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Miniversal deformations of matrices of bilinear forms

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ABSTRACT

Arnold [V.I. Arnold, On matrices depending on parameters, Russian Math. Surveys 26 (2) (1971) 29–43] constructed miniversal deformations of square complex matrices under similarity; that is, a simple normal form to which not only a given square matrix A but all matrices B close to it can be reduced by similarity transformations that smoothly depend on the entries of B . We construct miniversal deformations of matrices under congruence.

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1. Introduction

The reduction of a matrix to its Jordan form is an unstable operation: both the Jordan form and a reduction transformation depend discontinuously on the entries of the original matrix. Therefore, if the entries of a matrix are known only approximately, then it is unwise to reduce it to Jordan form. Furthermore, when investigating a family of matrices smoothly depending on parameters, then although each individual matrix can be reduced to its Jordan form, it is unwise to do so since in such an operation the smoothness relative to the parameters is lost.

For these reasons, Arnold [1] (see also [2,3]) constructed miniversal deformations of matrices under similarity; that is, a simple normal form to which not only a given square matrix A but all matrices

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B close to it can be reduced by similarity transformations that smoothly depend on the entries of B . Miniversal deformations were also constructed for:

- real matrices with respect to similarity [15] (see also [2,3]; this normal form was simplified in [18]);
- complex matrix pencils [12] (i.e., matrices of pairs of linear mappings $U \rightrightarrows V$; other normal forms of complex and real matrix pencils were constructed in [18,25], see also [19]);
- complex and real contragredient matrix pencils [18] (i.e., matrices of pairs of counter linear mappings $U \rightleftarrows V$);
- matrices of selfadjoint operators on a complex or real vector space with scalar product given by a skew-symmetric, or symmetric, or Hermitian nonsingular form, see [16,8,29,30] and [4, Appendix 6];
- matrices of linear operators on a unitary space [5]. Deformations of selfadjoint operators (Hermitian forms) on a unitary space are studied in [34].

All matrices that we consider are complex matrices.

In Section 2, we formulate Theorem 2.2 that gives miniversal deformations of matrices of bilinear forms; i.e., miniversal deformations of matrices with respect to *congruence transformations*

$$A \mapsto S^T A S, \quad S \text{ is nonsingular}$$

(and hence miniversal deformations of pairs consisting of a symmetric matrix and a skew-symmetric matrix since each square matrix can be expressed uniquely as their sum; see Remark 3.1). A more abstract form of Theorem 2.2, in the spirit of Arnold’s article [1], is given in Theorem 3.1 of Section 3.

We prove Theorem 3.1 in Sections 4–7. The proof is based on Lemma 4.2, which gives a method for constructing miniversal deformations. This lemma follows from a general theory of miniversal deformations. In Section 7.3 we give its constructive proof and find a congruence transformation that reduces a matrix to its miniversal deformation. Analogous interactive methods for constructing transforming matrices in the reduction to versal deformations of matrices under similarity and of matrix pencils under equivalence were developed in [17,26,27].

A preliminary version of this article appeared in 2007 preprint [13]; it was used in [14] for constructing the Hasse diagram of the closure ordering on the set of congruence classes of 3×3 matrices. The authors also recently obtained miniversal deformations of matrices of

- sesquilinear forms [11] (which allows to construct miniversal deformations of pairs (H_1, H_2) of Hermitian matrices because each square matrix can be expressed uniquely as their sum $H_1 + iH_2$),
- pairs of skew-symmetric forms [9], and
- pairs of symmetric forms [10].

2. The main theorem in terms of holomorphic matrix functions

Define the $n \times n$ matrices:

$$J_n(\lambda) := \begin{bmatrix} \lambda & 1 & & 0 \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{bmatrix}, \quad \Gamma_n := \begin{bmatrix} 0 & & & \ddots \\ & & -1 & \ddots \\ & & 1 & 1 \\ & -1 & -1 & \\ 1 & 1 & & 0 \end{bmatrix}.$$

We use the following canonical form of complex matrices for congruence.

Theorem 2.1 [21]. Each square complex matrix is congruent to a direct sum, determined uniquely up to permutation of summands, of matrices of the form

$$H_m(\lambda) := \begin{bmatrix} 0 & I_m \\ J_m(\lambda) & 0 \end{bmatrix} \quad (\lambda \neq 0, \lambda \neq (-1)^{m+1}), \quad \Gamma_n, \quad J_k(0) \tag{1}$$

in which $\lambda \in \mathbb{C}$ is determined up to replacement by λ^{-1} .

This canonical form was obtained in [21] basing on [31, Theorem 3] and was generalized to other fields in [24]; a direct proof that this form is canonical was given in [22,23].

Let

$$A_{\text{can}} = \bigoplus_i H_{p_i}(\lambda_i) \oplus \bigoplus_j \Gamma_{q_j} \oplus \bigoplus_l J_{r_l}(0), \quad r_1 \geq r_2 \geq \dots, \tag{2}$$

be the canonical form for congruence of an $n \times n$ matrix A . Then

$$S^T A S = A_{\text{can}} \tag{3}$$

for a nonsingular S . All matrices that are close to A are represented in the form $A + E$ in which $E \in \mathbb{C}^{n \times n}$ is close to 0_n .

Let $S(E)$ be an $n \times n$ matrix function that is holomorphic on a neighborhood of 0_n , which means that $S(E)$ is an $n \times n$ matrix whose entries are power series in n^2 entries of E , and these series are convergent in this neighborhood of 0_n . Let $S(0_n) = S$ in which S is from (3). We define the matrix function $\mathcal{D}(E)$ by

$$A_{\text{can}} + \mathcal{D}(E) = S(E)^T (A + E) S(E). \tag{4}$$

Then $\mathcal{D}(E)$ is holomorphic at 0_n and $\mathcal{D}(0_n) = 0_n$. Our purpose is to find a simple form of $\mathcal{D}(E)$ by choosing a suitable $S(E)$. In Theorem 2.2, we give $\mathcal{D}(E)$ with the minimal number of nonzero entries that can be attained by using transformations (4).

By a $(0,*)$ matrix we mean a matrix whose entries are 0 and *. Theorem 2.2 involves the following $(0,*)$ matrices, in which all stars are placed in one row or column:

- The $m \times n$ matrices

$$0^{\nearrow} := \begin{bmatrix} * & & & & \\ \vdots & & & & \\ * & & & & \\ & & 0 & & \end{bmatrix} \quad \text{if } m \leq n, \text{ or} \quad \begin{bmatrix} * & \dots & * \\ & & \\ & & 0 \\ & & & \end{bmatrix} \quad \text{if } m \geq n,$$

$$0^{\searrow} := \begin{bmatrix} * & & & & \\ 0 & & & & \\ * & & & & \\ & & 0 & & \\ & & & & \\ & & & & \\ \vdots & & & & \end{bmatrix} \quad \text{if } m \leq n, \text{ or} \quad \begin{bmatrix} * & 0 & * & 0 & \dots \\ & & & & \\ & & & 0 & \\ & & & & \end{bmatrix} \quad \text{if } m \geq n,$$

$$0^{\Downarrow} := \begin{bmatrix} 0 & & & & & \\ & * & & & & \\ 0 & & 0 & & & \\ & * & & & & \\ \vdots & & & & & \end{bmatrix} \text{ if } m \leq n, \text{ or } \begin{bmatrix} 0 & * & 0 & * & \dots \\ & & & & \\ & & 0 & & \\ & & & & \end{bmatrix} \text{ if } m \geq n$$

(if $m = n$ then we can use both the left and the right matrix).

- The matrices

$$0^{\nearrow}, 0^{\times}, 0^{\llcorner}; \quad 0^{\searrow}, 0^{\lessdot}, 0^{\lrcorner}; \quad 0^{\swarrow}, 0^{\gtrdot}, 0^{\triangleright}$$

are obtained from $0^{\nwarrow}, 0^{\vee}, 0^{\Downarrow}$ by the clockwise rotation through 90° ; respectively, 180° ; and 270° .

- The $m \times n$ matrices

$$0^\dagger := \begin{bmatrix} * & \dots & * \\ & 0 & \end{bmatrix} \text{ or } \begin{bmatrix} & 0 \\ * & \dots & * \end{bmatrix}$$

(0^\dagger can be taken in any of these forms), and

$$\mathcal{P}_{mn} := \begin{bmatrix} 0 & \dots & 0 & & \\ \vdots & \ddots & \vdots & & \\ 0 & \dots & 0 & 0 & * & \dots & * \end{bmatrix} \text{ in which } m \leq n \tag{5}$$

(\mathcal{P}_{mn} has $n - m - 1$ stars if $m < n$).

Let $A_{\text{can}} = A_1 \oplus A_2 \oplus \dots \oplus A_t$ be the decomposition (2), and let $\mathcal{D}(E)$ in (4) be partitioned conformably to the partition of A_{can} :

$$\mathcal{D} = \mathcal{D}(E) = \begin{bmatrix} \mathcal{D}_{11} & \dots & \mathcal{D}_{1t} \\ \vdots & \ddots & \vdots \\ \mathcal{D}_{t1} & \dots & \mathcal{D}_{tt} \end{bmatrix}. \tag{6}$$

Write

$$\mathcal{D}(A_i) := \mathcal{D}_{ii}, \quad \mathcal{D}(A_i, A_j) := (\mathcal{D}_{ji}, \mathcal{D}_{ij}) \text{ if } i < j. \tag{7}$$

Our main result is the following theorem, which we reformulate in a more abstract form in Theorem 3.1.

Theorem 2.2 [13]. *Let A be a square complex matrix, let A_{can} be its canonical matrix (2) for congruence, and let S be a nonsingular matrix such that $S^T A S = A_{\text{can}}$. Then all matrices $A + E$ that are sufficiently close*

to A can be simultaneously reduced by some transformation

$$A + E \mapsto S(E)^T(A + E)S(E), \quad \begin{array}{l} S(E) \text{ is nonsingular and} \\ \text{holomorphic at zero, } S(0) = S \end{array} \tag{8}$$

to the form $A_{can} + \mathcal{D}$ in which \mathcal{D} is a $(0,*)$ matrix whose stars represent entries that depend holomorphically on the entries of E , the number of stars in \mathcal{D} is minimal that can be achieved by transformations of the form (8), and the blocks of \mathcal{D} with respect to the partition (6) are defined in the notation (7) as follows:

(i) The diagonal blocks of \mathcal{D} are defined by

$$\mathcal{D}(H_m(\lambda)) = \begin{cases} \begin{bmatrix} 0 & 0 \\ 0 \swarrow & 0 \end{bmatrix} & \text{if } \lambda \neq \pm 1 \text{ (all blocks are } m \times m), \\ \begin{bmatrix} 0 \searrow & 0 \\ 0 \swarrow & 0 \llcorner \end{bmatrix} & \text{if } \lambda = 1 \text{ (} m \text{ is even by (1))}, \\ \begin{bmatrix} 0 \succ & 0 \\ 0 \swarrow & 0 \llcorner \end{bmatrix} & \text{if } \lambda = -1 \text{ (} m \text{ is odd by (1))}; \end{cases} \tag{9}$$

$$\mathcal{D}(\Gamma_n) = \begin{cases} 0 \succ & \text{if } n \text{ is even,} \\ 0 \searrow & \text{if } n \text{ is odd;} \end{cases} \tag{10}$$

$$\mathcal{D}(J_n(0)) = 0 \wedge. \tag{11}$$

(ii) The off-diagonal blocks of \mathcal{D} whose horizontal and vertical strips contain summands of A_{can} of the same type are defined by

$$\mathcal{D}(H_m(\lambda), H_n(\mu)) = \begin{cases} (0, 0) & \text{if } \lambda \neq \mu^{\pm 1}, \\ \left(\begin{bmatrix} 0 \swarrow & 0 \\ 0 & 0 \searrow \end{bmatrix}, 0 \right) & \text{if } \lambda = \mu^{-1} \neq \pm 1, \\ \left(\begin{bmatrix} 0 & 0 \nearrow \\ 0 \swarrow & 0 \end{bmatrix}, 0 \right) & \text{if } \lambda = \mu \neq \pm 1, \\ \left(\begin{bmatrix} 0 \swarrow & 0 \nearrow \\ 0 \swarrow & 0 \searrow \end{bmatrix}, 0 \right) & \text{if } \lambda = \mu = \pm 1; \end{cases} \tag{12}$$

$$\mathcal{D}(\Gamma_m, \Gamma_n) = \begin{cases} (0, 0) & \text{if } m - n \text{ is odd,} \\ (0 \swarrow, 0) & \text{if } m - n \text{ is even;} \end{cases} \tag{13}$$

$$\mathcal{D}(J_m(0), J_n(0)) = \begin{cases} (0 \wedge, 0 \wedge) & \text{if } m \geq n \text{ and } n \text{ is even,} \\ (0 \wedge + \mathcal{P}_{nm}, 0 \wedge) & \text{if } m \geq n \text{ and } n \text{ is odd.} \end{cases} \tag{14}$$

