2}} \leqslant C\|\phi\|_{H^{s_1,s_2}} + CT^{\mu/2}\|u\|_{X^{1/2,s_1,0}}\|u\|_{X^{1/2,s_1,s_2}}. \tag{6.10}$$

By combining two estimates (6.9) and (6.10) we obtain

$$\|L(u)\|_{Z_1} \leqslant C\big(\|\phi\|_{H^{s_1,0}} + \gamma_1\|\phi\|_{H^{s_1,s_2}}\big) + CT^{\mu/2}\|u\|^2_{Z_1}. \tag{6.11}$$

Since we can write $\partial_x(u^2) - \partial(v^2) = \partial_x[(u-v)(u+v)]$, in the same way we get

$$\|L(u) - L(v)\|_{X_T^{1/2,s_1,0}} \leqslant CT^{\mu/2}\|u-v\|_{X^{1/2,s_1,0}}\|u+v\|_{X^{1/2,s_1,0}}, \tag{6.12}$$

$$\begin{aligned}&\|L(u) - L(v)\|_{X_T^{1/2,s_1,s_2}}\\ &\quad\leqslant CT^{\mu/2}\big(\|u-v\|_{X^{1/2,s_1,0}}\|u+v\|_{X^{1/2,s_1,s_2}} + \|u+v\|_{X^{1/2,s_1,0}}\|u-v\|_{X^{1/2,s_1,s_2}}\big).\end{aligned} \tag{6.13}$$

Consequently, it results that

$$\|L(u) - L(v)\|_{Z_1} \leqslant CT^{\mu/2}\|u-v\|_{Z_1}\|u+v\|_{Z_1}. \tag{6.14}$$

Hence, setting $T_1 = (4C^2(\|\phi\|_{H^{s_1,0}} + \gamma_1\|\phi\|_{H^{s_1,s_2}}))^{-2/\mu}$ which yields, by definition of γ_1, to $T_1 = (8C^2\|\phi\|_{H^{s_1,0}})^{-2/\mu}$, we can deduce from (6.11) and (6.14) that L is strictly contractive on the ball of radius $r_1 = 2c(\|\phi\|_{H^{s_1,0}} + \gamma_1\|\phi\|_{H^{s_1,s_2}})$ in Z_1. This proves the existence of a unique solution u_1 to (6.1) in $X^{1/2,s_1,s_2}$ with $T_1 = T(\|\phi\|_{H^{s_1,0}})$.

Since $\phi \in H^{s_1,s_2}$, it follows that $\psi(.)W(.)\phi \in C([0,T_1], H^{s_1,s_2})$, moreover since $u_1 \in X^{1/2,s_1,s_2}$, we can deduce from Proposition 5.1 that $\partial_x(u_1^2) \in X_T^{-1/2+\delta,s_1-2\delta,s_2}$ and from (3.37) in Proposition 3.5 it results that

$$\int_0^t W(t-t')\partial_x\big(u_1^2(t')\big)\,dt' \in C\big([0,T_1], H^{s_1,s_2}\big).$$

Thus u_1 belongs $C([0,T_1], H^{s_1,s_2})$.

Step 2. Resolution of (6.1) in Z_2. Now proceeding exactly in the same way as above but in Z_2, we obtain that L is also strictly contractive on the ball of radius $r_1 = 2c(\|\phi\|_{H^{s_c^1,0}} + \gamma_2\|\phi\|_{H^{s_1,0}})$ in Z_2 with $T_2 = (4C^2(\|\phi\|_{H^{s_c^1,0}} + \gamma_2\|\phi\|_{H^{s_1,0}}))^{-1/\mu}$. Therefore by definition of γ_2, it follows that $T_2 = T(\|\phi\|_{H^{s_c^1,0}})$. Since obviously $H^{s_1,s_2} \subset H^{s_1,0}$, it follows that there exists a unique solution u_1 to (6.1) in $C([0,T_2], H^{s_1,0}) \cap X_T^{1/2,s_1,0}$ and $T_2 = T(\|\phi\|_{H^{s_c^1,0}})$, $s_c^1 \in\,]-1/2, 0]$. If we indicate by $T_* = T_{\max}$ the maximum time of the existence in Z_1 then by uniqueness, we have $u_1 = u_2$ on $[0, \min(T_2, T_*)[$ and this gives that $T_* \geqslant T(\|\phi\|_{H^{s_c^1,0}})$.

