Minimizing Nonconvex, Simple Integrals of Product Type

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Received September 13, 1999; revised February 7, 2000

We consider the problem of minimizing simple integrals of product type, i.e.

$$\min \left\{ \int_0^T g(x(t)) f(x'(t)) \, dt : x \in AC([0, T]), \ x(0) = x_0, \ x(T) = x_T \right\}, \quad (\mathcal{P})$$

where \( f: \mathbb{R} \to [0, \infty) \) is a possibly nonconvex, lower semicontinuous function with either superlinear or slow growth at infinity. Assuming that the relaxed problem \((\mathcal{P}^{**})\) obtained from \((\mathcal{P})\) by replacing \( f \) with its convex envelope \( f^{**} \) admits a solution, we prove attainment for \((\mathcal{P})\) for every continuous, positively bounded below the coefficient \( g \) such that (i) every point \( t \in \mathbb{R} \) is squeezed between two intervals where \( g \) is monotone and (ii) \( g \) has no strict local minima. This shows in particular that, for those \( f \) such that the relaxed problem \((\mathcal{P}^{**})\) has a solution, the class of coefficients \( g \) that yield existence to \((\mathcal{P})\) is dense in the space of continuous, positive functions on \( \mathbb{R} \). We discuss various instances of growth conditions on \( f \) that yield solutions to \((\mathcal{P}^{**})\) and we present examples that show that the hypotheses on \( g \) considered above for attainment are essentially sharp.

Key Words: nonconvex minimum problems; simple integrals; existence of solutions
This paper addresses the basic question of the calculus of variations, namely the existence of solutions, in the case of one dimensional, variational problems of the form

$$\min \left\{ \int_0^T g(x'(t)) \, dt : x \in AC([0, T]), \, x(0) = x_0, \, x(T) = x_T \right\}$$

where $f: \mathbb{R} \to [0, \infty]$ is a possibly nonconvex, lower semicontinuous function with either superlinear or slow growth at infinity and the coefficient $g: \mathbb{R} \to (0, \infty)$ is continuous and bounded away from zero.

The lack of convexity of $f$ affects the sequential lower semicontinuity of the integral with respect to the weak convergence in $AC([0, T])$ thus preventing the application of the direct method of the calculus of variations and the aim of this paper is to investigate which properties of the coefficient $g$ yield existence to $(\mathcal{P})$ regardless of the convexity of $f$.

Although the problem considered here has a comparatively simple structure, only a few attempts have been made to investigate the existence of solutions to $(\mathcal{P})$. With regard to these attempts, we mention [7] which considers autonomous integrals with a smooth Lagrangean $L(x(t), x'(t))$ that need not be product-like. The main assumption of the existence result of this paper reduces in this case to the requirement that $g$ be monotone on $\mathbb{R}$, thus ruling out those $g$ having strict local minima or maxima.

Although the Bolza-type example

$$\min \left\{ \int_0^T (1+|x(t)|)[1+(|x'(t)| - 1)^2] \, dt : x \in AC([0, T]), \, x(0) = x(T) = 0 \right\}$$

shows that solutions to $(\mathcal{P})$ are not expected to exist in general if $f$ fails to be convex and $g$ has strict local minima, there seems to be no evidence that finitely or even countably many strict maxima laying between intervals where $g$ is monotone might prevent $(\mathcal{P})$ from having a solution. In contrast, naive intuition suggests that a possible solution $x$ to $(\mathcal{P})$ should not spend a time of positive measure on any strict local maximum of $g$.

Still, the very same question considered here, namely the existence of solutions to $(\mathcal{P})$, has also been addressed recently in [5] from a somewhat different point of view: instead of looking for those continuous, positive, and bounded away from zero coefficients $g$ that yield solutions to $(\mathcal{P})$ under only the minimal requirements that $f$ be nonnegative, proper, lower semicontinuous and have superlinear growth at infinity, this paper...
investigates which properties of \( f \) other than convexity yield solutions to (\( \mathcal{P} \)) for all lower semicontinuous, positive, and bounded away from zero functions \( g \) or at least all but some exceptional ones. It may not be surprising—although it is positively not simple to prove—that the property that \( f \) be convex at zero, i.e. \( f^{**}(0) = f(0) \) is the answer. Indeed, assuming this and that \( f \) is an everywhere-finite, nonnegative, lower semicontinuous function with superlinear growth at infinity, [5] establishes the existence of solutions to (\( \mathcal{P} \)) for every lower semicontinuous, positive and bounded away from zero coefficient \( g \) whose level sets have negligible boundaries, and finally a forthcoming paper by A. Ornelas gets rid of this latter hypothesis, too.

As mentioned above, the aim of this paper is to contribute to the subject in the spirit of [7]. Indeed, we are going to prove (see Theorem 1.2) that, whenever \( f \) and \( g \) have the properties considered at the beginning of the paper and the relaxed problem (\( \mathcal{P}^{**} \)) obtained from (\( \mathcal{P} \)) by replacing \( f \) with its convex envelope \( f^{**} \), that is,

\[
\min_{x \in AC([0, T])} \int_0^T g(x(t)) f^{**}(x'(t)) \, dt : \begin{array}{l}
x(0) = x_0, \quad x(T) = x_T,
\end{array} \quad (\mathcal{P}^{**})
\]

admits a solution, the nonconvex problem (\( \mathcal{P} \)) has a solution too for every coefficient \( g \) such that

(a) every point of \( \mathbb{R} \) lies between two intervals where \( g \) is monotone;
(b) \( g \) has no strict local minima.

To be explicit, by (a) we mean that for every \( \zeta \in \mathbb{R} \) there is a \( \delta(\zeta) > 0 \) such that, on each interval \([\zeta - \delta(\zeta), \zeta]\) and \([\zeta, \zeta + \delta(\zeta)]\), \( g \) is either increasing or decreasing.

As simple model cases of functions \( g \) which the theorem applies to, consider

\[
g(\zeta) = 1 + \frac{1}{1 + \zeta^2}, \quad g(\zeta) = 1 + \max\{\sin \zeta, 0\}, \quad \zeta \in \mathbb{R},
\]

whereas \( g(\zeta) = 1 + \max\{\zeta \sin(1/\zeta), 0\} \) for \( \zeta > 0 \) and \( g(\zeta) = 1 \) for \( \zeta \leq 0 \) provides an example of a function satisfying (b) but failing (a). As a matter of curiosity, we remark also that, starting with the familiar Cantor–Vitali function on the unit interval, one can produce rather wild examples of continuous, periodic functions satisfying (a) and (b).
We refer the reader to Section 1 ahead for some comments on the hypotheses of Theorem 1.2. Here, we just mention that the class of all continuous, positive, and bounded away from zero functions \(g\) satisfying (a) and (b) is dense in the cone of continuous, positive functions on \(\mathbb{R}\) for the topology of uniform convergence on compact sets.

Finally, we remark that the existence result for the nonconvex problem (\(\mathcal{P}\)) we are going to prove is based on the assumption of attainment for the corresponding relaxed problem (\(\mathcal{P}^*\)), and thereby it can be applied to nonconvex problems featuring either superlinear or slow growth at infinity provided the associated relaxed problem admits a solution. Indeed, besides the standard case of functions \(f\) having superlinear growth at infinity (see Corollary 1.3) for which the existence of solutions for the corresponding relaxed problem (\(\mathcal{P}^*\)) follows immediately from the direct method of the calculus of variations, we consider also the case of functions \(f\) with slow growth at infinity (see Corollary 1.4) for which attainment for the relaxed problem (\(\mathcal{P}^*\)) can be obtained by applying the existence result of [2].

The remaining part of the paper is organized as follows. In the next section we introduce some notations, we recall some well-known preliminary results and we state the main result (Theorem 1.2) and prove its consequences (Corollaries 1.3 and 1.4). Finally, in the last section, Section 2, we give the proof of Theorem 1.2.

1. NOTATIONS AND STATEMENT OF THE MAIN RESULT

We begin by recalling some elementary definitions, notations, and results, mostly from convex analysis.

If \(A \subset \mathbb{R}^n\), we let \(\text{int}(A)\), \(\partial A\) and \(\text{co}(A)\) be the interior, the boundary and the convex hull of \(A\) respectively. If \(C \subset \mathbb{R}^n\) is a convex set, we denote its relative interior by \(\text{ri}(C)\) and we recall that the dimension of \(C\), denoted by \(\text{dim}(C)\), is the dimension of the affine space spanned by \(C\). We recall also that a convex subset \(F\) of \(C\) is said to be a face of \(C\) provided every closed line segment in \(C\) whose relative interior meets \(F\) has its endpoints in \(F\). The 0-dimensional faces of \(C\) are the extreme points of \(C\). Moreover, a face \(F\) of \(C\) is said to be proper if it is nonempty and different from \(C\) itself and, whenever \(C\) is also closed, its proper faces are closed subsets of \(\partial C\), and \(\partial C\) itself is the (disjoint) union of the relative interiors of the proper faces of \(C\) (see [8]).