(iii) The off-diagonal blocks of \mathcal{D} whose horizontal and vertical strips contain summands of A_{can} of different types are defined by

$$\mathcal{D}(H_m(\lambda), \Gamma_n) = \begin{cases} (0, 0) & \text{if } \lambda \neq (-1)^{n+1}, \\ ([0 \swarrow 0 \nearrow], 0) & \text{if } \lambda = (-1)^{n+1}; \end{cases} \tag{15}$$

$$\mathcal{D}(H_m(\lambda), J_n(0)) = \begin{cases} (0, 0) & \text{if } n \text{ is even,} \\ (0^\dagger, 0) & \text{if } n \text{ is odd;} \end{cases} \tag{16}$$

$$\mathcal{D}(\Gamma_m, J_n(0)) = \begin{cases} (0, 0) & \text{if } n \text{ is even,} \\ (0^\dagger, 0) & \text{if } n \text{ is odd.} \end{cases} \tag{17}$$

For each $A \in \mathbb{C}^{n \times n}$, the vector space

$$T(A) := \{C^T A + AC \mid C \in \mathbb{C}^{n \times n}\} \tag{18}$$

is the tangent space to the congruence class of A at the point A since

$$(I + \varepsilon C)^T A (I + \varepsilon C) = A + \varepsilon(C^T A + AC) + \varepsilon^2 C^T A C \tag{19}$$

for all $C \in \mathbb{C}^{n \times n}$ and $\varepsilon \in \mathbb{C}$.

The matrix \mathcal{D} from Theorem 2.2 was constructed such that

$$\mathbb{C}^{n \times n} = T(A_{\text{can}}) \oplus \mathcal{D}(\mathbb{C}) \tag{20}$$

in which $\mathcal{D}(\mathbb{C})$ is the vector space of all matrices obtained from \mathcal{D} by replacing its stars by complex numbers. Thus, the number of stars in \mathcal{D} is equal to the codimension of the congruence class of A_{can} ; it was independently calculated in [6]. The codimensions of *congruence classes of canonical matrices for *congruence were calculated in [7]. Simplest miniversal deformations of matrix pencils and contragredient matrix pencils [18], canonical matrices for *congruence [11], and canonical pairs of skew-symmetric matrices [10] were constructed by analogous methods.

Theorem 2.2 will be proved as follows: we first prove in Lemma 4.2 that each $(0, *)$ matrix that satisfies (20) can be taken as \mathcal{D} in Theorem 2.2, and then verify that \mathcal{D} from Theorem 2.2 satisfies (20).

Example 2.1. Let A be any 2×2 or 3×3 matrix. Then all matrices $A + E$ that are sufficiently close to A can be simultaneously reduced by transformations (8) to one of the following forms

$$\begin{aligned} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} * & * \\ * & * \end{bmatrix}, & \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ * & * \end{bmatrix}, & \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ * & 0 \end{bmatrix}, \\ & \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} * & 0 \\ * & * \end{bmatrix}, & \begin{bmatrix} 0 & 1 \\ \lambda & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ * & 0 \end{bmatrix} \ (\lambda \neq \pm 1), & \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} * & 0 \\ 0 & 0 \end{bmatrix}, \end{aligned}$$

or, respectively,

$$\begin{aligned} & \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}, & \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ * & * & * \\ * & * & * \end{bmatrix}, \\ & \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ * & 0 & 0 \\ * & * & * \end{bmatrix}, & \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ * & 0 & 0 \\ * & * & 0 \end{bmatrix}, \end{aligned}$$

$$\begin{aligned}
 & \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ & 0 \end{bmatrix} + \begin{bmatrix} * & 0 & 0 \\ * & * & 0 \\ * & * & * \end{bmatrix}, & \begin{bmatrix} 0 & 1 \\ \lambda & 0 \\ & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ * & 0 & 0 \\ * & * & * \end{bmatrix} \quad (\lambda \neq 0, \pm 1), \\
 & \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ * & 0 & * \\ * & 0 & * \end{bmatrix}, & \begin{bmatrix} 0 & -1 \\ 1 & 1 \\ & 0 \end{bmatrix} + \begin{bmatrix} * & 0 & 0 \\ 0 & 0 & 0 \\ * & * & * \end{bmatrix}, \\
 & \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ & 1 \end{bmatrix} + \begin{bmatrix} * & 0 & 0 \\ * & * & 0 \\ 0 & 0 & 0 \end{bmatrix}, & \begin{bmatrix} 0 & 1 \\ \mu & 0 \\ & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ * & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (\mu \neq \pm 1), \\
 & \begin{bmatrix} 0 & -1 \\ 1 & 1 \\ & 1 \end{bmatrix} + \begin{bmatrix} * & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ * & 0 & * \end{bmatrix}, \\
 & \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & -1 \\ 1 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ * & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
 \end{aligned}$$

Each of these matrices has the form $A_{\text{can}} + \mathcal{D}$ in which A_{can} is a direct sum of blocks of the form (1) (the zero entries outside of these blocks in A_{can} are not shown) and the stars in \mathcal{D} are complex numbers that tend to zero as E tends to 0. The number of stars is the smallest that can be attained by using transformations (8); it is equal to the codimension of the congruence class of A .

3. The main theorem in terms of miniversal deformations

The notion of a miniversal deformation of a matrix with respect similarity was given by Arnold [1] (see also [3, §30B]). It can be extended to matrices with respect to congruence as follows.

A deformation of a matrix $A \in \mathbb{C}^{n \times n}$ is a holomorphic map $\mathcal{A} : \Lambda \rightarrow \mathbb{C}^{n \times n}$ in which $\Lambda \subset \mathbb{C}^k$ is a neighborhood of $\vec{0} = (0, \dots, 0)$ and $\mathcal{A}(\vec{0}) = A$.

Let \mathcal{A} and \mathcal{B} be two deformations of A with the same parameter space \mathbb{C}^k . We consider \mathcal{A} and \mathcal{B} as equal if they coincide on some neighborhood of $\vec{0}$ (this means that each deformation is a germ). We say that \mathcal{A} and \mathcal{B} are equivalent if the identity matrix I_n possesses a deformation \mathcal{I} such that

$$\mathcal{B}(\vec{\lambda}) = \mathcal{I}(\vec{\lambda})^T \mathcal{A}(\vec{\lambda}) \mathcal{I}(\vec{\lambda})$$

for all $\vec{\lambda} = (\lambda_1, \dots, \lambda_k)$ in some neighborhood of $\vec{0}$.

Definition 3.1. A deformation $\mathcal{A}(\lambda_1, \dots, \lambda_k)$ of a square matrix A is called versal if every deformation $\mathcal{B}(\mu_1, \dots, \mu_l)$ of A is equivalent to a deformation of the form $\mathcal{A}(\varphi_1(\vec{\mu}), \dots, \varphi_k(\vec{\mu}))$ in which $\vec{\mu} = (\mu_1, \dots, \mu_l)$, all $\varphi_i(\vec{\mu})$ are power series that are convergent in a neighborhood of $\vec{0}$, and $\varphi_i(\vec{0}) = 0$. A versal deformation $\mathcal{A}(\lambda_1, \dots, \lambda_k)$ of A is called miniversal if there is no versal deformation that has less than k parameters.

For each $(0, *)$ matrix \mathcal{D} , we denote by $\mathcal{D}(\mathbb{C})$ the space of all matrices obtained from \mathcal{D} by replacing the stars with complex numbers (as in (20)) and by $\mathcal{D}(\mathcal{E})$ the parameter matrix obtained from \mathcal{D} by

replacing each (i, j) star with the parameter ε_{ij} . This means that

$$\mathcal{D}(\mathbb{C}) := \bigoplus_{(i,j) \in \mathcal{I}(\mathcal{D})} \mathbb{C}E_{ij}, \quad \mathcal{D}(\vec{\varepsilon}) := \sum_{(i,j) \in \mathcal{I}(\mathcal{D})} \varepsilon_{ij}E_{ij}, \tag{21}$$

in which every E_{ij} is the matrix unit (its (i, j) entry is 1 and the others are 0) and

$$\mathcal{I}(\mathcal{D}) \subseteq \{1, \dots, n\} \times \{1, \dots, n\}$$

is the set of indices of the stars in \mathcal{D} .

We say that a deformation of A is *simplest* if it has the form $A + \mathcal{D}(\vec{\varepsilon})$ in which \mathcal{D} is a $(0, *)$ matrix. Definition 3.1 of versality for a simplest deformation can be reformulated in the spirit of Section 2 as follows.

Definition 3.2. A simplest deformation $A + \mathcal{D}(\vec{\varepsilon})$ of a square matrix A is *versal* if there exists an $n \times n$ matrix $S(X)$ and a neighborhood $U \subset \mathbb{C}^{n \times n}$ of 0_n such that

- (i) the entries of $S(X)$ are power series in variables $x_{ij}, i, j = 1, \dots, n$ (they form the $n \times n$ matrix of unknowns $X = [x_{ij}]$),
- (ii) these series are convergent in U and $S(0_n) = I_n$,
- (iii) $S(E)^T(A + E)S(E) \in A + \mathcal{D}(\mathbb{C})$ for all $E \in U$.

Since each square matrix is congruent to its canonical matrix, it suffices to construct miniversal deformations of canonical matrices (2). Their miniversal deformations are given in the following theorem, which is another form of Theorem 2.2.

Theorem 3.1 [13]. Let A_{can} be a canonical matrix (2) for congruence. A simplest miniversal deformation of A_{can} can be taken in the form $A_{can} + \mathcal{D}(\vec{\varepsilon})$, where \mathcal{D} is the $(0, *)$ matrix partitioned into blocks \mathcal{D}_{ij} (as in (6)) that are defined by (9)–(17) in the notation (7).

Remark 3.1. Each square matrix A can be represented uniquely as

$$A = \mathcal{S} + \mathcal{C}, \quad \mathcal{S} \text{ is symmetric and } \mathcal{C} \text{ is skew-symmetric.} \tag{22}$$

A congruence of A corresponds to a simultaneous congruence of \mathcal{S} and \mathcal{C} . Thus, if A_{can} is a canonical matrix for congruence given in Theorem 2.1 and $A_{can} = \mathcal{S}_{can} + \mathcal{C}_{can}$ is its representation (22), then $(\mathcal{S}_{can}, \mathcal{C}_{can})$ is a canonical pair for simultaneous congruence of pairs of symmetric and skew-symmetric matrices. The pairs $(\mathcal{S}_{can}, \mathcal{C}_{can})$ were described in [23, Theorem 1.2(a)]. Theorem 3.1 admits to derive a miniversal deformation of $(\mathcal{S}_{can}, \mathcal{C}_{can})$; that is, to construct a normal form with minimal number of parameters to which all pairs $(\mathcal{S}, \mathcal{C})$ that are close to $(\mathcal{S}_{can}, \mathcal{C}_{can})$ and consist of symmetric and skew-symmetric matrices can be reduced by transformations

$$(\mathcal{S}, \mathcal{C}) \mapsto (S^T \mathcal{S} S, S^T \mathcal{C} S), \quad S \text{ is nonsingular,}$$

in which S smoothly depends on the entries of \mathcal{S} and \mathcal{C} . All one has to do is to express $A_{can} + \mathcal{D}(\vec{\varepsilon})$ as the sum of symmetric and skew-symmetric matrices.