The continuity of map $\phi \mapsto u$ from H^{s_1,s_2} to $X^{1/2,s_1,s_2}$ follows from classical argument, while the continuity from H^{s_1,s_2} to $C([0,T_1], H^{s_1,s_2})$ follows again from Proposition 3.5. The analyticity of the flow-map is a direct consequence of the implicit function theorem.

6.2. Uniqueness

The above contraction argument gives the uniqueness of the solution to the truncated integral equation (6.1). We give here the argument of [9] to deduce easily the uniqueness of the solution to the integral equation (2.14).

Let $u_1, u_2 \in X_T^{1/2,s_1,s_2}$ be two solutions of the integral equation (2.14) on the time interval $[0,T]$ and let $\tilde{u}_1 - \tilde{u}_2$ be an extension of $u_1 - u_2$ in $X^{1/2,s_1,s_2}$ such that

$$\|\tilde{u}_1 - \tilde{u}_2\|_{X^{1/2,s_1,s_2}} \leqslant 2\|u_1 - u_2\|_{X_\gamma^{1/2,s_1,s_2}}$$

with $0 < \gamma \leqslant T/2$. It results by Propositions 3.2 and 3.4 that

$$\begin{aligned}
&\|u_1 - u_2\|_{X_\gamma^{1/2,s_1,s_2}} \\
&\quad \leqslant \left\| \psi(t)\frac{\chi_{\mathbb{R}_+}(t)}{2}\int_0^t W(t-t')\partial_x\big(\psi_\gamma^2(t')\big(\tilde{u}_1(t') - \tilde{u}_2(t')\big)\big(u_1(t') + u_2(t')\big)\big)\,dt'\right\|_{X^{1/2,s_1,s_2}} \\
&\quad \leqslant C\big\|\partial_x\big(\psi_\gamma^2(t)\big(\tilde{u}_1(t) - \tilde{u}_2(t)\big)\big(u_1(t) + u_2(t)\big)\big)\big\|_{X^{-1/2+\delta,s_1-2\delta+\epsilon,s_2}} \\
&\quad \leqslant C\gamma^{\mu/2}\|\tilde{u}_1 - \tilde{u}_2\|_{X^{1/2,s_1,s_2}}\|u_1 + u_2\|_{X_T^{1/2,s_1,s_2}}
\end{aligned}$$

for some $\mu > 0$. Hence

$$\|u_1 - u_2\|_{X_\gamma^{1/2,s_1,s_2}} \leqslant 2C\gamma^{\mu/2}\big(\|u_1\|_{X_T^{1/2,s_1,s_2}} + \|u_2\|_{X_T^{1/2,s_1,s_2}}\big)\|u_1 - u_2\|_{X_\gamma^{1/2,s_1,s_2}}.$$

Taking $\gamma \leqslant (4C(\|u_1(t)\|_{X_T^{1/2,s_1,s_2}} + \|u_2(t)\|_{X_T^{1/2,s_1,s_2}}))^{-\mu/2}$, this forces $u_1 \equiv u_2$ on $[0,\gamma]$. Iterating this argument, one extends the uniqueness result on the whole time interval $[0,T]$.