Now, let \(f: \mathbb{R} \to [0, \infty]\) be a possibly extended-valued, nonnegative function. The epigraph of \(f\) is the subset of \(\mathbb{R} \times \mathbb{R}\) defined by

\[
\text{epi}(f) = \{(\xi, t) \in \mathbb{R} \times \mathbb{R} : f(\xi) \leq t\},
\]
and its projection on the first factor of $\mathbb{R} \times \mathbb{R}$, i.e.,

$$\text{dom}(f) = \{ \xi \in \mathbb{R} : f(\xi) < \infty \},$$

is the effective domain of $f$. We shall assume throughout the paper that $f: \mathbb{R} \to [0, \infty]$ is a proper and lower semicontinuous function. This means that $\text{epi}(f)$ is a nonempty, closed subset of $\mathbb{R} \times \mathbb{R}$.

Then, we recall the notion of subdifferentiability in the sense of convex analysis. We say that $f$ is subdifferentiable at a point $\xi \in \text{dom}(f)$ if there exists $d \in \mathbb{R}$ such that

$$f(\xi) \geq f(\xi) + d(\xi - \xi), \quad \xi \in \mathbb{R}. \quad (1.1)$$

Every such $d$ is a subgradient of $f$ at $\xi$ and the set of all such numbers $d$ is the subdifferential $\partial f(\xi)$ of $f$ at $\xi$. We extend the set-valued mapping $\partial f$ to whole $\mathbb{R}$ by setting $\partial f(\xi) = \emptyset$ whenever $\xi \notin \text{dom}(f)$ or no number $d$ satisfying (1.1) exists. Hence, $\partial f$ has closed, convex values. Moreover, whenever $f$ is also convex, $\partial f(\xi)$ is a nonempty, compact interval for every $\xi \in \text{int}(\text{dom}(f))$ and $f$ turns out to be locally Lipschitz continuous on $\text{int}(\text{dom}(f))$ so that $\partial f(\xi) = \{ f'(\xi) \}$ for almost every $\xi \in \text{int}(\text{dom}(f))$.

We recall also that the polar function of $f$ is the proper, lower semicontinuous, convex function $f^*: \mathbb{R} \to (-\infty, \infty]$ defined by

$$f^*(\xi) = \sup\{ \xi \xi - f(\xi) : \xi \in \mathbb{R} \}, \quad \xi \in \mathbb{R},$$

(see [4]) and that the bipolar function or convex envelope of $f$ is the polar $f^{**}: \mathbb{R} \to [0, \infty]$ of $f^*$. Thus, $f^{**}$ is a proper, lower semicontinuous, convex function and $f$ is convex if and only if $f^{**} = f$. However, this process of duality cannot be further iterated as the polar of the bipolar is the polar itself, i.e., $f^{***} = f^*$. Among the properties of $f^{**}$ that hold in this special one-dimensional setting, we recall that $\text{dom}(f^{**}) = \text{co}(\text{dom}(f))$, that $\text{epi}(f^{**})$ turns out to be the closure of the convex hull of $\text{epi}(f)$ (see [4]), and that the extreme points of $\text{epi}(f^{**})$ are contained in the boundary of $\text{epi}(f)$, namely $\xi \in \text{dom}(f)$ and $f(\xi) = t = f^{**}(\xi)$ whenever $f(\xi)$ is an extreme point of $\text{epi}(f^{**})$. We remark also that, $\text{epi}(f^{**})$ being a nonempty, closed, convex subset of $\mathbb{R} \times \mathbb{R}$, its proper faces are closed subsets of the boundary of $\text{epi}(f^{**})$ which are either extreme points or one-dimensional. Thus, these latter ones are (at most) countably many and any such face is said to be vertical if its projection on the first factor of $\mathbb{R} \times \mathbb{R}$ is a singleton. Note that $\text{epi}(f^{**})$ has at most two vertical faces. Note also that the projection $F'$ of a one-dimensional, nonvertical face $F$ of $\text{epi}(f^{**})$ is a non degenerate, closed interval whose interior is the projection of the relative interior of $F$ and that $f^{**}$ is affine on the projection of every such face of $\text{epi}(f^{**})$. Hence, $f^{**}$ is differentiable and $\partial f^{**}$ is single-valued and
constant on the interior of $F'$. Moreover, the set $\{f** < f\}$ is open and each of its connected components is contained in the projection of the relative interior of some one-dimensional, nonvertical face of $\text{epi}(f**)$.

Next, we recall that $f**$ is said to be strictly convex at infinity if the graph of $f**$ contains no nonvertical rays. In this case, this simply means that $f**$ is not eventually affine as $\xi \to -\infty$ and $\xi \to \infty$. It is plain that, whenever $f**$ enjoys this property, every nonvertical, proper face of $\text{epi}(f**)$ is compact.

Finally, we recall a very weak growth condition on $f**$ (see [1] and [3]) that will play a preminent rôle in the following. To this purpose, recall that whenever $d \in \mathbb{R}$ is a subgradient of $f**$ at some point $\xi \in \text{dom}(f**)$ the values of $f**(\xi)$ and $f^*(d)$ are related by

$$f**(\xi) + f^*(d) = d\xi$$

(see [4]) because of the equality $f*** = f^*$. Hence, writing (1.1) with $f**$ instead of $f$, it follows that the value at the origin of the supporting affine function at $f**$ through the point $(\xi, f**(\xi))$, i.e., $\zeta \in \mathbb{R} \to f**(\zeta) + d(\zeta - \xi)$, is given by $-f^*(d)$. Thereby, $f**$ satisfies

$$f**(\zeta) \geq d\zeta - f^*(d), \quad \zeta \in \mathbb{R}, \quad (1.2)$$

for every $d \in \partial f**(\zeta)$ and $\xi \in \text{dom}(f**)$, and equality holds for $\xi = \zeta$. In the following, we are interested in those proper, lower semicontinuous functions $f: \mathbb{R} \to [0, \infty]$ whose convex envelope $f**$ is subdifferentiable at every point of its effective domain and which have the further property that the mapping $Ef**: \mathbb{R} \to (-\infty, \infty]$ defined by

$$Ef**(\xi) = \{\sup \{-f^*(d) : d \in \partial f**(\zeta)\} \quad \xi \in \text{dom}(f**), \quad \zeta \notin \text{dom}(f**),$$

satisfies

$$Ef**(\xi) \to -\infty \quad \text{as} \quad |\xi| \to \infty. \quad (1.3)$$

This growth condition on $f$ is strictly weaker than superlinearity at infinity, i.e., the requirement that $f(\xi)/|\xi| \to \infty$ as $|\xi| \to \infty$. Indeed, (1.3) holds for every superlinear at infinity function $f$, whereas

$$f(\xi) = |\xi| - \log(1 + |\xi|), \quad \xi \in \mathbb{R},$$

provides a simple example of a convex function satisfying (1.3) and having linear growth at infinity. The properties of the class of functions satisfying (1.3) that we are interested in are gathered in the following proposition whose proof is self evident.
Proposition 1.1. Let $f: \mathbb{R} \to [0, \infty]$ be a proper and lower semicontinuous function such that $\partial f^{**}(\xi) \neq \emptyset$ for every $\xi \in \text{dom}(f^{**})$ and $\text{Ef}^{**}(\xi) \to -\infty$ as $|\zeta| \to \infty$. Then, 

(a) $f^{**}$ is strictly convex at infinity;
(b) $f^{**}(\xi) \geq c_0 |\xi| - c_1$ for every $\xi \in \mathbb{R}$ for some constants $c_0 > 0$ and $c_1 \geq 0$.

As for measure and functional theoretic notations, we denote the Lebesgue measure of a measurable subset $E$ of $\mathbb{R}$ by $|E|$. We recall that a point $t \in \mathbb{R}$ is a density point for $E$ if

$$\lim_{\varepsilon \to 0_+} \frac{1}{2\varepsilon} |E \cap (t-\varepsilon, t+\varepsilon)| = 1,$$

and that almost every point in $E$ has this property. Throughout the paper, we let $T$ be a positive number and we use standard notations for the Lebesgue space of integrable functions on $[0, T]$ and its norm. We recall also that, whenever $x \in L^1([0, T])$, a point $t_0 \in (0, T)$ is a Lebesgue point for $x$ if

$$\lim_{\varepsilon \to 0_+} \frac{1}{2\varepsilon} \int_{t_0-\varepsilon}^{t_0+\varepsilon} |x(t) - x(t_0)| \, dt = 0.$$

Lebesgue's differentiation theorem shows that almost every point $t \in (0, T)$ is a Lebesgue point for $x$. Moreover, we write $AC([0, T])$ for the space of absolutely continuous functions on $[0, T]$ which turns out to be a Banach space with respect to the Sobolev norm

$$\|x\|_{1,1} = \int_0^T \left[ |x(t)| + |x'(t)| \right] \, dt, \quad x \in AC([0, T]).$$

Now, we introduce the class of functionals we are going to consider in the following. For a proper, lower semicontinuous function $f: \mathbb{R} \to [0, \infty]$ and for a continuous function $g: \mathbb{R} \to (0, \infty)$, we consider the integral functional

$$I(x) = \int_0^T g(x(t)) f'(x'(t)) \, dt, \quad x \in AC([0, T]),$$

and the associated minimum problem

$$\min \{ I(x) : x \in AC([0, T]) \text{ with } x(0) = x_0 \text{ and } x(T) = x_T \} \quad (\mathcal{P})$$
with $x_0, x_T \in \mathbb{R}$. We consider also the functional
\[ I^{**}(x) = \int_0^T g(x(t)) f^{**}(x'(t)) \, dt, \quad x \in AC([0, T]), \]
and the associated minimum problem
\[ \min \{ I^{**}(x) : x \in AC([0, T]) \text{ with } x(0) = x_0 \text{ and } x(T) = x_T \}, \quad (\mathscr{P}^{**}) \]
which we loosely refer to as the relaxed functional and the relaxed minimum problem respectively. It is plain that $I^{**} \leq I$ on $AC([0, T])$ so that any solution $x$ to $(\mathscr{P}^{**})$ satisfying $f^{**}(x') = f(x')$ almost everywhere on $[0, T]$ is a solution to $(\mathscr{P})$ as well. Moreover, $I^{**}$ is sequentially weakly lower semicontinuous on the set of competing functions \( \{ x \in AC([0, T]) : x(0) = x_0 \text{ and } x(T) = x_T \} \).