4. A method for constructing miniversal deformations

In this section, we give a method for constructing simplest miniversal deformations; we use it in the proof of Theorem 3.1.

The deformation

$$U(\vec{\varepsilon}) := A + \sum_{i,j=1}^n \varepsilon_{ij} E_{ij}, \tag{23}$$

in which E_{ij} are the matrix units, is universal in the sense that every deformation $\mathcal{B}(\mu_1, \dots, \mu_l)$ of A has the form $U(\vec{\varphi}(\mu_1, \dots, \mu_l))$, in which $\varphi_{ij}(\mu_1, \dots, \mu_l)$ are power series that are convergent in a neighborhood of $\vec{0}$ and $\varphi_{ij}(\vec{0}) = 0$. Hence every deformation $\mathcal{B}(\mu_1, \dots, \mu_l)$ in Definition 3.1 can be replaced by $U(\vec{\varepsilon})$, which gives the following lemma.

Lemma 4.1. *The following two conditions are equivalent for any deformation $\mathcal{A}(\lambda_1, \dots, \lambda_k)$ of a matrix A :*

- (i) *The deformation $\mathcal{A}(\lambda_1, \dots, \lambda_k)$ is versal.*
- (ii) *The deformation (23) is equivalent to $\mathcal{A}(\varphi_1(\vec{\varepsilon}), \dots, \varphi_k(\vec{\varepsilon}))$ for some power series $\varphi_i(\vec{\varepsilon})$ that are convergent in a neighborhood of $\vec{0}$ and such that $\varphi_i(\vec{0}) = 0$.*

If U is a subspace of a vector space V , then each set $v + U$ with $v \in V$ is called an *affine subspace parallel to U* .

The proof of Theorem 3.1 is based on the following lemma, which gives a method of constructing miniversal deformations. A constructive proof of this lemma is given in Theorem 8.1.

Lemma 4.2. *Let $A \in \mathbb{C}^{n \times n}$ and let \mathcal{D} be a $(0, *)$ matrix of size $n \times n$. The following three statements are equivalent:*

- (i) *The deformation $A + \mathcal{D}(\vec{\varepsilon})$ of A (see (21)) is miniversal.*
- (ii) *The vector space $\mathbb{C}^{n \times n}$ decomposes into the direct sum*

$$\mathbb{C}^{n \times n} = T(A) \oplus \mathcal{D}(\mathbb{C})$$

in which $T(A)$ and $\mathcal{D}(\mathbb{C})$ are defined in (18) and (21).

- (iii) *Each affine subspace of $\mathbb{C}^{n \times n}$ parallel to $T(A)$ intersects $\mathcal{D}(\mathbb{C})$ at exactly one point.*

Proof. Let us define the action of the group $GL_n(\mathbb{C})$ of nonsingular $n \times n$ matrices on the space $\mathbb{C}^{n \times n}$ by

$$A^S := S^T A S, \quad A \in \mathbb{C}^{n \times n}, \quad S \in GL_n(\mathbb{C}). \tag{24}$$

The orbit A^{GL_n} of A under this action consists of all matrices that are congruent to A .

By (19), the space $T(A)$ is the tangent space to the orbit A^{GL_n} at the point A . Hence $\mathcal{D}(\vec{\varepsilon})$ is transversal to the orbit A^{GL_n} at the point A if

$$\mathbb{C}^{n \times n} = T(A) + \mathcal{D}(\mathbb{C})$$

(see definitions in [3, §29E]; two subspaces of a vector space are called *transversal* if their sum is the whole space).

This proves the equivalence of (i) and (ii) since a transversal (of the minimal dimension) to the orbit is a (mini)versal deformation; see [2, Section 1.6] or [32, Part V, Theorem 1.2]. The equivalence of (ii) and (iii) is obvious. \square

Recall that the orbits of canonical matrices (2) under the action (24) were also studied in [6, 14].

Corollary 4.1. *A simplest miniversal deformation of $A \in \mathbb{C}^{n \times n}$ can be constructed as follows. Let T_1, \dots, T_r be a basis of the space $T(A)$, and let E_1, \dots, E_{r^2} be the basis of $\mathbb{C}^{n \times n}$ consisting of all matrix units E_{ij} . Removing from the sequence $T_1, \dots, T_r, E_1, \dots, E_{r^2}$ every matrix that is a linear combination of*

the preceding matrices, we obtain a new basis $T_1, \dots, T_r, E_{i_1}, \dots, E_{i_k}$ of the space $\mathbb{C}^{n \times n}$. By Lemma 4.2, the deformation

$$\mathcal{A}(\varepsilon_1, \dots, \varepsilon_k) = A + \varepsilon_1 E_{i_1} + \dots + \varepsilon_k E_{i_k}$$

is miniversal.

For each $M \in \mathbb{C}^{m \times m}$ and $N \in \mathbb{C}^{n \times n}$, define the vector space

$$T(M, N) := \left\{ \underbrace{(S^T M + NR)}_{n\text{-by-}m}, \underbrace{(R^T N + MS)}_{m\text{-by-}n} \mid S \in \mathbb{C}^{m \times n}, R \in \mathbb{C}^{n \times m} \right\}. \tag{25}$$

Lemma 4.3. Let $A = A_1 \oplus \dots \oplus A_t$ be a block-diagonal matrix in which every A_i is $n_i \times n_i$. Let $\mathcal{D} = [D_{ij}]$ be a $(0, *)$ matrix of the same size and partitioned into blocks conformably to the partition of A . Then $A + \mathcal{D}(\vec{\varepsilon})$ is a simplest miniversal deformation of A for congruence if and only if

- (i) each affine subspace of $\mathbb{C}^{n_i \times n_i}$ parallel to $T(A_i)$ (which is defined in (18)) intersects $\mathcal{D}_{ii}(\mathbb{C})$ at exactly one point and
- (ii) each affine subspace of $\mathbb{C}^{n_j \times n_i} \oplus \mathbb{C}^{n_i \times n_j}$ parallel to $T(A_i, A_j)$ (which is defined in (25)) intersects $\mathcal{D}_{ji}(\mathbb{C}) \oplus \mathcal{D}_{ij}(\mathbb{C})$ at exactly one point.

Proof. By Lemma 4.2(iii), $A + \mathcal{D}(\vec{\varepsilon})$ is a simplest miniversal deformation of A if and only if for each $C \in \mathbb{C}^{n \times n}$ the affine subspace $C + T(A)$ contains exactly one $D \in \mathcal{D}(\mathbb{C})$; that is, for each C exactly one matrix in $\mathcal{D}(\mathbb{C})$ has the form

$$D = C + S^T A + AS \in \mathcal{D}(\mathbb{C}), \quad S \in \mathbb{C}^{n \times n}. \tag{26}$$

Let us partition D , C , and S into blocks conformably to the partition of A . By (26), for each i we have $D_{ii} = C_{ii} + S_{ii}^T A_i + A_i S_{ii}$, and for all i and j such that $i < j$ we have

$$\begin{bmatrix} D_{ii} & D_{ij} \\ D_{ji} & D_{jj} \end{bmatrix} = \begin{bmatrix} C_{ii} & C_{ij} \\ C_{ji} & C_{jj} \end{bmatrix} + \begin{bmatrix} S_{ii}^T & S_{ji}^T \\ S_{ij}^T & S_{jj}^T \end{bmatrix} \begin{bmatrix} A_i & 0 \\ 0 & A_j \end{bmatrix} + \begin{bmatrix} A_i & 0 \\ 0 & A_j \end{bmatrix} \begin{bmatrix} S_{ii} & S_{ij} \\ S_{ji} & S_{jj} \end{bmatrix}.$$

Thus, (26) is equivalent to the conditions

$$D_{ii} = C_{ii} + S_{ii}^T A_i + A_i S_{ii} \in \mathcal{D}_{ii}(\mathbb{C}) \quad \text{for } 1 \leq i \leq t \tag{27}$$

and

$$(D_{ji}, D_{ij}) = (C_{ji}, C_{ij}) + (S_{ij}^T A_i + A_j S_{ji}, S_{ji}^T A_j + A_i S_{ij}) \in \mathcal{D}_{ji}(\mathbb{C}) \oplus \mathcal{D}_{ij}(\mathbb{C}) \tag{28}$$

for $1 \leq i < j \leq t$. Hence, for each $C \in \mathbb{C}^{n \times n}$ there exists exactly one $D \in \mathcal{D}$ of the form (26) if and only if

- (i') for each $C_{ii} \in \mathbb{C}^{n_i \times n_i}$ there exists exactly one $D_{ii} \in \mathcal{D}_{ii}$ of the form (27) and
- (ii') for each $(C_{ji}, C_{ij}) \in \mathbb{C}^{n_j \times n_i} \oplus \mathbb{C}^{n_i \times n_j}$ there exists exactly one $(D_{ji}, D_{ij}) \in \mathcal{D}_{ji}(\mathbb{C}) \oplus \mathcal{D}_{ij}(\mathbb{C})$ of the form (28). \square

Corollary 4.2. In the notation of Lemma 4.3, $A + \mathcal{D}(\vec{\varepsilon})$ is a miniversal deformation of A if and only if each submatrix of $A + \mathcal{D}(\vec{\varepsilon})$ of the form

$$\begin{bmatrix} A_i + \mathcal{D}_{ii}(\vec{\varepsilon}) & \mathcal{D}_{ij}(\vec{\varepsilon}) \\ \mathcal{D}_{ji}(\vec{\varepsilon}) & A_j + \mathcal{D}_{jj}(\vec{\varepsilon}) \end{bmatrix} \quad \text{with } i < j$$

is a miniversal deformation of $A_i \oplus A_j$. A similar reduction to the case of canonical forms for congruence with two direct summands was used in [6] for the solution of the equation $XA + AX^T = 0$.

We are ready to prove Theorem 3.1. Let $A_{\text{can}} = A_1 \oplus A_2 \oplus \dots \oplus A_t$ be the canonical matrix (2), and let $\mathcal{D} = [\mathcal{D}_{ij}]_{i,j=1}^t$ be the $(0, *)$ matrix constructed in Theorem 3.1. Each A_i has the form $H_n(\lambda)$, or Γ_n , or $J_n(0)$, and so there are three types of diagonal blocks $\mathcal{D}(A_i) = \mathcal{D}_{ii}$ and six types of pairs of off-diagonal blocks $\mathcal{D}(A_i, A_j) = (\mathcal{D}_{ji}, \mathcal{D}_{ij})$, $i < j$; they were defined in (9)–(17). In the next three sections, we prove that all blocks of \mathcal{D} satisfy the conditions (i) and (ii) of Lemma 4.3.

5. Diagonal blocks of \mathcal{D}

Let us verify that the diagonal blocks of \mathcal{D} defined in part (i) of Theorem 2.2 satisfy the condition (i) of Lemma 4.3.