6.3. Global existence

By Proposition 5.2 $\partial_x(u^2) \in X^{-1/2+\delta,s_1+\epsilon-2\delta,s_2}$, therefore by Proposition 3.5 we obtain that

$$\int_0^t W(t-t')\partial_x\big(u^2(t')\big)\,dt' \in C\big([0,T], H^{s_1+\epsilon,s_2}\big).$$

Note that $W(.)\phi \in C(\mathbb{R}_+; H^{s_1,s_2}) \cap C(\mathbb{R}_+^*; H^{\infty,s_2})$. Hence

$$u \in C\big([0,T]; H^{s_1,s_2}\big) \cap C\big(]0,T]; H^{s_1+\epsilon,s_2}\big).$$

Recalling that $T = T(\|\phi\|_{H^{s_1^c,0}})$ with $s_1^c \in \,]-1/2, \min(0,s_1)]$ and using the uniqueness result, we deduce by induction that $u \in C(]0,T]; H^{\infty,s_2})$. This allows us to take the L^2-scalar product of (1.1) with u, which shows that $t \mapsto \|u(t)\|_{L^2}$ is nonincreasing on $]0,T]$. Since the time of local existence T only depends on $\|\phi\|_{H^{s_1^c,0}}$, this clearly gives that the solution is global in time.

7. Ill-posedness results for the (KPBII) equation

In this section, we prove the ill-posedness result for the (KPBII) equation stated in Theorem 2.2. We start by constructing a sequence of initial data $(\phi_n)_n$ which will ensure the nonregularity of the map $\phi \mapsto u$ from $H^{s,0}$ to $C([0,T], H^{s,0})$ for $s < -1/2$.

7.1. Proof of Theorem 2.2

Let u be a solution of (1.1). Then we have

$$u(\phi,t,x,y) = W(t)\phi(x,y) - \frac{1}{2}\int_0^t W(t-t')\partial_x\big(u^2(\phi,t',x,y)\big)\,dt', \tag{7.1}$$

suppose that the map is C^2. Since $u(0,t,x,y)=0$ it is easy to check that

$$\begin{aligned} u_1(t,x,y) &=: \frac{\partial u}{\partial \phi}(0,t,x,y)[h] = w(t)h,\\ u_2(t,x,y) &:= \frac{\partial^2 u}{\partial \phi^2}(0,t,x,y)[h,h]\\ &= -\int_0^t W(t-t')u_1(t',x,y)\partial_x\big(u_1(t',x,y)\big)\,dt'\\ &= -\int_0^t W(t-t')\partial_x\big(W(t')h\big)^2\,dt'. \end{aligned} \tag{7.2}$$

Due to the assumption of C^2-regularity of the map and since that $u(0,t,x,y)=0$, we can write a formal Taylor expansion

$$u(h,t,x,y) = u_1(t,x,y)[h] + u_2(t,x,y)[h,h] + O\big(\|h\|^2_{H^{s,0}}\big), \tag{7.3}$$

and we must have

$$\big\|u_1(t,.,.)\big\|_{H^{s,0}} \lesssim \|h\|_{H^{s,0}}, \qquad \big\|u_2(t,.,.)\big\|_{H^{s,0}} \lesssim \|h\|^2_{H^{s,0}}. \tag{7.4}$$

Taking the partial Fourier transform with respect to (x,y), it results that

$$\begin{aligned} \mathcal{F}_{x\mapsto\xi,y\mapsto\eta}\big(u_2(t,.,.)\big) = \int_0^t &\exp\big(-|t-t'|\xi^2\big)\exp\big(i(t-t')\big(\xi^3-\eta^2/\xi\big)\big)\\ &\times (i\xi)\times\big[\mathcal{F}_{x\mapsto\xi,y\mapsto\eta}\big(u_1(t')*u_1(t')\big)(\xi,\eta)\big]\,dt'. \end{aligned} \tag{7.5}$$