After these preliminaries, we can state the main result of the paper. Roughly speaking, it proves that, under mild assumptions on $f$ and $g$, the existence of a solution to the relaxed problem $(\mathscr{P}^{**})$ implies the existence of a solution to the nonconvex problem $(\mathscr{P})$. As such, its scope of applicability essentially depends on the availability of existence results for $(\mathscr{P}^{**})$, and we shall discuss below (see Corollaries 1.3 and 1.4) two instances of growth conditions on $f$ that yield solutions to $(\mathscr{P}^{**})$.

**Theorem 1.2.** Let $f : \mathbb{R} \to [0, \infty]$ be a proper and lower semicontinuous function such that
\begin{enumerate}
\item[(1.4)] $f^{**}$ is strictly convex at infinity;
\item[(1.5)] $f^{**}(\xi) \geq c_0 |\xi| - c_1$ for every $\xi \in \mathbb{R}$ for some constants $c_0 > 0$ and $c_1 \geq 0$;
\end{enumerate}
and let $g : \mathbb{R} \to (0, \infty)$ be a continuous function such that
\begin{enumerate}
\item[(1.6)] for every $\zeta \in \mathbb{R}$, there is $\delta(\zeta) > 0$ such that $g$ is monotone on each interval $[\zeta - \delta(\zeta), \zeta]$ and $[\zeta, \zeta + \delta(\zeta)]$;
\item[(1.7)] $g$ has no strict local minima.
\end{enumerate}
Assume also that $(\mathscr{P}^{**})$ has a solution. Then, $(\mathscr{P})$ has a solution too.

As to the role played by the various hypotheses in the theorem above, we recall that (1.4) implies that the projections of the one-dimensional, nonvertical faces of $\text{epi}(f^{**})$ are compact intervals. Thus, whenever $z$ is a solution to $(\mathscr{P}^{**})$ such that $f^{**}(z(t_0)) < f(z(t_0))$ for some point $t_0 \in (0, T)$, $z(t_0)$ can be written as a convex combination of values $y$ and
where $f^{**}$ and $f$ coincide. This, together with the qualitative behavior of $g$ described by (1.6) and (1.7), allows us to modify $z$ around $t_0$ so as to define a new solution $y$ to ($\mathcal{P}^{**}$) whose derivative is either $x$ or $\beta$ almost everywhere on the set where $y$ is different from $z$ itself. Finally, the remaining hypothesis (1.5) will be used to glue some of these solutions $y$ so as to find a further solution $x$ to ($\mathcal{P}^{**}$) such that $f^{**}(x') = f(x')$ almost everywhere on $[0, T]$, thus proving attainment for ($\mathcal{P}$). As regards the hypotheses (1.6) and (1.7) that identify the class of coefficients $g$ that yield attainment for ($\mathcal{P}$) whenever ($\mathcal{P}^{**}$) has a solution, they have different status. Indeed, (1.7) cannot be dropped, otherwise the theorem may fail as shown by the example presented in the Introduction. In contrast, (1.6) is connected with the technique of the proof and rules out the possibility that $g$ oscillates too wildly around some point.

It is easy to check that the class of continuous functions satisfying (1.6) consists of all functions $g$ that admit a locally finite covering of $\mathbb{R}$ consisting of nondegenerate, nonoverlapping, closed intervals where $g$ is monotone. For any such $g$, the sets of strict local minima and maxima are discrete and hence closed subsets of $\mathbb{R}$, and, in view of (1.7), we agree to write

$$M_g = \{ \xi \in \mathbb{R} : \xi \text{ is a strict local maximum of } g \}$$  

whenever $g$ satisfies (1.6) and (1.7). Moreover, every such $g$ is locally monotone on $\mathbb{R} \setminus M_g$ and we point out that the class of all continuous functions on $\mathbb{R}$ satisfying (1.6) and (1.7) which are also bounded below by positive constants is easily seen to be dense in the cone of positive functions in $C(\mathbb{R})$ endowed with the usual Fréchet space structure. Thus, assuming $f$ is such that ($\mathcal{P}^{**}$) admits a solution, Theorem 1.2 can be viewed also as a result ensuring the existence of a dense class of continuous, positive coefficients $g$ that yield existence to ($\mathcal{P}$).

Finally, we end this section by presenting two instances of growth hypotheses on $f$ ensuring the existence of solutions to ($\mathcal{P}^{**}$) and hence to ($\mathcal{P}$) by Theorem 2.2. The first one is the familiar case of functions $f$ having superlinear growth at infinity, whereas the second, a simple application of the existence result of [2], applies to problems featuring functions $f$ with slow growth at infinity. We wish to remark that both results apply to non-convex problems featuring one-sided constraints on the derivative, like $x' \geq 0$ or $x' > 0$ a.e. on $[0, T]$.

**Corollary 1.3.** Let $f : \mathbb{R} \to [0, \infty)$ be a proper and lower semicontinuous function such that

$$f(\xi)/|\xi| \to \infty \quad \text{as} \quad |\xi| \to \infty;$$  

(1.9)
and let \( g: \mathbb{R} \to (0, \infty) \) be a continuous function such that

\[
g(\zeta) \geq g_0 \quad \text{for every} \quad \zeta \in \mathbb{R} \quad \text{for some constant} \quad g_0 > 0;
\]

(1.10)

and (1.6) and (1.7) hold. Then, \((\mathcal{P})\) has a solution for every pair of boundary data \( x_0, x_T \in \mathbb{R} \).

**Proof.** See Theorem 2.2 on p. 250 in [4].

**Corollary 1.4.** Let \( f: \mathbb{R} \to [0, \infty] \) be a proper and lower semicontinuous function such that

\[
\text{dom}(f^{**}) \text{ is a cone; (1.11)}
\]

\[
\partial f^{**}(\zeta) \neq \emptyset \quad \text{for every} \quad \zeta \in \text{dom}(f^{**}); \quad (1.12)
\]

\[
Ef^{**}(\zeta) \to -\infty \quad \text{as} \quad ||\zeta|| \to \infty; \quad (1.13)
\]

and let \( g: \mathbb{R} \to (0, \infty) \) be a continuous function such that (1.10), (1.6), and (1.7) hold. Then, \((\mathcal{P})\) has a solution for every pair of boundary data \( x_0, x_T \in \mathbb{R} \).

Recall that a cone in \( \mathbb{R} \) is either \( \mathbb{R} \) itself or any open or closed half line starting at zero.

**Proof of Corollary 1.4.** We are going to prove that, whenever \( I^{**} \) is not identically \( \infty \) on the set of feasible functions \( \mathcal{A} = \{ x \in AC([0, T]) : x(0) = x_0 \text{ and } x(T) = x_T \} \), a minimum problem equivalent to the relaxed problem \((\mathcal{P}^{**})\) satisfies the hypotheses of the existence result of [2]. Thereby, the relaxed problem \((\mathcal{P}^{**})\) admits a solution and the conclusion follows from Proposition 1.1 and Theorem 1.2.