5.1. Diagonal blocks $\mathcal{D}(H_n(\lambda))$

Due to Lemma 4.3(i), it suffices to prove that each $2n$ -by- $2n$ matrix $A = [A_{ij}]_{i,j=1}^2$ can be reduced to exactly one matrix of the form (9) by adding

$$\begin{bmatrix} S_{11}^T & S_{21}^T \\ S_{12}^T & S_{22}^T \end{bmatrix} \begin{bmatrix} 0 & I_n \\ J_n(\lambda) & 0 \end{bmatrix} + \begin{bmatrix} 0 & I_n \\ J_n(\lambda) & 0 \end{bmatrix} \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} = \begin{bmatrix} S_{21}^T J_n(\lambda) + S_{21} & S_{11}^T + S_{22} \\ S_{22}^T J_n(\lambda) + J_n(\lambda) S_{11} & S_{12}^T + J_n(\lambda) S_{12} \end{bmatrix}$$

in which $S = [S_{ij}]_{i,j=1}^2$ is an arbitrary $2n$ -by- $2n$ matrix. Taking $S_{22} = -A_{12}$ and the other $S_{ij} = 0$, we obtain a new matrix A with $A_{12} = 0$. To preserve A_{12} , we hereafter must take S with $S_{11}^T + S_{22} = 0$. Therefore, we can add $S_{21}^T J_n(\lambda) + S_{21}$ to the (new) A_{11} , $S_{12}^T + J_n(\lambda) S_{12}$ to A_{22} , and $-S_{11} J_n(\lambda) + J_n(\lambda) S_{11}$ to A_{21} . Using these additions, we can reduce A to the form (9) on the strength of the following 3 lemmas.

Lemma 5.1. *Adding $SJ_n(\lambda) + S^T$ with a fixed λ and an arbitrary S , we can reduce each $n \times n$ matrix to exactly one matrix of the form*

$$\begin{cases} 0 & \text{if } \lambda \neq \pm 1, \\ 0 \succcurlyeq & \text{if } \lambda = 1, \\ 0 \succ & \text{if } \lambda = -1. \end{cases} \tag{29}$$

Proof. Let $A = [a_{ij}]$ be an arbitrary $n \times n$ matrix. We will reduce it along its skew diagonals

$$\begin{array}{ccccccc} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{array}$$

starting from the upper left corner; that is, in the following order:

$$a_{11}, (a_{21}, a_{12}), (a_{31}, a_{22}, a_{13}), \dots, a_{nn}. \tag{30}$$

We reduce A by adding $\Delta A := SJ_n(\lambda) + S^T$ in which $S = [s_{ij}]$ is any $n \times n$ matrix. For instance, if $n = 4$ then

$$\Delta A = \begin{bmatrix} \lambda s_{11} + 0 + s_{11} & \lambda s_{12} + s_{11} + s_{21} & \lambda s_{13} + s_{12} + s_{31} & \lambda s_{14} + s_{13} + s_{41} \\ \lambda s_{21} + 0 + s_{12} & \lambda s_{22} + s_{21} + s_{22} & \lambda s_{23} + s_{22} + s_{32} & \lambda s_{24} + s_{23} + s_{42} \\ \lambda s_{31} + 0 + s_{13} & \lambda s_{32} + s_{31} + s_{23} & \lambda s_{33} + s_{32} + s_{33} & \lambda s_{34} + s_{33} + s_{43} \\ \lambda s_{41} + 0 + s_{14} & \lambda s_{42} + s_{41} + s_{24} & \lambda s_{43} + s_{42} + s_{34} & \lambda s_{44} + s_{43} + s_{44} \end{bmatrix}.$$

Case 1: $\lambda \neq \pm 1$. We reduce A to 0 by induction: Assume that the first $t - 1$ skew diagonals of A in the sequence (30) are zero. To preserve them, we must and will take the first $t - 1$ skew diagonals of S equal to zero. If the t th skew diagonal of S is (x_1, \dots, x_r) , then we can add

$$(\lambda x_1 + x_r, \lambda x_2 + x_{r-1}, \lambda x_3 + x_{r-2}, \dots, \lambda x_r + x_1) \tag{31}$$

to the t th skew diagonal of A . Each vector $(c_1, \dots, c_r) \in \mathbb{C}^r$ is represented in the form (31) since the corresponding system of linear equations

$$\lambda x_1 + x_r = c_1, \quad \lambda x_2 + x_{r-1} = c_2, \quad \dots, \quad \lambda x_r + x_1 = c_r$$

has a nonzero determinant for all $\lambda \neq \pm 1$. We make the t th skew diagonal of A equal to zero.

Case 2: $\lambda = 1$. We say that a vector $(v_1, v_2, \dots, v_r) \in \mathbb{C}^r$ is *symmetric* if it is equal to (v_r, \dots, v_2, v_1) , and *skew-symmetric* if it is equal to $(-v_r, \dots, -v_2, -v_1)$. Let us consider the equality

$$(x_1, x_2, \dots, x_r) + (0, y_2, \dots, y_r) = (a_1, a_2, \dots, a_r) \tag{32}$$

in which $\vec{x} = (x_1, \dots, x_r)$ is symmetric and $\vec{y} = (y_2, \dots, y_r)$ is skew-symmetric. The following two statements hold:

- (a) If r is odd, then for each a_1, \dots, a_r there exist unique \vec{x} and \vec{y} satisfying (32).
- (b) If r is even, then for each a_1, \dots, a_{r-1} there exist unique a_r, \vec{x} , and \vec{y} satisfying (32), and for each a_2, \dots, a_r there exist unique a_1, \vec{x} , and \vec{y} satisfying (32).

Indeed, if $r = 2k + 1$, then (32) takes the form

$$(x_1, \dots, x_k, x_{k+1}, x_k, \dots, x_1) + (0, y_2, \dots, y_{k+1}, -y_{k+1}, \dots, -y_2) = (a_1, \dots, a_{2k+1}),$$

and so it can be rewritten as follows:

$$\left[\begin{array}{ccc|ccc} 1 & & & 0 & & \\ & 1 & & 1 & \ddots & \\ & & \ddots & & \ddots & 0 \\ & & & 1 & & 1 \\ \hline & & & 1 & 0 & -1 \\ & \ddots & \ddots & & \ddots & \\ 1 & 0 & & -1 & & \end{array} \right] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{k+1} \\ y_2 \\ \vdots \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{k+1} \\ a_{k+2} \\ \vdots \\ a_{2k+1} \end{bmatrix}.$$

The matrix of this system is nonsingular since we can add the columns of the second vertical strip to the corresponding columns of the first vertical strip and reduce it to the form

$$\left[\begin{array}{cc|ccc} 1 & & & & 0 \\ 1 & 1 & & & 1 \ \ddots \\ & \ddots & \ddots & & \ddots \\ & & 1 & 1 & 0 \\ \hline & & & & 1 \\ & 0 & & & -1 \\ & & & & \ddots \\ & & & & -1 \end{array} \right]$$

with nonsingular diagonal blocks. This proves (a).

If $r = 2k$, then (32) takes the form

$$(x_1, \dots, x_k, x_k, \dots, x_1) + (0, y_2, \dots, y_k, 0, -y_k, \dots, -y_2) = (a_1, \dots, a_{2k}),$$

and so it can be rewritten as follows:

$$\left[\begin{array}{cc|ccc} 1 & & & & 0 \\ & 1 & & & 1 \ \ddots \\ & & \ddots & & \ddots \\ & & & 1 & 1 \\ \hline & & & 1 & 0 \\ & & \ddots & & \ddots \\ & & & & -1 \\ & 1 & & 0 & \ddots \\ & & & & \ddots \\ 1 & & & & -1 \end{array} \right] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \\ y_2 \\ \vdots \\ y_k \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \\ a_{k+1} \\ a_{k+2} \\ \vdots \\ a_{2k} \end{bmatrix}.$$

The matrix of this system is $2k$ -by- $(2k - 1)$ and can be reduced as follows. For $i = 1, \dots, k - 1$, we add the i th column of the second vertical strip to the i th column of the first vertical strip or subtract it from the $(i + 1)$ st column of the first vertical strip and obtain

$$\left[\begin{array}{cc|ccc} 1 & & & & 0 \\ 1 & 1 & & & 1 \ \ddots \\ & \ddots & \ddots & & \ddots \\ & & 1 & 1 & 0 \\ \hline & & & & 1 \\ & 1 & & & 0 \\ & & & & \ddots \\ & & & & -1 \\ & & & & \ddots \\ & & & & -1 \end{array} \right] \text{ or } \left[\begin{array}{cc|ccc} 1 & & & & 0 \\ & 0 & & & 1 \ \ddots \\ & & \ddots & & \ddots \\ & & & 0 & 1 \\ \hline & & & 1 & 0 \\ & & \ddots & & \ddots \\ & & & 1 & -1 \\ & 1 & & 0 & \ddots \\ & & & & \ddots \\ 1 & 1 & & & -1 \end{array} \right].$$

The first matrix without the first row and the second matrix without the last row are nonsingular. This proves (b).

Since $\lambda = 1$, we can add $\Delta A = SJ_n(1) + S^T$ to A . The matrix S is an arbitrary of size $n \times n$, write it in the form $S = B + C$ in which

$$B := \frac{S + S^T}{2} \quad \text{and} \quad C := \frac{S - S^T}{2} \tag{33}$$

are its symmetric and skew-symmetric parts. Then

$$SJ_n(1) + S^T = S + SJ_n(0) + S^T = 2B + (B + C)J_n(0),$$

and so we can add to A any matrix

$$\Delta A = 2B + (B + C)J_n(0) \tag{34}$$

in which $B = [b_{ij}]$ is symmetric and $C = [c_{ij}]$ is skew-symmetric.

We reduce A to the form $0 \succcurlyeq$ along the skew diagonals (30) as follows. Taking $b_{11} = -a_{11}/2$, we make the $(1, 1)$ entry of A equal to zero. Reasoning by induction, we fix $t \in \{1, \dots, n - 1\}$ and assume that

- the first $t - 1$ skew diagonals of A have been reduced to the form $0 \succcurlyeq$ (that is, these diagonals coincide with the corresponding skew diagonals of some matrix of the form $0 \succcurlyeq$) and these skew diagonals are uniquely determined by the initial matrix A ;
- if $t \leq n$ and S preserves the first $t - 1$ skew diagonals of A (i.e., the first $t - 1$ skew diagonals of (34) are zero) then the first $t - 1$ skew diagonals of B are zero.

Let $t \leq n$. Then the t th skew diagonal of (34) has the form

$$(b_1, b_2, \dots, b_r) + (0, c_2, \dots, c_r) \tag{35}$$

in which (b_1, b_2, \dots, b_r) is an arbitrary symmetric vector (it is the t th skew diagonal of $2B$) and (c_2, c_3, \dots, c_r) is an arbitrary skew-symmetric vector (it is the $(t - 1)$ st skew diagonal of C). The statements (a) and (b) imply that we can make the t th skew diagonal of A as in $0 \succcurlyeq$ by adding (35). Moreover, this skew diagonal is uniquely determined, and to preserve it the t th skew diagonal of B must be zero.

For instance, if $t = 2 \leq n$, then we add (b_1, b_1) and reduce the second skew diagonal of A to the form $(*, 0)$ or $(0, *)$. If $t = 3 \leq n$, then we add $(b_1, b_2, b_1) + (0, c_2, -c_2)$ and make the third skew diagonal of A equal to 0.

Let $t > n$. Let us take S in which the first $t - 1$ skew diagonals are equal to 0. Then the t th skew diagonal of (34) has the form

$$(b_1, b_2, \dots, b_r) + (c_1, c_2, \dots, c_r) \tag{36}$$

in which (b_1, b_2, \dots, b_r) is the t th skew diagonal of $2B$ is symmetric and (c_1, c_2, \dots, c_r) is the $(t - 1)$ st skew diagonal of C without the last entry. Thus, (b_1, b_2, \dots, b_r) is any symmetric, c_1 is arbitrary, and (c_2, c_3, \dots, c_r) is any skew-symmetric. Adding (36), we reduce the t th skew diagonal of A to 0.