Note that

$$
\begin{aligned}
&F_{x\mapsto\xi,y\mapsto\eta}\big(u_1(t')*u_1(t')\big)(\xi,\eta)\\
&\quad= F_{x\mapsto\xi,y\mapsto\eta}\big(w(t')\phi\big)*F_{x\mapsto\xi,y\mapsto\eta}\big(w(t')\phi\big)\\
&\quad= \int_{\mathbb{R}^2}\hat{\phi}(\xi-\xi_1,\eta-\eta_1)\hat{\phi}(\xi_1,\eta_1)\exp\big(-\big(\xi_1^2+(\xi-\xi_1)^2\big)t'\big)\\
&\qquad\times\exp\left(it\left(\xi_1^3-\frac{\eta_1^2}{\xi_1}+(\xi-\xi_1)^2-\frac{(\eta-\eta_1)^2}{\xi-\xi_1}\right)\right)d\xi_1\,d\eta_1. \qquad (7.6)
\end{aligned}
$$

Now let $P(\xi,\eta)=\xi^3-\eta^2/\xi$. Hence

$$
\begin{aligned}
&\mathcal{F}_{x\mapsto\xi,y\mapsto\eta}\big(u_2(t,.,.)\big)\\
&\quad= \int_{\mathbb{R}^2}\hat{\phi}(\xi-\xi_1,\eta-\eta_1)\hat{\phi}(\xi_1,\eta_1)(i\xi)e^{-t\xi^2}e^{itP(\xi,\eta)}\int_0^t e^{it'(\xi_1^2+(\xi-\xi_1)^2-\xi^2)}\\
&\qquad\times e^{it'(P(\xi_1,\eta_1)+P(\xi-\xi_1,\eta-\eta_1)-P(\xi,\eta))}\,dt'\,d\xi_1\,d\eta_1. \qquad (7.7)
\end{aligned}
$$

Let $\chi(\xi,\xi_1,\eta,\eta_1)=P(\xi_1,\eta_1)+P(\xi-\xi_1,\eta-\eta_1)-P(\xi,\eta)$. A simple calculation shows that

$$
\chi(\xi,\xi_1,\eta,\eta_1)=3\xi\xi_1(\xi-\xi_1)+\frac{(\eta\xi_1-\eta_1\xi)^2}{\xi\xi_1(\xi-\xi_1)}.
$$

Therefore we can deduce that

$$
\begin{aligned}
\mathcal{F}_{x\mapsto\xi,y\mapsto\eta}\big(u_2(t,.,.)\big)&=(i\xi)e^{itP(\xi,\eta)}\int_{\mathbb{R}^2}\hat{\phi}(\xi_1,\eta_1)\hat{\phi}(\xi-\xi_1,\eta-\eta_1)\\
&\qquad\times\frac{e^{-t(\xi_1^2+(\xi-\xi_1)^2)}e^{it\chi(\xi,\xi_1,\eta,\eta_1)}-e^{-\xi^2t}}{-2\xi_1(\xi-\xi_1)+i\chi(\xi,\xi_1,\eta,\eta_1)}\,d\xi_1\,d\eta_1, \qquad (7.8)
\end{aligned}
$$

it follows that

$$
\begin{aligned}
\big\|u_2(t)\big\|_{H^{s,0}}^2&:=\int_{\mathbb{R}^2}\big(1+|\xi|^2\big)^s\big|\mathcal{F}_{x\mapsto\xi,y\mapsto\eta}\big(u_2(t,.,.)(\xi,\eta)\big)\big|^2\,d\xi\,d\eta\\
&=\int_{\mathbb{R}^2}|\xi|^2\big(1+|\xi|^2\big)^s\bigg|\int_{\mathbb{R}^2}\hat{\phi}(\xi_1,\eta_1)\hat{\phi}(\xi-\xi_1,\eta-\eta_1)\\
&\qquad\times\frac{e^{-t(\xi_1^2+(\xi-\xi_1)^2)}e^{it\chi(\xi,\xi_1,\eta,\eta_1)}-e^{-\xi^2t}}{-2\xi_1(\xi-\xi_1)+i\chi(\xi,\xi_1,\eta,\eta_1)}\,d\xi_1\,d\eta_1\bigg|^2\,d\xi\,d\eta. \qquad (7.9)
\end{aligned}
$$