Therefore, assume that \( I^{**} \) attains a finite value on \( \mathcal{A} \) and note that, due to 1.11, the unique, feasible, affine function \( \bar{x}(t) = x_0 + (x_T - x_0) t / T \), \( 0 \leq t \leq T \), yields a finite value to \( I^{**} \). Let \( I(\bar{x}) \) be the corresponding sublevel set of \( I^{**} \), i.e., \( I(\bar{x}) = \{ x \in \mathcal{A} : I^{**}(x) \leq I^{**}(\bar{x}) \} \), so that the minimum problem \( \min \{ I^{**}(x) : x \in I(\bar{x}) \} \) is obviously equivalent to \((\mathcal{P}^{**})\). Moreover, the hypotheses (1.12) and (1.13) together with Proposition 1.1 imply that every \( x \in I(\bar{x}) \) satisfies \( |x(t)| \leq M \) for every \( 0 \leq t \leq T \) with

\[
M = \min \{ |x_0|, |x_T| \} + \frac{1}{g_0 c_0} \left[ I^{**}(\bar{x}) + g_0 c_1 T \right]
\]

(this estimate can be improved if \( \text{dom}(f^{**}) \) is a half line) whence the equivalence of \((\mathcal{P}^{**})\) and

\[
\min \{ I^{**}(x) : x \in I(\bar{x}) \text{ and } |x(t)| \leq M \text{ for } 0 \leq t \leq T \} \quad (\mathcal{P}^{**})
\]

follows.
Now, we check that Theorem 2 of [2] applies to $(\mathcal{P}_{\alpha_0}^{**})$. Indeed, the validity of the basic hypotheses is obvious and the principal hypotheses require the existence of some $\varepsilon > 0$ such that

$$\text{ess inf}\{ |x'(t)| : 0 \leq t \leq T \} < \varepsilon,$$  \hspace{1cm} (1.14)

$$\lim_{\rho \to \infty} \sup_{\mathcal{P}} \{ g(\zeta)[f^{**}(\zeta) - d_\varepsilon] : |\zeta| \leq M, \zeta \in \text{dom}(f^{**}), |\zeta| > \rho, d \in \partial f^{**}(\zeta) \}$$

$$< \inf_{\mathcal{P}} \{ g(\zeta)[f^{**}(\zeta) - d_\varepsilon] : |\zeta| \leq M, \zeta \in \text{dom}(f^{**}), |\zeta| < \varepsilon, d \in \partial f^{**}(\zeta) \}.$$  \hspace{1cm} (1.15)

A routine check shows that (1.14) holds with

$$\varepsilon > \frac{1}{g_0 c_0} [I^{**}(\bar{x}) + g_0 c_1 T]$$

because of (1.12), (1.13), and Proposition 1.1. Again, a smaller value of $\varepsilon$ can be found when $\text{dom}(f^{**})$ is a half line.

As regards (1.15), we claim that its left hand side is $-\infty$. Indeed, $Ef^{**}$ is negative for every large enough $\zeta$ in $\text{dom}(f^{**})$ because of (1.12) and (1.13), whence

$$\sup_{\mathcal{P}} \{ g(\zeta)[f^{**}(\zeta) - d_\varepsilon] : |\zeta| \leq M, \zeta \in \text{dom}(f^{**}), |\zeta| > \rho, d \in \partial f^{**}(\zeta) \}$$

$$< g_0 \sup_{\mathcal{P}} \{ Ef^{**}(\zeta) : \zeta \in \text{dom}(f^{**}), |\zeta| > \rho \}$$

follows for every large enough $\rho$. As the right-hand side goes to $-\infty$ as $\rho \to \infty$ by (1.13), the claim is proved.

Thus, it is enough to prove that the right-hand side of (1.15) is finite and we break the remaining part of the proof according to the possible shape of $\text{dom}(f^{**})$.

Indeed, if $\text{dom}(f^{**}) = \mathbb{R}$, it is easy to check that

$$\inf_{\mathcal{P}} \{ g(\zeta)[f^{**}(\zeta) - d_\varepsilon] : |\zeta| \leq M, \zeta \in \text{dom}(f^{**}), |\zeta| < \varepsilon, d \in \partial f^{**}(\zeta) \}$$

$$> \inf_{\mathcal{P}} \{ g(\zeta) \min\{-f^{*}(d_1), -f^{*}(d_2)\} : |\zeta| \leq M \}.$$  \hspace{1cm} (1.16)

where $d_1 = \min \partial f^{**}(-\varepsilon)$ and $d_2 = \max \partial f^{**}(\varepsilon)$. If $\text{dom}(f^{**})$ is either $[0, \infty)$ or $(0, \infty)$, the same kind of reasoning yields
\[ \inf \{ g(\xi)[f^{**}(\xi) - d_2^\xi] : |\xi| < M, \xi \in \text{dom}(f^{**}), |\xi| < \varepsilon, d \in \partial f^{**}(\xi) \} \geq \inf \{ g(\xi)[ -f^*(d_2)] : |\xi| < M \}, \]

where \( d_2 \) is defined as before and the remaining cases \( \text{dom}(f^{**}) = (-\infty,0] \) or \( (-\infty,0) \) are analogous. In all cases, the right-hand side is finite and this establishes (1.15).

Therefore, Theorem 2 in [2] implies that \( (\mathcal{P}_{eq}^{**}) \) and hence \( (\mathcal{P}^{**}) \) admit a solution, and this completes the proof. \[ \square \]

2. PROOF OF THE MAIN RESULT

The aim of this section is to prove Theorem 1.2. The proof is based on considering a solution \( z \) to the relaxed problem \( (\mathcal{P}^{**}) \) and on associating with it a family of comparison functions that are still solutions to \( (\mathcal{P}^{**}) \) and whose derivatives belong to the set where \( f^{**} = f \) almost everywhere on the set where they are different from \( z \) itself. Then, a covering argument allows us to select and glue some of these comparison functions in order to find a new solution \( x \) to \( (\mathcal{P}^{**}) \) satisfying \( f^{**}(x) = f(x) \) almost everywhere on \([0,T] \), thus proving attainment for \( (\mathcal{P}) \). The construction of the comparison functions is described in the following lemma.

**Lemma 3.1.** Let \( z \in AC([0,T]) \) and assume \( E \) is a measurable subset of \((0,T)\) such that

\[ E \subseteq \{ t \in (0,T) : z \text{ is differentiable at } t \text{ and } \alpha < z'(t) < \beta \} \]

and \( |E| > 0 \) for some \(-\infty < \alpha < \beta < \infty\). Then, for every \( s \in E \) such that \( s \) is a density point for \( E \),

\[ (2.1) \]

and for every \( \delta = \delta(s) > 0 \), there exist \( \varepsilon_0 = \varepsilon_0(s) > 0 \), two families of compact subintervals \( \{ K^+_{\delta,s} : 0 < \varepsilon \leq \varepsilon_0 \} \) of \((0,T)\), and two families of functions \( \{ z^+_{\delta,s} : 0 < \varepsilon \leq \varepsilon_0 \} \) in \( AC([0,T]) \) such that, setting

\[ J^+_{\kappa,s} = \left( s - \frac{\varepsilon}{\beta - z'(s)}, s + \frac{\varepsilon}{z'(s) - \alpha} \right), \quad J^-_{\kappa,s} = \left( s - \frac{\varepsilon}{\beta - z'(s)}, s + \frac{\varepsilon}{z'(s) - \alpha} \right), \]

(2.2)

for every \( \varepsilon > 0 \), the properties
\[ J_{s,T}^+ \subset K_{s,s}^+ \subset J_{s,s}^+ \subset (0, T); \]  
\[ z_{s,s}^+ = z \quad \text{on} \quad [0, T] \cap \text{int}(K_{s,s}^+); \]  
\[ z(t) < z_{s,s}^+(t) < z(t) + \delta \quad \text{for every} \quad t \in \text{int}(K_{s,s}^+); \]  
\[ z(t) - \delta < z_{s,s}^-(t) < z(t) \quad \text{for every} \quad t \in \text{int}(K_{s,s}^-); \]  
\[ \varepsilon \geq z_{s,s}^+(t) - [z(s) + z'(s)(t-s)] \geq \varepsilon/2 \quad \text{for every} \quad t \in J_{s,s}^+; \]  
\[ -\varepsilon/2 \geq z_{s,s}^-(t) - [z(s) + z'(s)(t-s)] \geq -\varepsilon \quad \text{for every} \quad t \in J_{s,s}^-; \]  
\[ (z_{s,s}^+(t))' \in \{ \alpha, \beta \} \quad \text{for a.e.} \quad t \in K_{s,s}^+; \]  
\[ \int_{K_{s,s}^+} h(z_{s,s}^+(t))(z_{s,s}^+(t))' \, dt = \int_{K_{s,s}^+} h(z(t))(z'(t)) \, dt, \quad h \in \mathcal{C}(\mathbb{R}), \]  
(2.3)  
(2.4)  
(2.5)  
(2.6)  
(2.7)  
(2.8)  
(2.9)  
(2.10)  
(2.11)  
(2.12)
As all points of the connected components containing $s$ are contained in $(0,\infty)$ and $a_s^-(t)$ is negative on $J_s^-$ and they both vanish at the endpoints of $J_s^+$ and $J_s^-$ respectively. Moreover, $(a_s^+)'(t) \in \{ x - z'(s), \beta - z'(s) \}$ for every $t \neq s$ and $e > 0$.