Case 3: $\lambda = -1$. We can add $SJ_n(-1) + S^T$ to A . Write S in the form $B + C$, in which B and C are defined in (33). Then

$$\Delta A = SJ_n(-1) + S^T = -S + SJ_n(0) + S^T = -2C + (B + C)J_n(0).$$

We reduce A to the form $0 \succ$ along the skew diagonals (30) as follows. The $(1, 1)$ entry of ΔA is 0; so we cannot change a_{11} . Reasoning by induction, we fix $t \in \{1, \dots, n - 1\}$ and assume that

- the first $t - 1$ skew diagonals of A have been reduced to the form $0 \succ$ and these diagonals are uniquely determined by the initial matrix A ;
- if $t \leq n$ and S preserves the first $t - 1$ skew diagonals of A (i.e., the first $t - 1$ skew diagonals of (34) are zero) then the first $t - 1$ skew diagonals of C are zero.

If $t \leq n$, then we can add to the t th skew diagonal of A any vector

$$(c_1, c_2, \dots, c_t) + (0, b_2, \dots, b_t)$$

in which (c_1, \dots, c_t) is skew-symmetric (it is the t th skew diagonal of $-2C$) and (b_2, \dots, b_t) is symmetric (it is the $(t - 1)$ st skew diagonal of B). We make the t th skew diagonal of A as in $0 \succ$. For instance, if $t = 2 \leq n$, then we add $(c_1, -c_1) + (0, b_2)$ and make the second skew diagonal of A equal to zero. If $t = 3 \leq n$, then we add $(c_1, 0, -c_1) + (0, b_2, b_2)$ and reduce the third skew diagonal of A to the form $(*, 0, 0)$ or $(0, 0, *)$.

Let $t > n$. Let us take S in which the first $t - 1$ skew diagonals are equal to 0. Then we can add to the t th skew diagonal of A any vector

$$(c_1, c_2, \dots, c_r) + (b_1, b_2, \dots, b_r)$$

in which (c_1, \dots, c_r) is skew-symmetric (it is the t th skew diagonal of $-2C$), b_1 is arbitrary, and (b_2, \dots, b_{r-1}) is symmetric (it is the $(t - 1)$ st skew diagonal of B without the first and the last elements). We make the t th skew diagonal of A equal to zero. \square

Lemma 5.2. Adding $J_n(\lambda)S + S^T$, we can reduce each $n \times n$ matrix to exactly one matrix of the form

$$\begin{cases} 0 & \text{if } \lambda \neq \pm 1, \\ 0^{\ll} & \text{if } \lambda = 1, \\ 0^{\lt} & \text{if } \lambda = -1. \end{cases} \tag{37}$$

Proof. By Lemma 5.1, for each $n \times n$ matrix B there exists R such that $M := B + R J_n(\lambda) + R^T$ has the form (29). Then

$$M^T = B^T + J_n(\lambda)^T R^T + R.$$

Write

$$Z := \begin{bmatrix} 0 & & 1 \\ & \ddots & \\ 1 & & 0 \end{bmatrix}.$$

Because $Z J_n(\lambda)^T Z = J_n(\lambda)$, we have

$$Z M^T Z = Z B^T Z + J_n(\lambda)(Z R Z)^T + Z R Z.$$

This ensures Lemma 5.2 since $Z B^T Z$ is arbitrary and $Z M^T Z$ is of the form (37). \square

Lemma 5.3. Adding $S J_n(\lambda) - J_n(\lambda)S$, we can reduce each $n \times n$ matrix to exactly one matrix of the form 0^{\swarrow} .

Proof. Let $A = [a_{ij}]$ be an arbitrary $n \times n$ matrix. Adding

$$S J_n(\lambda) - J_n(\lambda) S = S J_n(0) - J_n(0) S$$

$$= \begin{bmatrix} s_{21} - 0 & s_{22} - s_{11} & s_{23} - s_{12} & \dots & s_{2n} - s_{1,n-1} \\ \dots & \dots & \dots & \dots & \dots \\ s_{n-1,1} - 0 & s_{n-1,2} - s_{n-2,1} & s_{n-1,3} - s_{n-2,2} & \dots & s_{n-1,n} - s_{n-2,n-1} \\ s_{n1} - 0 & s_{n2} - s_{n-1,1} & s_{n3} - s_{n-1,2} & \dots & s_{nn} - s_{n-1,n-1} \\ 0 - 0 & 0 - s_{n1} & 0 - s_{n2} & \dots & 0 - s_{n,n-1} \end{bmatrix},$$

we reduce A along the diagonals

$$a_{n1}, (a_{n-1,1}, a_{n2}), (a_{n-2,1}, a_{n-1,2}, a_{n3}), \dots, a_{1n}$$

to the form $0 \swarrow$. \square

5.2. Diagonal blocks $\mathcal{D}(\Gamma_n)$

Due to Lemma 4.3(i), it suffices to prove that each $n \times n$ matrix A can be reduced to exactly one matrix of the form (10) by adding $\Delta A := S^T \Gamma_n + \Gamma_n S$. Write Γ_n as the sum of its symmetric and skew-symmetric parts: $\Gamma_n = \Gamma_n^{(s)} + \Gamma_n^{(c)}$, where

$$\Gamma_n^{(s)} = \begin{bmatrix} 0 & & \dots \\ & 0 & \dots \\ & 0 & 1 \\ 0 & -1 & \\ 0 & 1 & 0 \end{bmatrix} \text{ and } \Gamma_n^{(c)} = \begin{bmatrix} 0 & & \dots \\ & -1 & \\ & 1 & \\ -1 & & \\ 1 & & 0 \end{bmatrix} \text{ if } n \text{ is even,}$$

$$\Gamma_n^{(s)} = \begin{bmatrix} 0 & & \dots \\ & -1 & \\ & 1 & \\ -1 & & \\ 1 & & 0 \end{bmatrix} \text{ and } \Gamma_n^{(c)} = \begin{bmatrix} 0 & & \dots \\ & 0 & \dots \\ & 0 & 1 \\ 0 & -1 & \\ 0 & 1 & 0 \end{bmatrix} \text{ if } n \text{ is odd.}$$

Then the symmetric and skew-symmetric parts of ΔA are

$$\Delta A^{(s)} = S^T \Gamma_n^{(s)} + \Gamma_n^{(s)} S, \quad \Delta A^{(c)} = S^T \Gamma_n^{(c)} + \Gamma_n^{(c)} S$$

in which $S = [s_{ij}]$ is any $n \times n$ matrix.

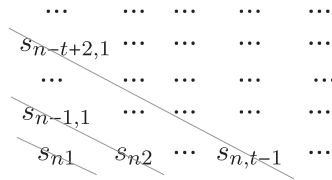
Case 1: n is even. Then

$$\Delta A^{(s)} = \begin{bmatrix} 0 + 0 & s_{n1} + 0 & -s_{n-1,1} + 0 & \dots & s_{21} + 0 \\ 0 + s_{n1} & s_{n2} + s_{n2} & -s_{n-1,2} + s_{n3} & \dots & s_{22} + s_{nn} \\ 0 - s_{n-1,1} & s_{n3} - s_{n-1,2} & -s_{n-1,3} - s_{n-1,3} & \dots & s_{23} - s_{n-1,n} \\ \dots & \dots & \dots & \dots & \dots \\ 0 + s_{21} & s_{nn} + s_{22} & -s_{n-1,n} + s_{23} & \dots & s_{2n} + s_{2n} \end{bmatrix},$$

$$\Delta A^{(c)} = \begin{bmatrix} s_{n1} - s_{n1} & -s_{n-1,1} - s_{n2} & s_{n-2,1} - s_{n3} & \dots & -s_{11} - s_{nn} \\ s_{n2} + s_{n-1,1} & -s_{n-1,2} + s_{n-1,2} & s_{n-2,2} + s_{n-1,3} & \dots & -s_{12} + s_{n-1,n} \\ s_{n3} - s_{n-2,1} & -s_{n-1,3} - s_{n-2,2} & s_{n-2,3} - s_{n-2,3} & \dots & -s_{13} - s_{n-2,n} \\ \dots & \dots & \dots & \dots & \dots \\ s_{nn} + s_{11} & -s_{n-1,n} + s_{12} & s_{n-2,n} + s_{13} & \dots & -s_{1n} + s_{1n} \end{bmatrix}.$$

We reduce $A = [a_{ij}]$ to the form $0 \succ$ along its skew diagonals (30) as follows. The $(1, 1)$ entry of $\Delta A = \Delta A^{(s)} + \Delta A^{(c)}$ is zero, and so the $(1, 1)$ entry of A is not changed. Reasoning by induction, we fix $t \in \{1, \dots, n - 1\}$ and assume that

- the first t skew diagonals of A have been reduced to the form $0 \succ$ and they are uniquely determined by the initial A ;
- the addition of ΔA preserves the first t skew diagonals of A if and only if the first $t - 1$ diagonals of S starting from the lower left diagonal



are zero and its t th diagonal

$$(s_{n-t+1,1}, s_{n-t+2,2}, s_{n-t+3,3}, \dots, s_{n-2,t-2}, s_{n-1,t-1}, s_{n,t})$$

is symmetric if t is odd and skew-symmetric if t is even.

Write

$$(v_1, v_2, \dots, v_t) := \left((-1)^{t-1} s_{n-t+1,1}, \dots, s_{n-2,t-2}, -s_{n-1,t-1}, s_{n,t} \right);$$

this vector is symmetric for all t .

The $(t + 1)$ st skew diagonal of $\Delta A^{(s)}$ is

$$(0, v_t, v_{t-1}, \dots, v_2, v_1) + (v_1, v_2, \dots, v_{t-1}, v_t, 0). \tag{38}$$

If t is odd, then every symmetric vector of dimension $t + 1$ is represented in the form (38). If t is even, then (38) without the first and the last elements is an arbitrary symmetric vector of dimension $t - 1$ and the first (and the last) element of (38) is fully determined by the other elements. Since the $(t + 1)$ st skew diagonal of $\Delta A^{(c)}$ is an arbitrary skew-symmetric vector of dimension $t + 1$, this means that the $(t + 1)$ st skew diagonal of A is reduced to zero if t is odd, and to the form $(*, 0, \dots, 0)$ or $(0, \dots, 0, *)$ if t is even. To preserve it, we hereafter must take those S in which the $(t + 1)$ st skew diagonal of $\Delta A^{(c)}$ is zero; this means that the $(t + 1)$ st diagonal of S is symmetric if $t + 1$ is odd and skew-symmetric if $t + 1$ is even.

Thus, the first n skew diagonals in A have the form of the corresponding diagonals in $0 \succ$.

The $(n + 1)$ st skew diagonal of $\Delta A^{(s)}$ has the form

$$(v_n, v_{n-1}, \dots, v_2) + (v_2, \dots, v_{n-1}, v_n)$$

(compare with (38)) and every symmetric vector of dimension $n - 1$ is represented in this form. Hence, the $(n + 1)$ st skew diagonal of ΔA is an arbitrary vector of dimension $n - 1$ and we make the $(n + 1)$ st

skew diagonal of A equal to zero. Analogously, we make its $n + 2, n + 3, \dots$ skew diagonals equal to zero and reduce A to the form $0 \succ$.