We choose now a sequence of initial data $(\phi_N)_N$, $N > 0$, defined through its Fourier transform by[1]

$$\hat{\phi}_N(\xi,\eta) = N^{-3/2-s}(\chi_{D_{1,N}} + \chi_{D_{2,N}}), \tag{7.10}$$

where N is a positive parameter such that $N \gg 1$, and $D_{1,N}$, $D_{2,N}$ are the rectangles in $\mathbb{R}^2$ defined by

$$D_{1,N} = [N/2, N] \times \left[-6N^2, 6N^2\right], \qquad D_{2,N} = [N, 2N] \times \left[\sqrt{3}N^2, (\sqrt{3}+1)N^2\right].$$

It is simple to see that $\|\phi_N\|_{H^{s,0}} \sim 1$. Let us denote by $u_{2,N}$ the sequence of the second iteration u_2 associated with ϕ_N. Setting

$$K(t,\xi,\xi_1,\eta,\eta_1) = \frac{e^{-t(\xi_1^2+(\xi-\xi_1)^2)}e^{it\chi(\xi,\xi_1,\eta,\eta_1)} - e^{-\xi^2 t}}{-2\xi_1(\xi-\xi_1) + i\chi(\xi,\xi_1,\eta,\eta_1)},$$

$\|u_{2,N}(t)\|^2_{H^{s,0}}$ can be split into three parts:

$$\left\|u_{2,N}(t)\right\|^2_{H^{s,0}} = C\left(\left|f_1(t)\right| + \left|f_2(t)\right| + \left|f_3(t)\right|\right),$$

where

$$\left|f_1(t)\right|^{1/2} = CN^{-3-2s}\Bigg[\int_{\mathbb{R}^2} |\xi|^2\left(1+|\xi|^2\right)^s \times \Bigg|\int_{\substack{(\xi_1,\eta_1)\in D_{1,N}\\(\xi-\xi_1,\eta-\eta_1)\in D_{1,N}}} K(t,\xi,\xi_1,\eta,\eta_1)\,d\xi_1\,d\eta_1\Bigg|^2 d\xi\,d\eta\Bigg]^{1/2},$$

$$\left|f_2(t)\right|^{1/2} = CN^{-3-2s}\Bigg[\int_{\mathbb{R}^2} |\xi|^2\left(1+|\xi|^2\right)^s \times \Bigg|\int_{\substack{(\xi_1,\eta_1)\in D_{2,N}\\(\xi-\xi_1,\eta-\eta_1)\in D_{2,N}}} K(t,\xi,\xi_1,\eta,\eta_1)\,d\xi_1\,d\eta_1\Bigg|^2 d\xi\,d\eta\Bigg]^{1/2},$$

$$\left|f_3(t)\right|^{1/2} = CN^{-3-2s}\Bigg[\int_{\mathbb{R}^2} |\xi|^2\left(1+|\xi|^2\right)^s \Bigg|\int_{k(\xi,\eta)} K(t,\xi,\xi_1,\eta,\eta_1)\,d\xi_1\,d\eta_1\Bigg|^2 d\xi\,d\eta\Bigg]^{1/2},$$

where

[1] The forthcoming analysis still works in the case of taking a real sequence ϕ_N^1 such that $\hat{\phi}_N^1(\xi,\eta) = \hat{\phi}_N(|\xi|,\eta)$.

$$\begin{aligned} k(\xi,\eta) &= \{(\xi_1,\eta_1)\colon\ (\xi-\xi_1,\eta-\eta_1)\in D_{1,N},\ (\xi_1,\eta_1)\in D_{2,N}\} \\ &\quad \cup \{(\xi_1,\eta_1)\colon\ (\xi_1,\eta_1)\in D_{1,N},\ (\xi-\xi_1,\eta-\eta_1)\in D_{2,N}\} \\ &:= k^1(\xi,\eta)\cup k^2(\xi,\eta). \end{aligned} \tag{7.11}$$