Then, we choose $e_0 = \varepsilon_0(s)$ such that $0 < \varepsilon_0 < \min \{ \rho/2p, \delta/2 \}$ and

$$ 0 < \varepsilon \leq \varepsilon_0 \Rightarrow 0 \leq \eta(e) \leq 1/4, \quad (2.12) $$

and we check that

$$
\begin{align*}
\begin{cases}
    z(t) > [z(s) + z'(s)(t-s)] + a_s^+(t) & t \in \partial J_{s+}^+, \\
    z(t) < [z(s) + z'(s)(t-s)] + a_s^-(t) & t \in \partial J_{s-}^-, \quad (2.13+) \\
    z(t) \geq [z(s) + z'(s)(t-s)] + a_s^+(t) & t \in \partial J_{s+}^+, \quad (2.13-) \\
    z(t) \leq [z(s) + z'(s)(t-s)] + a_s^-(t) & t \in \partial J_{s-}^-, \\
\end{cases}
\end{align*}
$$

for every $0 < \varepsilon \leq \varepsilon_0$. As to $(2.13^+)$, let $0 < \varepsilon \leq \varepsilon_0$ and note that $a_s^+(t)$ takes the value $-\varepsilon$ at the endpoints of $J_{s+}^+$ and that $a_s^+(t) \geq \varepsilon/2$ for every $t \in J_{s+}^+$.

As all points of $J_{s+}^+$ and $J_{s+}^{2e}$ are within $2p\varepsilon$ from $s$, the definition of $\eta$ and (2.12) yield

$$
\begin{align*}
    z(t) - [z(s) + z'(s)(t-s)] - a_s^+(t) \geq -\eta(e) + \varepsilon \geq 3\varepsilon/4 & t \in \partial J_{s+}^+, \\
    z(t) - [z(s) + z'(s)(t-s)] + a_s^+(t) \leq \eta(e) - \varepsilon/2 \leq -\varepsilon/4 & t \in J_{s+}^+. \\
\end{align*}
$$

The same kind of computation yields $(2.13^-)$.

Now, for $0 < \varepsilon \leq \varepsilon_0$, we define $K_{s+}^\pm$ and $K_{s-}^\pm$ to be the closures of the connected components containing $s$ of the open sets $\{ t \in (0, T) : z(t) < [z(s) + z'(s)(t-s)] + a_s^+(t) \}$ and $\{ t \in (0, T) : z(t) > [z(s) + z'(s)(t-s)] + a_s^-(t) \}$ respectively. By (2.13), all intervals $K_{s+}^\pm$ are compact neighborhoods of $s$ contained in $(0, T)$ which satisfy (2.3). The corresponding functions

$$
\begin{align*}
    z_{s+}^\pm(t) = \begin{bmatrix} [z(s) + z'(s)(t-s)] + a_s^+(t) \\ z(t) \end{bmatrix} & t \in K_{s+}^\pm, \\
    \quad & t \in [0, T) \backslash K_{s+}^\pm.
\end{align*}
$$

Hence, $\eta(e) \to 0$ as $e \to 0^+$. Consider also the piecewise affine functions $a_{\varepsilon^+, \varepsilon}^\pm, \varepsilon > 0$, defined by

$$
\begin{align*}
    a_{\varepsilon^+, \varepsilon}^+(t) &= \begin{bmatrix} [\beta - z'(s)](t-s) + \varepsilon \\ [\alpha - z'(s)](t-s) + \varepsilon \end{bmatrix} & t \leq s, \\
    & \begin{bmatrix} [\alpha - z'(s)](t-s) + \varepsilon \\ [\beta - z'(s)](t-s) - \varepsilon \end{bmatrix} & t \geq s, \\
    a_{\varepsilon^-, \varepsilon}^-(t) &= \begin{bmatrix} [\alpha - z'(s)](t-s) - \varepsilon \\ [\beta - z'(s)](t-s) - \varepsilon \end{bmatrix} & t \leq s, \\
    & \begin{bmatrix} [\beta - z'(s)](t-s) - \varepsilon \\ [\alpha - z'(s)](t-s) + \varepsilon \end{bmatrix} & t \geq s.
\end{align*}
$$
The membership of all functions \( z_{\alpha_k}^\pm \) in \( AC([0, T]) \) as well as the validity of (2.4), (2.7), and (2.8) are obvious, so we are left with check (2.5\( ^+ \)), (2.5\( ^- \)), and (2.6\( ^+ \)), (2.6\( ^- \)).

As for (2.5\( ^+ \)), note that \( |a_{\alpha_k, \epsilon}^+(t)| \leq \epsilon \) for every \( t \in J_{\alpha_k}^+ \), and every \( \epsilon > 0 \). Hence, we have
\[
0 < z_{\alpha_k}^+(t) - z(t) \leq |z(t) - [z(s) + z'(s)(t-s)]| + |a_{\alpha_k}^+(t)| < \delta/2 + \epsilon \leq \delta
\]
for every \( t \in \text{int}(K_{\alpha_k}^+) \) and \( 0 < \epsilon \leq \epsilon_0 \) because of (2.3), (2.10), and the choice of \( \epsilon_0 \). A specular argument proves (2.5\( ^- \)). At last, the definition of the functions \( z_{\alpha_k}^\pm \) and (2.3) show that \( z_{\alpha_k}^\pm(t) - [z(s) + z'(s)(t-s)] = a_{\alpha_k}^\pm(t) \) for every \( t \in J_{\alpha_k}^\pm \). Thus, (2.6\( ^+ \)) and (2.6\( ^- \)) follow immediately from the very definitions of \( a_{\alpha_k}^\pm \) and \( J_{\alpha_k}^\pm \) and this completes the proof. 

We can now prove Theorem 1.2.

**Proof of Theorem 1.2.** We begin by noting that, because of (1.4), the open set \( \{f^{**} < f\} \) is covered by (at most) countably many, nonempty, open, bounded, and pairwise disjoint intervals, say \( \{{\alpha_k, \beta_k}\}_k \), such that
\[
\begin{align*}
f^{**}(\xi) &= q_k + d_k \xi, \quad \alpha_k \leq \xi \leq \beta_k, \\
f^{**}(\xi) &= f(\xi), \quad \xi \in \{\alpha_k, \beta_k\},
\end{align*}
\]
(2.14)
where \( \{d_k\} = \partial f^{**}(\xi) \) for every \( \alpha_k < \xi < \beta_k \) and \( q_k = -f^*(d_k) \). Moreover, (1.6) and (1.7) imply that the set defined by (1.8) is discrete, and hence closed, and that \( g \) is locally monotone on \( R \setminus M_g \). Now, let \( z \in AC([0, T]) \) be a solution to (\( \mathcal{P}^{**} \)), and assume that \( I^{**}(z) < \infty \); otherwise there is nothing else to prove. We are going to prove that \( z \) can be modified so as to find a new solution \( x \) to (\( \mathcal{P}^{**} \)) such that
\[
f^{**}(x'(t)) = f(x'(t)) \quad \text{for a.e.} \quad t \in [0, T],
\]
(2.15)
thus showing that \( x \) is a solution to (\( \mathcal{P} \)) as well. The proof goes through the following three steps.

**Step 1.** First, we prove that, whenever some set
\[
E_k = \{ t \in (0, T) : \alpha_k < z'(t) < \beta_k \text{ and } z(t) \in R \setminus M_g \}
\]
(2.16)
has positive measure, we can use Lemma 2.1 to associate with almost every point \( s \in E_k \) a family of new solutions \( \{z_{\alpha_k, \epsilon}^\pm : 0 < \epsilon \leq \epsilon_0(s)\} \) to (\( \mathcal{P}^{**} \)) such that the sets \( K_{\alpha_k} \) defined as the closures of \( \{z_{\alpha_k, \epsilon}^\pm \} \) for every \( 0 < \epsilon \leq \epsilon_0(s) \)
are compact, nondegenerate intervals around $s$ that shrink nicely to $\{s\}$ itself as $\varepsilon \to 0_+$ and are such that the properties
\[
\begin{align*}
  z(t), z_+, z_-(t) &\in \mathbb{R}\setminus M_g & \text{for every } t \in K_{s,\varepsilon}; \\
  \sup\{|z_+(t) - z(t)| : 0 \leq t \leq T\} &\leq \delta; \\
  f^\star(\{z_+(t)\}) &\neq f(z(\varepsilon)) & \text{for } t \in K_{s,\varepsilon};
\end{align*}
\] (2.17a)
hold for every $0 < \varepsilon \leq \varepsilon_0(s)$ where $\delta = \delta(s) > 0$ is such that $g$ is monotone on the interval $[z(s) - 2\delta, z(s) + 2\delta]$.

**Step 2.** Then we use the modified solutions of the previous step and a covering argument to define a new solution $y$ to ({$P^\star$}) such that
\[
f^\star(y(t)) = f(y(t)) \text{ for a.e. } t \in y^{-1}(\mathbb{R}\setminus M_g). \quad (2.18)
\] (2.17b)

**Step 3.** Finally, we show that $y$ can once more be modified so as to find a new solution $x$ to ($P^\star$) satisfying (2.15).