Case 2: n is odd. Then

$$\Delta A^{(s)} = \begin{bmatrix} s_{n1} + s_{n1} & -s_{n-1,1} + s_{n2} & s_{n-2,1} + s_{n3} & \dots & s_{11} + s_{nn} \\ s_{n2} - s_{n-1,1} & -s_{n-1,2} - s_{n-1,2} & s_{n-2,2} - s_{n-1,3} & \dots & s_{1n} - s_{n-1,n} \\ s_{n3} + s_{n-2,1} & -s_{n-1,3} + s_{n-2,2} & s_{n-2,3} + s_{n-2,3} & \dots & s_{13} + s_{n-2,n} \\ \dots & \dots & \dots & \dots & \dots \\ s_{nn} + s_{11} & -s_{n-1,n} + s_{12} & s_{n-2,n} + s_{13} & \dots & s_{1n} + s_{1n} \end{bmatrix},$$

$$\Delta A^{(c)} = \begin{bmatrix} 0 + 0 & s_{n1} + 0 & -s_{n-1,1} + 0 & \dots & -s_{21} + 0 \\ 0 - s_{n1} & s_{n2} - s_{n2} & -s_{n-1,2} - s_{n3} & \dots & -s_{22} - s_{nn} \\ 0 + s_{n-1,1} & s_{n3} + s_{n-1,2} & -s_{n-1,3} + s_{n-1,3} & \dots & -s_{23} + s_{n-1,n} \\ \dots & \dots & \dots & \dots & \dots \\ 0 + s_{21} & s_{nn} + s_{22} & -s_{n-1,n} + s_{23} & \dots & -s_{2n} + s_{2n} \end{bmatrix}.$$

We reduce A along its skew diagonals (30). The first skew diagonal of $\Delta A^{(s)}$ is arbitrary; we make the first entry of A equal to zero.

Let $t < n$. Assume that

- the first t skew diagonals of A have been reduced to the form $0 \succ$ and they are uniquely determined by the initial A ;
- the addition of ΔA preserves these diagonals if and only if the first $t - 1$ diagonals of S , starting from the lower left diagonal, are zero and the t th diagonal (u_1, \dots, u_t) of S is symmetric if t is even and skew-symmetric if t is odd.

Then the vector

$$(v_1, v_2, \dots, v_t) := \left((-1)^{t-1} u_1, \dots, u_{t-2}, -u_{t-1}, u_t \right)$$

is skew-symmetric for all t .

The $(t + 1)$ st skew diagonal of $\Delta A^{(c)}$ is

$$(0, v_t, v_{t-1}, \dots, v_2, v_1) - (v_1, v_2, \dots, v_{t-1}, v_t, 0). \tag{39}$$

If t is even, then every skew-symmetric vector of dimension $t + 1$ is represented in the form (39). If t is odd, then (39) without the first and the last elements is an arbitrary skew-symmetric vector of dimension $t - 1$ and the first (and the last) element of (39) is fully determined by the other elements. Since the $(t + 1)$ st skew diagonal of $\Delta A^{(s)}$ is an arbitrary symmetric vector of dimension $t + 1$, this means that the $(t + 1)$ st skew diagonal of A reduces to 0 if t is even, and to the form $(*, 0, \dots, 0)$ or $(0, \dots, 0, *)$ if t is odd. To preserve it, we hereafter must take those S in which the $(t + 1)$ st skew diagonal of $\Delta A^{(s)}$ is zero; this means that the $(t + 1)$ st diagonal of S is symmetric if $t + 1$ is even and skew-symmetric if $t + 1$ is odd.

The first n skew diagonals in A have the form of the corresponding diagonals in $0 \succ$. The $(n + 1)$ st skew diagonal in $\Delta A^{(c)}$ has the form

$$(v_n, v_{n-1}, \dots, v_2) - (v_2, \dots, v_{n-1}, v_n)$$

(compare with (39)) and every skew-symmetric vector is represented in this form. Hence, the $(n + 1)$ st skew diagonal of ΔA is an arbitrary vector of dimension $n - 1$ and we make the $(n + 1)$ st skew diagonal

of A equal to zero. Analogously, we make its $n + 2, n + 3, \dots$ skew diagonals equal to zero and reduce A to the form $0 \times$.

5.3. Diagonal blocks $\mathcal{D}(J_n(0))$

Due to Lemma 4.3(i), it suffices to prove that each $n \times n$ matrix A can be reduced to exactly one matrix of the form (11) by adding

$$\Delta A := S^T J_n(0) + J_n(0) S = \begin{bmatrix} 0 + s_{21} & s_{11} + s_{22} & s_{21} + s_{23} & \dots & s_{n-1,1} + s_{2n} \\ 0 + s_{31} & s_{12} + s_{32} & s_{22} + s_{33} & \dots & s_{n-1,2} + s_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ 0 + s_{n1} & s_{1,n-1} + s_{n2} & s_{2,n-1} + s_{n3} & \dots & s_{n-1,n-1} + s_{nn} \\ 0 + 0 & s_{1n} + 0 & s_{2n} + 0 & \dots & s_{n-1,n} + 0 \end{bmatrix} \tag{40}$$

in which $S = [s_{ij}]$ is any $n \times n$ matrix. Thus,

$$\Delta A = [b_{ij}], \quad b_{ij} := s_{j-1,i} + s_{i+1,j} \quad (s_{0i} := 0, \quad s_{n+1,j} := 0),$$

and so all entries of ΔA have the form $s_{kl} + s_{l+1,k+1}$. The transitive closure of $(k, l) \sim (l + 1, k + 1)$ is an equivalence relation on the set $\{1, \dots, n\} \times \{1, \dots, n\}$.

Represent ΔA as the sum

$$\Delta A = B_{n1} + B_{n-1,1} + \dots + B_{11} + B_{12} + \dots + B_{1n}$$

of matrices that correspond to the equivalence classes and are defined as follows. Each B_{1j} ($j = 1, 2, \dots, n$) is obtained from ΔA by replacing with 0 all of its entries except for

$$s_{1j} + s_{j+1,2}, s_{j+1,2} + s_{3,j+2}, s_{3,j+2} + s_{j+3,4}, \dots \tag{41}$$

and each B_{i1} ($i = 2, 3, \dots, n$) is obtained from ΔA by replacing with 0 all of its entries except for

$$0 + s_{i1}, s_{i1} + s_{2,i+1}, s_{2,i+1} + s_{i+2,3}, s_{i+2,3} + s_{4,i+3}, s_{4,i+3} + s_{i+4,5}, \dots; \tag{42}$$

the pairs of indices in (41) and in (42) are equivalent:

$$(1, j) \sim (j + 1, 2) \sim (3, j + 2) \sim (j + 3, 4) \sim \dots$$

and

$$(i, 1) \sim (2, i + 1) \sim (i + 2, 3) \sim (4, i + 3) \sim (i + 4, 5) \sim \dots$$

We call the entries (41) and (42) the *main entries* of B_{1j} and B_{i1} ($i > 1$). The matrices $B_{n1}, \dots, B_{11}, B_{12}, \dots, B_{1n}$ have no common s_{ij} .

An arbitrary sequence of complex numbers can be represented in the form (41). The entries (42) are linearly dependent only if the last entry in this sequence has the form $s_{kn} + 0$ (see (40)); then $(k, n) = (2p, i - 1 + 2p)$ for some p , and so $i = n + 1 - 2p$. Thus the following sequences (42) are linearly dependent:

$$0 + s_{n-1,1}, s_{n-1,1} + s_{2n}, s_{2n} + 0; \\ 0 + s_{n-3,1}, s_{n-3,1} + s_{2,n-2}, s_{2,n-2} + s_{n-1,3}, s_{n-1,3} + s_{4n}, s_{4n} + 0; \dots$$

One of the main entries of each of the matrices $B_{n-1,1}, B_{n-3,1}, B_{n-5,1}, \dots$ is the linear combination of the other main entries of this matrix, which are arbitrary. The main entries of the other matrices B_{i1} and B_{1j} are arbitrary. Adding B_{i1} and B_{1j} , we reduce A to the form $0 \hat{\times}$.

6. Off-diagonal blocks of \mathcal{D} that correspond to summands of A_{can} of the same type

Now we verify the condition (ii) of Lemma 4.3 for those off-diagonal blocks of \mathcal{D} (defined in Theorem 2.2(ii)) whose horizontal and vertical strips contain summands of A_{can} of the same type.

6.1. Pairs of blocks $\mathcal{D}(H_m(\lambda), H_n(\mu))$

Due to Lemma 4.3(ii), it suffices to prove that each pair (B, A) of $2n \times 2m$ and $2m \times 2n$ matrices can be reduced to exactly one pair of the form (12) by adding

$$(S^T H_m(\lambda) + H_n(\mu)R, R^T H_n(\mu) + H_m(\lambda)S), \quad S \in \mathbb{C}^{2m \times 2n}, \quad R \in \mathbb{C}^{2n \times 2m}.$$

Taking $R = 0$ and $S = -H_m(\lambda)^{-1}A$, we reduce A to 0. To preserve $A = 0$ we hereafter must take S and R such that $R^T H_n(\mu) + H_m(\lambda)S = 0$; that is,

$$S = -H_m(\lambda)^{-1}R^T H_n(\mu),$$

and so we can add

$$\Delta B := -H_n(\mu)^T R H_m(\lambda)^{-T} H_m(\lambda) + H_n(\mu)R$$

to B .

Write $P := -H_n(\mu)^T R$, then $R = -H_n(\mu)^{-T} P$ and

$$\Delta B = P \begin{bmatrix} J_m(\lambda) & 0 \\ 0 & J_m(\lambda)^{-T} \end{bmatrix} - \begin{bmatrix} J_n(\mu)^{-T} & 0 \\ 0 & J_n(\mu) \end{bmatrix} P. \tag{43}$$

Let us partition $B, \Delta B$, and P into $n \times m$ blocks:

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}, \quad \Delta B = \begin{bmatrix} \Delta B_{11} & \Delta B_{12} \\ \Delta B_{21} & \Delta B_{22} \end{bmatrix}, \quad P = \begin{bmatrix} X & Y \\ Z & T \end{bmatrix}.$$

By (43),

$$\Delta B_{11} = XJ_m(\lambda) - J_n(\mu)^{-T}X, \quad \Delta B_{12} = YJ_m(\lambda)^{-T} - J_n(\mu)^{-T}Y,$$

$$\Delta B_{21} = ZJ_m(\lambda) - J_n(\mu)Z, \quad \Delta B_{22} = TJ_m(\lambda)^{-T} - J_n(\mu)T.$$

These equalities show that we can reduce each block B_{ij} separately by adding ΔB_{ij} .

(i) First we reduce B_{11} by adding $\Delta B_{11} = XJ_m(\lambda) - J_n(\mu)^{-T}X$.

If $\lambda \neq \mu^{-1}$, then ΔB_{11} is an arbitrary $n \times m$ matrix since $J_m(\lambda)$ and $J_n(\mu)^{-T}$ have no common eigenvalues; we make $B_{11} = 0$.

Let $\lambda = \mu^{-1}$. Then

$$J_n(\mu)^{-T} = \begin{bmatrix} \mu^{-1} & & & & & & 0 \\ & -\mu^{-2} & \mu^{-1} & & & & \\ & \mu^{-3} & -\mu^{-2} & \mu^{-1} & & & \\ & \ddots & \ddots & \ddots & \ddots & & \\ & \ddots & \ddots & & \mu^{-3} & -\mu^{-2} & \mu^{-1} \end{bmatrix} = \begin{bmatrix} \lambda & & & & & & 0 \\ & -\lambda^2 & \lambda & & & & \\ & \lambda^3 & -\lambda^2 & \lambda & & & \\ & \ddots & \ddots & \ddots & \ddots & & \\ & \ddots & \ddots & & \lambda^3 & -\lambda^2 & \lambda \end{bmatrix}.$$

Adding

$$\begin{aligned} \Delta B_{11} &= XJ_m(0) - (J_n(\mu)^{-T} - \lambda I_n)X \\ &= \begin{bmatrix} 0 & x_{11} & \dots & x_{1,m-1} \\ 0 & x_{21} & \dots & x_{2,m-1} \\ 0 & x_{31} & \dots & x_{3,m-1} \\ \dots & \dots & \dots & \dots \end{bmatrix} + \lambda^2 \begin{bmatrix} 0 & \dots & 0 \\ x_{11} & \dots & x_{1m} \\ x_{21} & \dots & x_{2m} \\ \dots & \dots & \dots \end{bmatrix} - \lambda^3 \begin{bmatrix} 0 & \dots & 0 \\ 0 & \dots & 0 \\ x_{11} & \dots & x_{1m} \\ \dots & \dots & \dots \end{bmatrix} + \dots, \end{aligned}$$

we reduce B_{11} to the form 0^{\nearrow} along the skew diagonals starting from the upper left corner.