Therefore, obviously

$$\|u_{2,N}(t)\|^2_{H^{s,0}} \geqslant C|f_3|.$$

Since we can write $\xi=\xi_1+(\xi-\xi_1)$ and $\eta=\eta_1+(\eta-\eta_1)$, it follows that

$$\begin{aligned} \|u_2(t)\|^2_{H^{s,0}} &\geqslant CN^{-4s-6}\int_{3N/2}^{3N}\int_{(\sqrt{3}-6)N^2}^{(\sqrt{3}+7)N^2}|\xi|^2\big(1+|\xi|^2\big)^s \\ &\quad\times\left|\int_{k(\xi,\eta)}\frac{e^{-t(\xi_1^2+(\xi-\xi_1)^2)}e^{it\chi(\xi,\xi_1,\eta,\eta_1)}-e^{-\xi^2t}}{-2\xi_1(\xi-\xi_1)+i\chi(\xi,\xi_1,\eta,\eta_1)}\,d\xi_1\,d\eta_1\right|^2 d\xi\,d\eta. \end{aligned} \tag{7.12}$$

We need to find a lower bound for the right-hand side of (7.12). Thus it is necessary to evaluate the contribution of the function $\chi(\xi,\xi_1,\eta,\eta_1)$ in $k(\xi,\eta)$. This in the aim of the following lemma which is inspired by [10].

Lemma 7.1. *Let $(\xi_1,\eta_1)\in k^1(\xi,\eta)$ or $(\xi_1,\eta_1)\in k^2(\xi,\eta)$. For $N\gg 1$ we have*

$$|\chi(\xi,\xi_1,\eta,\eta_1)|\lesssim N^3.$$

Proof. By definition of the function $\chi(\xi,\xi_1,\eta,\eta_1)$ we can write

$$|\chi(\xi,\xi_1,\eta,\eta_1)|\leqslant|\chi_1(\xi,\xi_1,\eta,\eta_1)|+|6\xi\xi_1(\xi-\xi_1)|, \tag{7.13}$$

where

$$\chi_1(\xi,\xi_1,\eta,\eta_1)=3\xi\xi_1(\xi-\xi_1)-\frac{(\eta\xi_1-\eta_1\xi)^2}{\xi\xi_1(\xi-\xi_1)}.$$

Now let $(\xi_1,\eta_1)\in k^1(\xi,\eta)$, i.e. $(\xi-\xi_1,\eta-\eta_1)\in D_{1,N}$ and $(\xi_1,\eta_1)\in D_{2,N}$.

Let $\xi\in\mathbb{R}$ such that $(\xi-\xi_1)\in[N/2,N]$ and we fix $(\xi_1,\eta_1)\in D_{2,N}$. We will seek $\eta^*(\xi,\xi_1,\eta_1)$ such that $\chi_1(\xi,\xi_1,\eta^*(\xi,\xi_1,\eta_1),\eta_1)=0$ and $|\eta^*(\xi,\xi_1,\eta_1)-\eta_1|\leqslant 6N^2$. Indeed, we choose

$$\eta^*(\xi,\xi_1,\eta_1)=\eta_1+\frac{(\xi-\xi_1)(\eta_1-\sqrt{3}\xi\xi_1)}{\xi_1}.$$

Thus

$$|\eta^*(\xi,\xi_1,\eta_1)-\eta_1|\leqslant\frac{|\xi-\xi_1|}{|\xi_1|}\big|\eta_1-\sqrt{3}\xi_1^2-\sqrt{3}\xi_1(\xi-\xi_1)\big|.$$