**Proof of Step 1.** Assume that some set $E_k$ defined by (2.16) has positive measure and, to simplify the notations, drop the index $k$ everywhere, i.e. write $E = E_k$, $\alpha = \alpha_k$, and so on. Note in particular that (2.14) reduces to $f^\star(\xi) = q + \delta\xi$ for $\xi \in \alpha, \beta$ and $f^\star(\xi) = f(\xi)$ for $\xi \in \{\alpha, \beta\}$. Then, choose $s \in E$ such that
\[
\begin{align*}
  s &\text{ is a density point for } E; \\
  s &\text{ is a Lebesgue point for } z_i & f^\star &\text{ for } i \in \{\alpha, \beta\}. \quad (2.19)
\end{align*}
\] (2.20)

Note that almost every point $s \in E$ satisfies (2.19) and (2.20) and that this latter condition together with 3.7 and the very definition of $E$ implies that
\[
\begin{align*}
  \frac{1}{2\varepsilon} \int_{s-\varepsilon}^{s+\varepsilon} |f^\star(z'(t)) - [q + 2\varepsilon z'(t)]| \, dt &\to 0 \quad \text{as } \varepsilon \to 0_+. \quad (2.21)
\end{align*}
\] (2.16)

Moreover, recalling (1.6), let $\delta = \delta(s) > 0$ such that $g$ is monotone on the interval $[z(s) - 2\delta, z(s) + 2\delta]$.

Then, apply Lemma 2.1 to $z$, $s \in E$, and $\delta$ be as above, and thereby let $\{z^\pm_{s,\varepsilon} : 0 < \varepsilon \leq \varepsilon_0(s)\}$ and $\{K^\pm_{s,\varepsilon} : 0 < \varepsilon \leq \varepsilon_0(s)\}$ be the corresponding functions and intervals. We are going to choose the functions $z^\pm_{s,\varepsilon}$ and the intervals $K^\pm_{s,\varepsilon}$ among $z^\pm_{s,\varepsilon}$ and $K^\pm_{s,\varepsilon}$. For this purpose, recall that each interval $K^\pm_{s,\varepsilon}$ is the closure of $\{z_{s,\varepsilon} \neq z\}$ and note that, by (2.3) and (2.2), we can choose $\varepsilon_0 = \varepsilon_0(s) > 0$ small enough to have $|z(t) - z(s)| \leq \delta$ for every $t \in K^\pm_{s,\varepsilon}$ and $0 < \varepsilon \leq \varepsilon_0$. Hence, $|z_{s,\varepsilon}(t) - z(s)| \leq 2\delta$ for the same $t$ and $\varepsilon$ by either (2.5 -) or (2.5 +) whence (2.17a) follows, regardless of the choice of $+$ or $-$. Moreover, (2.17b) follows from either (2.5 +) or (2.5 -) and (2.4),
whereas (2.7) and (2.14) yield (2.17c). Summing up, every modified function $z_{x,\varepsilon}^\pm$ remains in the interval around $z(x)$ where $g$ is monotone on the set where it is different from $z$ itself and, almost everywhere on the same set, its derivative is pushed in the set where $f^{**}$ and $f$ coincide.

Now, we claim that we can choose $+$ or $-$ in such a way that the value of $I^{**}$ computed at the corresponding functions $z_{x,\varepsilon}^+$ or $z_{x,\varepsilon}^-$ remains unaffected for every small enough $\varepsilon$. To see this, we note that the equality

$$
\int_{[0,T]} g(z_{x,\varepsilon}^\pm(t)) f^{**}((z_{x,\varepsilon}^\pm)'(t)) dt = \int_{[0,T]} g(z(t)) f^{**}(z'(t)) dt
$$

holds for every $0 < \varepsilon \leq \varepsilon_0$ because of (2.4) and we address ourselves to computing

$$
\int_{K_{x,\varepsilon}^+} g(z_{x,\varepsilon}^+(t)) f^{**}((z_{x,\varepsilon}^+)'(t)) dt, \quad 0 < \varepsilon \leq \varepsilon_0.
$$

By (2.7) and (2.14), we find

$$
\int_{K_{x,\varepsilon}^+} g(z_{x,\varepsilon}^+(t)) f^{**}((z_{x,\varepsilon}^+)'(t)) dt = f^{**} \left( \frac{1}{\int_{K_{x,\varepsilon}^+} g(z_{x,\varepsilon}^+(t)) dt} \int_{K_{x,\varepsilon}^+} g(z_{x,\varepsilon}^+(t))(z_{x,\varepsilon}^+)'(t) dt \right)
$$

for every $\varepsilon$. Then, we write the argument of $f^{**}$ at the right-hand side of the previous equality as

$$
\frac{1}{\int_{K_{x,\varepsilon}^+} g(z(t)) dt} \int_{K_{x,\varepsilon}^+} g(z(t)) z'(t) dt
$$

$$
+ \left\{ \frac{1}{\int_{K_{x,\varepsilon}^+} g(z_{x,\varepsilon}^+(t)) dt} \int_{K_{x,\varepsilon}^+} g(z_{x,\varepsilon}^+(t))(z_{x,\varepsilon}^+)'(t) dt \right. \right.

$$

$$
- \left. \left. \frac{1}{\int_{K_{x,\varepsilon}^+} g(z(t)) dt} \int_{K_{x,\varepsilon}^+} g(z(t)) z'(t) dt \right\},
$$

and we note that

$$
\frac{1}{\int_{K_{x,\varepsilon}^+} g(z(t)) dt} \int_{K_{x,\varepsilon}^+} g(z(t)) z'(t) dt \in (\alpha, \beta)
$$
for every small enough \( \varepsilon \) since

\[
\left| \int_{K_{x_s}^+} \frac{1}{g(z(t))} \left[ g(z(t)) z'(t) dt - z'(s) \right] \right| \to 0 \quad \text{as} \quad \varepsilon \to 0_+ \tag{2.23}
\]

because of (2.20) and (2.3). Hence, recalling (2.14) and that the argument of \( f^{**} \) at the right-hand side of (2.22) is in \([x, \beta]\) and applying Jensen's inequality, we obtain

\[
\int_{K_{x_s}^+} \frac{1}{g(z(t))} \left[ g(z(t)) \left( z_{x_s}^+ (t) \right)' \right] dt \\
\leq \int_{K_{x_s}^+} \frac{1}{g(z(t))} \left[ g(z(t)) f^{**}(z'(t)) \right] dt \\
+ d \left[ \int_{K_{x_s}^+} \frac{1}{g(z(t))} \left[ g(z(t)) \left( z_{x_s}^+ (t) \right)' \right] dt \right] \\
- \int_{K_{x_s}^+} g(z(t)) z'(t) dt
\]

for every small enough \( \varepsilon \). Writing

\[
\int_{K_{x_s}^+} g(z(t)) dt = 1 + \int_{K_{x_s}^+} \left[ g(z(t)) - g(z(t)) \right] dt \\
\int_{K_{x_s}^+} g(z(t)) dt , \quad 0 < \varepsilon \leq \varepsilon_0,
\]

and recalling (2.8), we see that (2.22) and (2.24) yield

\[
\int_{K_{x_s}^+} g(z(t)) f^{**}(z_{x_s}^+ (t)) dt \\
\leq \int_{K_{x_s}^+} g(z(t)) f^{**}(z'(t)) dt + \int_{K_{x_s}^+} \left[ f^{**}(z'(t)) - dz'(t) \right] dt \\
\times \int_{K_{x_s}^+} \left[ g(z(t)) - g(z(t)) \right] dt
\]

for every small enough \( \varepsilon \) again.

Now, recalling (2.5+) and (2.5-), that \( |z(t) - z(s)| \leq \delta \) for every \( t \in K_{x_s}^\pm \), and every \( \varepsilon \), and that \( g \) is monotone on the interval \([z(s) - 2\delta, z(s) + 2\delta] \), we conclude that the integrals

\[
\int_{K_{x_s}^+} \left[ g(z(t)) - g(z(t)) \right] dt \quad \text{and} \quad \int_{K_{x_s}^+} \left[ g(z(t)) - g(z(t)) \right] dt
\]
have opposite signs (possibly zero) for every $\varepsilon$. We are thus left to prove that the functions

$$R^\pm(\varepsilon) = \frac{\int_{K^\pm s} g(z(t))[f^{**}(z'(t)) - dz'(t)] dt}{\int_{K^\pm s} g(z(t)) dt}, \quad 0 < \varepsilon \leq \varepsilon_0,$$

have constant sign for every small enough $\varepsilon$, regardless of the choice of $+$ or $-$. Indeed, (1.2) and (2.14) show that $R^\pm(\varepsilon) = g$ holds for every $\varepsilon$ so that $R^\pm(\varepsilon)$ is nonnegative for every $\varepsilon$ whenever $q \geq 0$. If $q < 0$, (2.21), (2.3), and the very same argument of (2.23) show that $R^\pm(\varepsilon) \to q$ as $\varepsilon \to 0_+$. Therefore, choosing $\sigma = \sigma(s) \in \{+,-\}$ accordingly, we conclude that $I^{**}(z^0_\varepsilon) \leq I^{**}(z)$ for every small enough $\varepsilon$ so that, upon possibly redefining $\varepsilon_0$ and setting $z^0_\varepsilon = z^0_\varepsilon$, and $K^0_\varepsilon = K^0_\varepsilon$, the proof of the step is complete.