(ii) Let us reduce B_{12} by adding $\Delta B_{12} = YJ_m(\lambda)^{-T} - J_n(\mu)^{-T}Y$.
 If $\lambda \neq \mu$, then ΔB_{12} is arbitrary; we make $B_{12} = 0$.
 Let $\lambda = \mu$. Write $F := J_n(0)$. Since

$$J_n(\lambda)^{-1} = (\lambda I_n + F)^{-1} = \lambda^{-1}I_n - \lambda^{-2}F + \lambda^{-3}F^2 - \dots,$$

we have

$$\begin{aligned} \Delta B_{12} &= Y(J_m(\lambda)^{-T} - \lambda^{-1}I_m) - (J_n(\lambda)^{-T} - \lambda^{-1}I_n)Y \\ &= -\lambda^{-2} \begin{bmatrix} y_{12} & \dots & y_{1m} & 0 \\ y_{22} & \dots & y_{2m} & 0 \\ y_{32} & \dots & y_{3m} & 0 \\ \dots & \dots & \dots & \dots \end{bmatrix} + \lambda^{-2} \begin{bmatrix} 0 & \dots & 0 \\ y_{11} & \dots & y_{1m} \\ y_{21} & \dots & y_{2m} \\ \dots & \dots & \dots \end{bmatrix} + \dots \end{aligned}$$

We reduce B_{12} to the form 0^{\nearrow} along its diagonals starting from the upper right corner.

(iii) Let us reduce B_{21} by adding $\Delta B_{21} = ZJ_m(\lambda) - J_n(\mu)Z$.
 If $\lambda \neq \mu$, then ΔB_{21} is arbitrary; we make $B_{21} = 0$.
 If $\lambda = \mu$, then

$$\begin{aligned} \Delta B_{21} &= Z(J_m(\lambda) - \lambda I_m) - (J_n(\lambda) - \lambda I_n)Z \\ &= \begin{bmatrix} 0 & z_{11} & \dots & z_{1,m-1} \\ \dots & \dots & \dots & \dots \\ 0 & z_{n-1,1} & \dots & z_{n-1,m-1} \\ 0 & z_{n1} & \dots & z_{n,m-1} \end{bmatrix} - \begin{bmatrix} z_{21} & \dots & z_{2m} \\ \dots & \dots & \dots \\ z_{n1} & \dots & z_{nm} \\ 0 & \dots & 0 \end{bmatrix}; \end{aligned}$$

we reduce B_{21} to the form 0^{\swarrow} along its diagonals starting from the lower left corner.

(iv) Finally, reduce B_{22} by adding $\Delta B_{22} = TJ_m(\lambda)^{-T} - J_n(\mu)T$.
 If $\lambda \neq \mu^{-1}$, then ΔB_{22} is arbitrary; we make $B_{22} = 0$.
 If $\lambda = \mu^{-1}$, then

$$\begin{aligned} \Delta B_{22} &= T(J_m(\lambda)^{-T} - \mu I_m) - (J_n(\mu) - \mu I_n)T \\ &= -\mu^2 \begin{bmatrix} \dots & t_{1m} & 0 \\ \dots & \dots & \dots \\ \dots & t_{n-1,m} & 0 \\ \dots & t_{nm} & 0 \end{bmatrix} + \mu^3 \begin{bmatrix} \dots & t_{1m} & 0 & 0 \\ \dots & \dots & \dots & \dots \\ \dots & t_{n-1,m} & 0 & 0 \\ \dots & t_{nm} & 0 & 0 \end{bmatrix} - \dots - \begin{bmatrix} t_{21} & \dots & t_{2m} \\ \dots & \dots & \dots \\ t_{n1} & \dots & t_{nm} \\ 0 & \dots & 0 \end{bmatrix}; \end{aligned}$$

we reduce B_{22} to the form 0^{\searrow} along its skew diagonals starting from the lower right corner.

6.2. Pairs of blocks $\mathcal{D}(\Gamma_m, \Gamma_n)$

Due to Lemma 4.3(ii), it suffices to prove that each pair (B, A) of $n \times m$ and $m \times n$ matrices can be reduced to exactly one pair of the form (13) by adding

$$(S^T \Gamma_m + \Gamma_n R, R^T \Gamma_n + \Gamma_m S), \quad S \in \mathbb{C}^{m \times n}, \quad R \in \mathbb{C}^{n \times m}.$$

Taking $R = 0$ and $S = -\Gamma_m^{-1}A$, we reduce A to 0. To preserve $A = 0$ we hereafter must take S and R such that $R^T \Gamma_n + \Gamma_m S = 0$; that is, $S = -\Gamma_m^{-1}R^T \Gamma_n$, and so we can add

$$\Delta B := -\Gamma_n^T R \Gamma_m^{-T} \Gamma_m + \Gamma_n R$$

to B .

Write $P := \Gamma_n^T R$, then

$$\Delta B = -P(\Gamma_m^{-T} \Gamma_m) + (\Gamma_n \Gamma_n^{-T})P.$$

Since

$$\Gamma_n = \begin{bmatrix} 0 & & & \ddots \\ & & -1 & \ddots \\ & & 1 & 1 \\ & -1 & -1 & \\ 1 & 1 & & 0 \end{bmatrix}, \quad \Gamma_n^{-1} = (-1)^{n+1} \begin{bmatrix} \vdots & \vdots & \vdots & \vdots & \ddots \\ -1 & -1 & -1 & -1 & \\ 1 & 1 & 1 & & \\ -1 & -1 & & & \\ 1 & & & & 0 \end{bmatrix},$$

we have

$$\Gamma_m^{-T} \Gamma_m = (-1)^{m+1} \begin{bmatrix} 1 & 2 & * \\ & 1 & \ddots \\ & & \ddots & 2 \\ 0 & & & 1 \end{bmatrix}$$

and

$$\Gamma_n \Gamma_n^{-T} = (-1)^{n+1} \begin{bmatrix} 1 & & & 0 \\ -2 & 1 & & \\ & \ddots & \ddots & \\ * & & -2 & 1 \end{bmatrix}. \tag{44}$$

If $n - m$ is odd, then

$$(-1)^{n+1} \Delta B = 2P + P \begin{bmatrix} 0 & 2 & * \\ & 0 & \ddots \\ & & \ddots & 2 \\ 0 & & & 0 \end{bmatrix} + \begin{bmatrix} 0 & & & 0 \\ -2 & 0 & & \\ & \ddots & \ddots & \\ * & & -2 & 0 \end{bmatrix} P$$

and we reduce B to 0 along its skew diagonals starting from the upper left corner.

If $m - n$ is even, then

$$(-1)^{n+1} \Delta B = -P \begin{bmatrix} 0 & 2 & & * \\ & 0 & \ddots & \\ & & \ddots & 2 \\ 0 & & & 0 \end{bmatrix} + \begin{bmatrix} 0 & & & 0 \\ -2 & 0 & & \\ & \ddots & \ddots & \\ * & & -2 & 0 \end{bmatrix} P$$

and we reduce B to the form 0^\sphericalangle along its skew diagonals starting from the upper left corner.

6.3. Pairs of blocks $\mathcal{D}(J_m(0), J_n(0))$ with $m \geq n$

Due to Lemma 4.3(ii), it suffices to prove that each pair (B, A) of $n \times m$ and $m \times n$ matrices with $m \geq n$ can be reduced to exactly one pair of the form (14) by adding the matrices

$$\Delta A = R^T J_n(0) + J_m(0)S, \quad \Delta B^T = J_m(0)^T S + R^T J_n(0)^T$$

to A and B^T (it is convenient for us to reduce the transpose of B).

Write $S = [s_{ij}]$ and $R^T = [-r_{ij}]$ (they are m -by- n). Then

$$\Delta A = \begin{bmatrix} s_{21} - 0 & s_{22} - r_{11} & s_{23} - r_{12} & \dots & s_{2n} - r_{1,n-1} \\ \dots & \dots & \dots & \dots & \dots \\ s_{m-1,1} - 0 & s_{m-1,2} - r_{m-2,1} & s_{m-1,3} - r_{m-2,2} & \dots & s_{m-1,n} - r_{m-2,n-1} \\ s_{m1} - 0 & s_{m2} - r_{m-1,1} & s_{m3} - r_{m-1,2} & \dots & s_{mn} - r_{m-1,n-1} \\ 0 - 0 & 0 - r_{m1} & 0 - r_{m2} & \dots & 0 - r_{m,n-1} \end{bmatrix}$$

and

$$\Delta B^T = \begin{bmatrix} 0 - r_{12} & 0 - r_{13} & \dots & 0 - r_{1n} & 0 - 0 \\ s_{11} - r_{22} & s_{12} - r_{23} & \dots & s_{1,n-1} - r_{2n} & s_{1n} - 0 \\ \dots & \dots & \dots & \dots & \dots \\ s_{m-2,1} - r_{m-1,2} & s_{m-2,2} - r_{m-1,3} & \dots & s_{m-2,n-1} - r_{m-1,n} & s_{m-2,n} - 0 \\ s_{m-1,1} - r_{m2} & s_{m-1,2} - r_{m3} & \dots & s_{m-1,n-1} - r_{mn} & s_{m-1,n} - 0 \end{bmatrix}.$$

Adding ΔA , we can reduce A to the form 0^\sphericalangle ; for definiteness, we take A in the form

$$0^\sphericalangle := \begin{bmatrix} 0_{m-1,n} \\ * * \dots * \end{bmatrix}. \tag{45}$$

To preserve this form, we hereafter must take

$$s_{21} = \dots = s_{m1} = 0, \quad s_{ij} = r_{i-1,j-1} \quad (2 \leq i \leq m, 2 \leq j \leq n).$$

Write

$$(r_{00}, r_{01}, \dots, r_{0,n-1}) := (s_{11}, s_{12}, \dots, s_{1n}),$$

then

$$\Delta B^T = \begin{bmatrix} 0 - r_{12} & 0 - r_{13} & \dots & 0 - r_{1n} & 0 - 0 \\ r_{00} - r_{22} & r_{01} - r_{23} & \dots & r_{0,n-2} - r_{2n} & r_{0,n-1} - 0 \\ 0 - r_{32} & r_{11} - r_{33} & \dots & r_{1,n-2} - r_{3n} & r_{1,n-1} - 0 \\ 0 - r_{42} & r_{21} - r_{43} & \dots & r_{2,n-2} - r_{4n} & r_{2,n-1} - 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 - r_{m2} & r_{m-2,1} - r_{m3} & \dots & r_{m-2,n-2} - r_{mn} & r_{m-2,n-1} - 0 \end{bmatrix}.$$

If r_{ij} and $r_{i'j'}$ belong to the same diagonal of ΔB^T , then $i - j = i' - j'$. Hence, the diagonals of ΔB^T have no common r_{ij} , and so we can reduce the diagonals of B^T independently.

The first n diagonals of ΔB^T starting from the upper right corner are

$$\begin{aligned} &0, \quad (-r_{1n}, r_{0,n-1}), \quad (-\underline{r_{1,n-1}}, r_{0,n-2} - r_{2n}, \underline{r_{1,n-1}}), \\ &(-r_{1,n-2}, r_{0,n-3} - r_{2,n-1}, r_{1,n-2} - r_{3n}, r_{2,n-1}), \\ &(-\underline{r_{1,n-3}}, r_{0,n-4} - r_{2,n-2}, \underline{r_{1,n-3}} - r_{3,n-1}, r_{2,n-2} - r_{4n}, \underline{r_{3,n-1}}), \dots \end{aligned}$$

(we underline linearly dependent entries in each diagonal), adding them we make the first n diagonals of B^T as in O^\times .