We recall that $\eta_1 \in [\sqrt{3}N^2, (\sqrt{3}+1)N^2]$ and $\xi_1 \in [N, 2N]$. Therefore it follows that

$$\sqrt{3}\xi_1^2 \in \left[\sqrt{3}N^2, 4\sqrt{3}N^2\right],$$

and we have

$$\left|\eta_1 - \sqrt{3}N^2\right| \leqslant 3\sqrt{3}N^2.$$

Since $|\xi_1| \leqslant 2N$ and $|\xi - \xi_1| \geqslant N/2$, it results that

$$\left|\eta^*(\xi, \xi_1, \eta_1) - \eta_1\right| \leqslant 1/4\left(3\sqrt{3}N^2 + 2\sqrt{3}N^2\right) \leqslant 6N^2.$$

Now by the mean value theorem we can write

$$\chi_1(\xi, \xi_1, \eta, \eta_1) = \chi_1\left(\xi, \xi_1, \eta^*(\xi, \xi_1, \eta_1), \eta_1\right) + \left(\eta - \eta^*(\xi, \xi_1, \eta_1)\right)\frac{\partial \chi_1}{\partial \eta}(\xi, \xi_1, \bar{\eta}, \eta_1),$$

where $\bar{\eta} \in [\eta, \eta^*(\xi, \xi_1, \eta_1)]$. Therefore

$$\left|\chi_1(\xi, \xi_1, \eta, \eta_1)\right| = \left|\eta - \eta^*(\xi, \xi_1, \eta_1)\right|\left|\frac{2\xi_1(\bar{\eta}\xi_1 - \eta_1\xi)}{\xi\xi_1(\xi - \xi_1)}\right|.$$

Since $|\eta - \eta^*(\xi, \xi_1, \eta_1)| \leqslant |\eta - \eta_1| + |\eta_1 - \eta^*(\xi, \xi_1, \eta_1)| \leqslant CN^2$, it follows that

$$\begin{aligned}
\left|\chi_1(\xi, \xi_1, \eta, \eta_1)\right| &\lesssim |\xi_1|\left|\eta - \eta^*(\xi, \xi_1, \eta_1)\right|\left|\frac{(\bar{\eta} - \eta_1)\xi_1 - \eta_1(\xi - \xi_1)}{\xi\xi_1(\xi - \xi_1)}\right| \\
&\lesssim N^3\left(\frac{|(\bar{\eta} - \eta_1)\xi_1|}{|\xi\xi_1(\xi - \xi_1)|} + \frac{|\eta_1(\xi - \xi_1)|}{|\xi\xi_1(\xi - \xi_1)|}\right) \\
&\lesssim N^3\left(\frac{(\sqrt{3}+1)N^2}{N^2} + C\frac{N^2}{N^2}\right) \\
&\lesssim N^3,
\end{aligned}$$

by the relation of (7.13) it results that

$$\left|\chi(\xi, \xi_1, \eta, \eta_1)\right| \leqslant CN^3 + 6|\xi||\xi_1||\xi - \xi_1|.$$

Since one has $|\xi| \sim |\xi_1| \sim |\xi - \xi_1| \sim N$, it follows that

$$\left|\chi(\xi, \xi_1, \eta, \eta_1)\right| \leqslant CN^3.$$

Now, in the other case where $(\xi_1, \eta_1) \in k^2(\xi, \eta)$, i.e. $(\xi_1, \eta_1) \in D_{1,N}$ and $(\xi - \xi_1, \eta - \eta_1) \in D_{2,N}$, follows from the first case since we can write $(\xi_1, \eta_1) = (\xi - (\xi - \xi_1), \eta - (\eta - \eta_1)) \in D_{1,N}$ and that

$$\chi_1(\xi, \xi_1, \eta, \eta_1) = \chi_1(\xi, \xi - \xi_1, \eta, \eta - \eta_1).$$