Proof of Step 2. We assume the open (in $[0,T]$) set $U = \{t \in [0,T]: z(t) \in \mathbb{R} \setminus M_g\}$ is nonempty otherwise the conclusion trivially holds with $y = z$. We set also

$$E_k = t \in (0,T): z \text{ is differentiable at } t \text{ and } z_\varepsilon < z'(t) < z_\varepsilon \cap U$$

$$E = \bigcup_k E_k,$$

and we agree to discard all indexes $k$ corresponding to negligible sets $E_k$. Thus,

$$f^{**}(z'(t)) = f(z'(t)) \quad \text{ for a.e. } t \in U \setminus E. \quad (2.25)$$

Now, we apply the construction of the previous step to each set $E_k$. Thereby, with almost every point $s \in E_k$ we associate two families of compact, non degenerate subintervals $K_{s,k} : 0 < \varepsilon \leq \varepsilon_0(s)$ of $(0,T)$ and two families of solutions $\{z_\varepsilon : 0 < \varepsilon \leq \varepsilon_0(s)\}$ to $(\mathcal{P}^{**})$ such that (2.17) holds. We remark that every such interval $K_{s,k}$ is contained in $U$ by construction.

Now, we are left to prove that we can select and glue together some of these functions $z_\varepsilon$, so as to find a new solution $y$ to $(\mathcal{P}^{**})$ satisfying (2.18). To this purpose, let $E'$ be the full measure subset of $E$ consisting of all points $s$ which the construction of Step 1 applies to, namely all points $s \in E_k$ which are density points for $E_k$ as well as Lebesgue points for $z'$ and $f^{**} \circ z'$. Recalling 3.3 and the way the intervals $K_{s,k}$ were defined, we see that the intervals $\{K_{s,k} : 0 < \varepsilon \leq \varepsilon_0(s) \text{ and } s \in E' \}$ constitute a Vitali covering of $E'$ (see [6, Remark 2, p. 25]). Hence, Vitali's covering theorem yields (at most) countably many points $s_\varepsilon \in E'$ and numbers $\varepsilon_\varepsilon \in (0, \varepsilon_0(s_\varepsilon))$ such
that the corresponding intervals $K_h = K_{\alpha_h}$ are pairwise disjoint subsets of $U$ that cover $E'$, and hence $E$ as well, up to a null set; i.e.,

$$
\left| E \setminus \left( \bigcup_h K_h \right) \right| = 0. \tag{2.26}
$$

Let also $z_h = z_{\alpha_h} \in AC([0, T])$ be the corresponding functions. Each function $z_h$ is a solution to $(\not\not\not\not\not)$ so that the equality

$$
\int_{K_h} g(z_h(t)) f^*(z'(t)) \, dt = \int_{K_h} g(z(t)) f^*(z'(t)) \, dt \quad \text{for every } h \tag{2.27}
$$

follows from (2.4). Moreover, $z_h(t) \in \mathbb{R} \setminus M_\delta$ for every $t \in K_h$ because of (2.17a).

Then, set

$$
y(t) = z(t) + \sum_h [z_h(t) - z(t)], \quad t \in [0, T].
$$

We claim that the series converges strongly in $AC([0, T])$. Indeed, the functions $z_h - z$ are in $AC([0, T])$ and their supports $K_h$ are compact, pairwise disjoint subintervals of $(0, T)$ so that the series defining $y$ is actually a finite sum for every $t$ and its partial sums are bounded by $\delta$ because of (2.17b). Hence, the series converges strongly in $L^1([0, T])$ by Lebesgue’s dominated convergence theorem. As to the derivatives, recalling (2.17b), choose $g_0 > 0$ such that $g(z(t)), g(z_h(t)) \geq g_0 > 0$ for every $t \in [0, T]$ and $h$ so that

$$
\sum_h \int_{K_h} |z'_h(t)| \, dt \leq \frac{1}{c_0 g_0} \left[ I^*(z) + c_1 g_0 T \right]
$$

follows from (1.5) and (2.27). Hence,

$$
\sum_h \int_0^T |z'_h(t) - z'(t)| \, dt = \sum_h \int_{K_h} |z'_h(t) - z'(t)| \, dt \\
\leq \sum_h \int_{K_h} |z'_h(t)| \, dt + \int_0^T |z'(t)| \, dt \\
\leq \frac{1}{c_0 g_0} \left[ I^*(z) + c_1 g_0 T \right] + \int_0^T |z'(t)| \, dt < \infty,
$$

i.e., the series of the derivatives converges strongly in $L^1([0, T])$ and this proves the claim.
Now, it is plain that $y$ is feasible for $(\mathcal{P}^{**})$ and, adding up (2.27) for every $h$, we see that $y$ is a solution to $(\mathcal{P}^{**})$. As $K_h \subseteq U$ for every $h$ and

$$y'(t) = \begin{cases} z'(t) & \text{for a.e. } t \in [0, T] \backslash \bigcup_h K_h, \\ z'_h(t) & \text{for a.e. } t \in K_h, \end{cases}$$

we obtain from (2.17c), (2.26), and (2.25) that $f^{**}(y') = f(y')$ almost everywhere on $U$. Finally, it is easy to check that $y^{-1}(\mathbb{R}\setminus M_g) = z^{-1}(\mathbb{R}\setminus M_g) = U$ so that the conclusion follows.

**Proof of Step 3.** Let $y$ be the solution to $(\mathcal{P}^{**})$ constructed in Step 2 and set $V = \{ t \in [0, T] : y(t) \in \mathbb{R}\setminus M_g \}$. Hence, $f^{**}(y') = f(y')$ almost everywhere on $V$. The proof will be accomplished by constructing a new solution $x$ to $(\mathcal{P}^{**})$ such that the set $\{ x \neq y \}$ covers $[0, T]\setminus V$ up to a null set and the equality $f^{**}(x') = f(x')$ holds almost everywhere on $\{ x \neq y \}$.

To this purpose, we assume that $V$ does not coincide with $[0, T]$, otherwise the conclusion trivially holds with $x = y$ and we note that the discreteness of $M_g$ implies that there are at most finitely many points $m \in M_g$ corresponding to nonempty sets $y^{-1}(m)$, say, $\{ m_j : j = 1, \ldots, j_0 \}$ for some $j_0 \geq 1$. Accordingly, on account of (1.6) and (1.7), we choose $\delta > 0$ such that

$$|m - m_j| \geq 4\delta \quad \text{for every } m \in M_g \setminus \{ m_j \} \text{ and } j = 1, \ldots, j_0; \quad (2.28a)$$

$$g(m_j) > g(\xi) \quad \text{for } 0 < |\xi - m_j| \leq 2\delta \text{ and } j = 1, \ldots, j_0; \quad (2.28b)$$

$$g \text{ is increasing on } [m_j - 2\delta, m_j] \text{ and decreasing on } [m_j, m_j + 2\delta] \text{ for } j = 1, \ldots, j_0. \quad (2.28c)$$

Hence, the open (in $[0, T]$) sets $W_j = \{ t \in [0, T] : |y(t) - m_j| < \delta \}$ are pairwise disjoint neighbourhoods of the closed sets $C_j = y^{-1}(m_j)$ and $\bigcup_j C_j = [0, T]\setminus V$. Moreover, each set $C_j$ is a level set of $y$ so that $y'$ vanishes almost everywhere on it.

Then, we turn to $f^{**}$. As $f$ is nonnegative by assumption, $f^{**}$ is nonnegative, too. Hence, $0 \leq f^{**}(0) \leq f(0)$ and we break the remaining part of the proof into three cases according as to the mutual values of $f^{**}(0)$ and $f(0)$.

**Case 1.** $0 \leq f^{**}(0) = f(0)$. As $y'$ vanishes almost everywhere on $[0, T]\setminus V$, the conclusion trivially holds with $x = y$.

**Case 2.** $0 < f^{**}(0) < f(0)$. In this case, we show that each set $C_j$ is negligible. Thus, $V$ coincides with $[0, T]$ up to a null set and the conclusion follows by choosing $x = y$ again.
Indeed, in view of the inequality \( f^*(0) < f(0) \) and (2.14), let \( k \) be such that \( x_k < \beta \) so that, dropping the index \( k \) everywhere, (2.14) reduces to

\[
\begin{align*}
  f^*(\xi) &= q + d \xi, \quad x \leq \xi \leq \beta, \\
  f^*(\xi) &= f(\xi), \quad \xi \in \{x, \beta\},
\end{align*}
\]

(2.29)

where \( \{d\} = \partial f^*(\xi) \) for every \( x < \xi < \beta \) and \( q = -f^*(d) \). Thus, \( q = f^*(0) > 0 \).

Now, assume by contradiction that some set \( C \) has positive measure and set

\[
C = \{ t \in (0, T) \colon y(t) = m_j, y \text{ is differentiable at } t \text{ and } y'(t) = 0 \}. \tag{2.30}
\]

Hence, \( C \) and \( C_j \) coincide up to a null set. Also set \( m = m_j \) and \( W = W_j \).