This completes the proof of the lemma. □

Let us now end the proof of Theorem 2.2. Note that for any $\xi \in [3N/2, 3N]$ and $[(\sqrt{3}-6)N^2, (\sqrt{3}+7)N^2]$, we have $\operatorname{mes}(K(\xi,\eta)) \geqslant CN^3$. We recall that we have

$$\|u_{2,N}(t)\|_{H^{s,0}}^2 \geqslant CN^{-4s-6} \int_{3N/2}^{3N} \int_{(\sqrt{3}-6)N^2}^{(\sqrt{3}+7)N^2} |\xi|^2 \left(1+|\xi|^2\right)^s \times \left| \int_{k(\xi,\eta)} \frac{e^{-\xi^2 t}[e^{(-2\xi_1(\xi-\xi_1)+i\chi(\xi,\xi_1,\eta,\eta_1))t}-1]}{-2\xi_1(\xi-\xi_1)+i\chi(\xi,\xi_1,\eta,\eta_1)}\, d\xi_1\, d\eta_1 \right|^2 d\xi\, d\eta. \tag{7.14}$$

Now, we choose a sequence of times $(t_N)_N$ defined by

$$t_N = \frac{1}{N^{3+\epsilon_0}}, \quad 0 < \epsilon_0 \ll 1 \text{ (fixed)}.$$

For $N \gg 1$ it is clear

$$e^{-\xi^2 t_N} \sim e^{-N^2 t_N} \sim e^{-\frac{1}{N^{1+\epsilon_0}}} > C. \tag{7.15}$$

Moreover, by Lemma 7.1, it follows that $|-2\xi_1(\xi-\xi_1)+i\chi(\xi,\xi_1,\eta,\eta_1)| \leqslant N^2+N^3 \leqslant CN^3$. Hence

$$\begin{aligned} \left| \frac{e^{(-2\xi_1(\xi-\xi_1)+i\chi(\xi,\xi_1,\eta,\eta_1))t_N}-1}{-2\xi_1(\xi-\xi_1)+i\chi(\xi,\xi_1,\eta,\eta_1)} \right| &= |t_N| + O\left(|t_N|^2 N^3\right) \\ &= \frac{1}{N^{3+\epsilon_0}} + O\left(\frac{1}{N^{3+2\epsilon_0}}\right). \end{aligned} \tag{7.16}$$

By combining the relations (7.15), (7.16), we obtain

$$\begin{aligned} &\left| \int_{k(\xi,\eta)} \frac{e^{-\xi^2 t}[e^{(-2\xi_1(\xi-\xi_1)+i\chi(\xi,\xi_1,\eta,\eta_1))t_N}-1]}{-2\xi_1(\xi-\xi_1)+i\chi(\xi,\xi_1,\eta,\eta_1)}\, d\xi_1\, d\eta_1 \right| \\ &\quad \geqslant C \operatorname{mes}\left(k(\xi,\eta)\right) \times \frac{1}{N^{3+\epsilon_0}} \geqslant CN^{-\epsilon_0}. \end{aligned} \tag{7.17}$$

By virtue of (7.14), it results that

$$\begin{aligned} \|u_{2,N}(t_N)\|_{H^{s,0}}^2 &\geqslant CN^{-4s-6} \int_{3N/2}^{3N} \int_{(\sqrt{3}-6)N^2}^{(\sqrt{3}+7)N^2} |\xi|^2 \left(1+|\xi|^2\right)^s d\xi\, d\eta \times N^{-2\epsilon_0} \\ &\geqslant CN^{-6-4s} N^{2s} N^2 N^3 N^{-2\epsilon_0} \\ &\geqslant CN^{-1-2\epsilon_0-2s}, \end{aligned}$$

and hence

$$1 \sim \|\phi_N\|_{H^{s,0}}^2 \geqslant \|u_{2,N}(t_N)\|_{H^{s,0}} \geqslant N^{-1/2-\epsilon_0-s}.$$

This leads to a contradiction for $N \gg 1$, since we have $-1/2-\epsilon_0-s>0$ for $s<-1/2-\epsilon_0$. This completes the proof of Theorem 2.2.

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