Then, consider a point \( s \in C \) such that

\[
s \text{ is a density point for } C; \tag{2.31}
\]

\[
s \text{ is a Lebesgue point for } y' \text{ and } f^* \circ y'. \tag{2.32}
\]

and the families of intervals \( \{K^+_{x,s} : 0 < x \leq \varepsilon_0(s)\} \) and functions \( \{y^\pm_{x,s} : 0 < x \leq \varepsilon_0(s)\} \) associated with \( \delta \) are defined in (2.28) by Lemma 2.1. Also let \( J^\pm_{x,s} \) be the open intervals defined by (2.2) and assume that \( \varepsilon_0 = \varepsilon_0(s) \) is such that

\[
0 < \varepsilon_0 \leq 2\delta \quad \text{and} \quad J^\pm_{x,s} \subset W. \tag{2.33}
\]

Hence, all intervals \( K^\pm_{x,s} \) are contained in \( W \) by (2.3) and \( |y^\pm_{x,s}(t) - m| \leq 2\delta \) for every \( t \in K^\pm_{x,s} \) and every \( 0 < x \leq \varepsilon_0 \) by either (2.5\( ^+ \)) or (2.5\( ^- \)) and the very definition of \( W \). Moreover, all functions \( y^\pm_{x,s} \) are feasible for \( f^* \).

Next, recalling that \( m \) is a strict local maximum of \( g \), choose a decreasing sequence \( (\varepsilon_h)_h \) in \( (0, \varepsilon_0] \) such that \( \varepsilon_h \to 0_+ \) and set

\[
\eta_h = \frac{1}{\varepsilon_h} \sup \{|y(t) - m| : |t - s| < 2p\varepsilon_h\} \quad \text{for every } h,
\]

where \( p \) is defined by (2.11) with \( y \) instead of \( z \). Obviously, \( \eta_h \to 0_+ \) since \( y \) is differentiable at \( s \) with \( y'(s) = 0 \) by (2.30) and, moreover, \( 0 \leq \eta_h \varepsilon_h \leq \delta \) by (2.33). Finally, upon possibly extracting a subsequence that we still label as \( (\varepsilon_h)_h \), we can assume that the minimum between \( g(m - \eta_h \varepsilon_h) \) and
\(g(m + \eta h \epsilon_k)\) is actually achieved for every \(h\) by terms with the same sign inside, say \(g(m + \eta h \epsilon_k)\), so that
\[
0 \leq g(m) - g(m + \eta h \epsilon_k) = \max\{g(m) - g(m - \eta h \epsilon_k), g(m) - g(m + \eta h \epsilon_k)\}
\]  
(2.34)
holds for every \(h\).

To simplify the notations, set \(y_h = y_{h, \epsilon_k}^\star\), \(K_h = K_{h, \epsilon_k}^\star\) for every \(h\) and \(J_h = J_{h, \epsilon_k}^\star\), for \(\epsilon > 0\). Note that (2.3) reduces to
\[
J_{h, \epsilon_k} \subset K_h \subset J_{2h, \epsilon_k}.  
\]  
(2.35)
Each function \(y_h\) is feasible for \((P^{**})\) and the equality
\[
\int_{[0, T]} K_h g(y_h(t)) f^{**}(y'_h(t)) dt = \int_{[0, T]} K_h g(y(t)) f^{**}(y'(t)) dt  
\]  
(2.36)
follows from (2.4). The very same computations of Step 1 yield that
\[
\int_{K_h} g(y_h(t)) f^{**}(y'_h(t)) dt 
\leq \int_{K_h} g(y(t)) f^{**}(y'(t)) dt + R_h \int_{K_h} [g(y_h(t)) - g(y(t))] dt  
\]  
(2.37)
for every \(h\) with
\[
R_h = \frac{\int_{K_h} g(y(t)) [f^{**}(y'(t)) - dy'(t)] dt}{\int_{K_h} g(y(t)) dt}. 
\]
All \(R_h\) are positive since \(R_h \geq q = f^{**}(0) > 0\) by (1.2) and we claim that the integral
\[
\int_{K_h} [g(y_h(t)) - g(y(t))] dt
\]
is eventually negative so that a contradiction follows from (2.36) and (2.37).

To see this, set
\[
A_h^+ = \frac{1}{|K_h|} \int_{K_h} [g(m) - g(y_h(t))] dt\]
and
\[
A_h^- = \frac{1}{|K_h|} \int_{K_h} [g(m) - g(y(t))] dt. 
\]
for every \( h \) so that the claim reduces to proving that eventually \( A^1_h - A^2_h > 0 \). Indeed, recalling (2.35) and that \( g \) is decreasing on \([m, m+2\delta]\), and noting that (2.6+) reduces to

\[
2\delta \geq \varepsilon_h \geq y(t) - m \geq \varepsilon_h/2, \quad t \in J_{y/2},
\]

because \( s \in C \) and because of (2.33), we find that

\[
A^1_h \geq \frac{1}{|J_{y/2}|} \int_{J_{y/2}} [g(m) - g(y(t))] \, dt
\]

\[
\geq \frac{1}{|J_{y/2}|} \int_{J_{y/2}} [g(m) - g(m + \varepsilon_h/2)] \, dt
\]

\[
\geq \frac{1}{4} [g(m) - g(m + \varepsilon_h/2)]
\]

for every \( h \) since \(|J_{y/2}|/|J_{y/2}| = 1/4 \) by (2.2). As for \( A^2_h \), note that

\[
A^2_h = \frac{1}{|K_h|} \int_{K_h \setminus C} [g(m) - g(y(t))] \, dt
\]

for every \( h \) and that \( m - \eta_h \varepsilon_h \leq y(t) \leq m + \eta_h \varepsilon_h \) for \( t \in K_h \) by (2.35) and the very definition of \( \eta_h \). Hence,

\[
0 \leq g(m) - g(y(t)) \leq \max\{g(m) - g(m - \eta_h \varepsilon_h), g(m) - g(m + \eta_h \varepsilon_h)\}
\]

\[
= g(m) - g(m + \eta_h \varepsilon_h)
\]

for every \( t \in K_h \) and every \( h \) by (2.28c) and (2.34), whence

\[
0 \leq A^2_h \leq \frac{|K_h \setminus C|}{|K_h|} [g(m) - g(m + \eta_h \varepsilon_h)]
\]

for every \( h \). Since \( \eta_h \to 0 \), it follows that \( g(m) - g(m + \varepsilon_h/2) \geq g(m) - g(m + \eta_h \varepsilon_h) \to 0 \) eventually by (2.28c). As the ratio \(|K_h \setminus C|/|K_h|\) goes to zero because of (2.9) with \( E = C \), the conclusion follows.

**Case 3.** \( 0 = \mathcal{J}^**(0) < \mathcal{J}(0) \). In this last case, we prove that to every non-negligible set \( C_j \) there corresponds a solution \( x_j \) to \((\mathcal{J}^{**})\) such that \( C_j \subset \{x_j \neq y_j\} \subset W_j \) (up to a null set) and \( \mathcal{J}^{**}(x_j) = f(x_j) \) almost everywhere on \( \{x_j \neq y_j\} \). As the open (in \([0, T]\)) sets \( W_j \) are finitely many and pairwise disjoint, the conclusion follows by setting

\[
x(t) = y(t) + \sum_j [x_j(t) - y(t)], \quad t \in [0, T].
\]
To this purpose, as in Case 2, let $k$ be such that $x_k < 0 < \beta_h$ so that, dropping the index $k$ everywhere, (2.29) holds with $q = f^{**}(0) = 0$. As $f^{**}$ is nonnegative and $x < 0 < \beta$, we conclude that $d = 0$ and that $f^{**}$ vanishes on $[x, \beta]$.

Now, let $C_j$ have positive measure and let $C$ be defined by (2.30). Arguing as in Case 2, with each point $s \in C$ satisfying (2.31) and (2.32), we associate the families of functions $\{y_{s,e}^\pm : 0 < e < e_0(s)\}$ and intervals $\{K_{s,e}^\pm : 0 < e \leq e_0(s)\}$ and we assume again that $e_0 = e_0(s) > 0$ is such that (2.33) holds. As $f^{**}$ vanishes on $[x, \beta]$, it follows from (2.7) that $f^{**}((y_{s,e}^\pm)') = 0$ almost everywhere on $K_{s,e}^\pm$. Hence, recalling that $y$ and $y_{s,e}$ agree off the interval $K_{s,e}^\pm$ by (2.4), and that both $f^{**}$ and $g$ are nonnegative, we conclude that all functions $\{y_{s,e}^\pm : 0 < e \leq e_0(s)\}$ are solutions to $(\mathcal{P}^{**})$. Finally, a covering argument similar to that of Step 2 yields a solution $x_j$ to $(\mathcal{P}^{**})$ such that $C_j \subset \{x_j \neq y\}$ up to a null set, $\{x_j \neq y\} \subset W$ and $f^{**}(x_j) = f(x) \text{ almost everywhere on } \{x_j \neq y\}$. This completes the proof.

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