The $(n + 1)$ st diagonal of ΔB^T is

$$\begin{cases} (r_{00} - r_{22}, r_{11} - r_{33}, \dots, r_{n-2,n-2} - r_{nm}) & \text{if } m = n, \\ (r_{00} - r_{22}, r_{11} - r_{33}, \dots, r_{n-2,n-2} - r_{nm}, r_{n-1,n-1}) & \text{if } m > n. \end{cases}$$

Adding it, we make the $(n + 1)$ st diagonal of B^T equal to zero.

If $m > n + 1$, then the $(n + 2)$ nd, \dots , m th diagonals of ΔB^T are

$$\begin{aligned} &(-r_{32}, r_{21} - r_{43}, r_{32} - r_{54}, \dots, r_{n,n-1}), \\ &\dots \\ &(-r_{m-n+1,2}, r_{m-n,1} - r_{m-n+2,3}, r_{m-n+1,2} - r_{m-n+3,4}, \dots, r_{m-2,n-1}). \end{aligned}$$

Each of these diagonals contains n elements. If n is even, then the length of each diagonal is even and its elements are linearly independent; we make the corresponding diagonals of B^T equal to zero. If n is odd, then the length of each diagonal is odd and the set of its odd-numbered elements is linearly dependent; we make all elements of the corresponding diagonals of B^T equal to zero except for their last elements (they correspond to the stars of \mathcal{P}_{nm} defined in (5)).

It remains to reduce the last $n - 1$ diagonals of B^T (the last $n - 2$ diagonals if $m = n$). The corresponding diagonals of ΔB^T are

$$\begin{aligned} &-r_{m2}, \\ &(-r_{m-1,2}, r_{m-2,1} - r_{m3}), \\ &(-r_{m-2,2}, r_{m-3,1} - r_{m-1,3}, r_{m-2,2} - r_{m4}), \\ &(-r_{m-3,2}, r_{m-4,1} - r_{m-2,3}, r_{m-3,2} - r_{m-1,4}, r_{m-2,3} - r_{m5}), \\ &\dots \\ &(-r_{m-n+3,2}, r_{m-n+2,1} - r_{m-n+4,3}, \dots, r_{m-2,n-3} - r_{m,n-1}), \end{aligned}$$

and, only if $m > n$,

$$(-r_{m-n+2,2}, r_{m-n+1,1} - r_{m-n+3,3}, \dots, r_{m-2,n-2} - r_{mn}).$$

Adding these diagonals, we make the corresponding diagonals of B^T equal to zero. To preserve the zero diagonals, we hereafter must take $r_{m2} = r_{m4} = r_{m6} = \dots = 0$ and arbitrary $r_{m1}, r_{m3}, r_{m5}, \dots$.

Recall that A has the form 0^\downarrow defined in (45). Since $r_{m1}, r_{m3}, r_{m5}, \dots$ are arbitrary, we can reduce A to the form

$$\begin{bmatrix} 0_{m-1,n} \\ * 0 * 0 \dots \end{bmatrix}$$

by adding ΔA ; these additions preserve the already reduced B . If $m = n$, then we can alternatively reduce A to the form

$$\begin{bmatrix} \dots\dots\dots \\ 0 0 \dots 0 \\ * 0 \dots 0 \\ 0 0 \dots 0 \\ * 0 \dots 0 \end{bmatrix}$$

preserving the form 0^\wedge of B .

7. Off-diagonal blocks of \mathcal{D} that correspond to summands of A_{can} of distinct types

Finally, we verify the condition (ii) of Lemma 4.3 for those off-diagonal blocks of \mathcal{D} (defined in Theorem 2.2(iii)) whose horizontal and vertical strips contain summands of A_{can} of different types.

7.1. Pairs of blocks $\mathcal{D}(H_m(\lambda), \Gamma_n)$

Due to Lemma 4.3(ii), it suffices to prove that each pair (B, A) of $n \times 2m$ and $2m \times n$ matrices can be reduced to exactly one pair of the form (15) by adding

$$(S^T H_m(\lambda) + \Gamma_n R, R^T \Gamma_n + H_m(\lambda) S), \quad S \in \mathbb{C}^{2m \times n}, \quad R \in \mathbb{C}^{n \times 2m}.$$

Taking $R = 0$ and $S = -H_m(\lambda)^{-1} A$, we reduce A to 0. To preserve $A = 0$, we hereafter must take S and R such that $R^T \Gamma_n + H_m(\lambda) S = 0$; that is,

$$S = -H_m(\lambda)^{-1} R^T \Gamma_n.$$

Hence, we can add

$$\Delta B = \Gamma_n R - \Gamma_n^T R H_m(\lambda)^{-T} H_m(\lambda)$$

to B . Write $P = \Gamma_n^T R$, then

$$\Delta B = \Gamma_n \Gamma_n^{-T} P - P (J_m(\lambda) \oplus J_m(\lambda)^{-T}).$$

Divide B and P into two blocks of size $n \times m$:

$$B = [M N], \quad P = [U V].$$

We can add to M and N the matrices

$$\Delta M := \Gamma_n \Gamma_n^{-T} U - U J_m(\lambda), \quad \Delta N := \Gamma_n \Gamma_n^{-T} V - V J_m(\lambda)^{-T}.$$

If $\lambda \neq (-1)^{n+1}$ then by (44) the eigenvalues of $\Gamma_n \Gamma_n^{-T}$ and the eigenvalues of $J_m(\lambda)$ and $J_m(\lambda)^{-T}$ are distinct, and we make $M = N = 0$.

If $\lambda = (-1)^{n+1}$ then

$$\Delta M = (-1)^n \begin{bmatrix} 0 & & & 0 \\ 2 & 0 & & \\ & \ddots & \ddots & \\ * & & 2 & 0 \end{bmatrix} U - U \begin{bmatrix} 0 & 1 & & 0 \\ & 0 & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{bmatrix},$$

$$\Delta N = (-1)^n \begin{bmatrix} 0 & & & 0 \\ 2 & 0 & & \\ & \ddots & \ddots & \\ * & & 2 & 0 \end{bmatrix} V + V \begin{bmatrix} 0 & & & 0 \\ 1 & 0 & & \\ & \ddots & \ddots & \\ * & & 1 & 0 \end{bmatrix}.$$

We reduce M to the form $0 \nearrow$ along its skew diagonals starting from the upper left corner, and N to the form $0 \searrow$ along its diagonals starting from the upper right corner.

7.2. Pairs of blocks $\mathcal{D}(H_m(\lambda), J_n(0))$

Due to Lemma 4.3(ii), it suffices to prove that each pair (B, A) of $n \times 2m$ and $2m \times n$ matrices can be reduced to exactly one pair of the form (16) by adding

$$(S^T H_m(\lambda) + J_n(0)R, R^T J_n(0) + H_m(\lambda)S), \quad S \in \mathbb{C}^{2m \times n}, \quad R \in \mathbb{C}^{n \times 2m}.$$

Taking $R = 0$ and $S = -H_m(\lambda)^{-1}A$, we reduce A to 0. To preserve $A = 0$ we hereafter must take S and R such that $R^T J_n(0) + H_m(\lambda)S = 0$; that is,

$$S = -H_m(\lambda)^{-1}R^T J_n(0).$$

Hence we can add

$$\begin{aligned} \Delta B &:= J_n(0)R - J_n(0)^T R H_m(\lambda)^{-T} H_m(\lambda) \\ &= J_n(0)R - J_n(0)^T R (J_m(\lambda) \oplus J_m(\lambda)^{-T}) \end{aligned}$$

to B .

Divide B and R into two blocks of size $n \times m$:

$$B = [M \ N], \quad R = [U \ V].$$

We can add to M and N the matrices

$$\Delta M := J_n(0)U - J_n(0)^T U J_m(\lambda), \quad \Delta N := J_n(0)V - J_n(0)^T V J_m(\lambda)^{-T}.$$

We reduce M as follows. Let $(u_1, u_2, \dots, u_n)^T$ be the first column of U . Then we can add to the first column \vec{b}_1 of M the vector

$$\begin{aligned} \Delta \vec{b}_1 &:= (u_2, \dots, u_n, 0)^T - \lambda(0, u_1, \dots, u_{n-1})^T \\ &= \begin{cases} 0 & \text{if } n = 1, \\ (u_2, u_3 - \lambda u_1, u_4 - \lambda u_2, \dots, u_n - \lambda u_{n-2}, -\lambda u_{n-1})^T & \text{if } n > 1. \end{cases} \end{aligned}$$

The elements of this vector are linearly independent if n is even, and they are linearly dependent if n is odd. We reduce \vec{b}_1 to zero if n is even, and to the form $(*, 0, \dots, 0)^T$ or $(0, \dots, 0, *)^T$ if n is odd. Then we successively reduce the other columns transforming M to 0 if n is even and to the form 0_{mm}^\uparrow if n is odd.

We reduce N in the same way starting from the last column.

7.3. Pairs of blocks $\mathcal{D}(\Gamma_m, J_n(0))$

Due to Lemma 4.3(ii), it suffices to prove that each pair (B, A) of $n \times m$ and $m \times n$ matrices can be reduced to exactly one pair of the form (17) by adding

$$(S^T \Gamma_m + J_n(0)R, R^T J_n(0) + \Gamma_m S), \quad S \in \mathbb{C}^{m \times n}, \quad R \in \mathbb{C}^{n \times m}.$$

Taking $R = 0$ and $S = -\Gamma_m^{-1}A$, we reduce A to 0. To preserve $A = 0$ we hereafter must take S and R such that $R^T J_n(0) + \Gamma_m S = 0$; that is, $S = -\Gamma_m^{-1}R^T J_n(0)$. Hence, we can add

$$\begin{aligned} \Delta B &:= J_n(0)R - J_n(0)^T R \Gamma_m^{-T} \Gamma_m \\ &= \begin{bmatrix} r_{21} & \dots & r_{2m} \\ \dots & \dots & \dots \\ r_{n1} & \dots & r_{nm} \\ 0 & \dots & 0 \end{bmatrix} - (-1)^{m+1} \begin{bmatrix} 0 & \dots & 0 \\ r_{11} & \dots & r_{1m} \\ \dots & \dots & \dots \\ r_{n-1,1} & \dots & r_{n-1,m} \end{bmatrix} \begin{bmatrix} 1 & 2 & * \\ & 1 & \ddots \\ & & \ddots & 2 \\ 0 & & & 1 \end{bmatrix} \end{aligned}$$

to B . We move along the columns of B starting from the first column and reduce B to 0 if n is even and to 0^\uparrow if n is odd.

Appendix A. A transformation that reduces a matrix to its miniversal deformation

In this section, we fix an $n \times n$ complex matrix A and a $(0, *)$ matrix \mathcal{D} of the same size such that

$$\mathbb{C}^{n \times n} = T(A) + \mathcal{D}(\mathbb{C}), \tag{46}$$

in which (see (18) and (21))

$$T(A) = \{C^T A + AC \mid C \in \mathbb{C}^{n \times n}\}, \quad \mathcal{D}(\mathbb{C}) = \bigoplus_{(i,j) \in \mathcal{I}(\mathcal{D})} \mathbb{C} E_{ij},$$

and all E_{ij} are the matrix units. We prove that the deformation $A + \mathcal{D}(\vec{\varepsilon})$ of A defined in (21) is miniversal. To this end, we construct an $n \times n$ matrix $\mathcal{S}(X)$ satisfying the conditions (i)–(iii) of Definition 3.2.

For each $P = [p_{ij}] \in \mathbb{C}^{n \times n}$, we write

$$\|P\| := \sqrt{\sum |p_{ij}|^2}, \quad \|P\|_{\mathcal{D}} := \sqrt{\sum_{(i,j) \notin \mathcal{I}(\mathcal{D})} |p_{ij}|^2}.$$

Note that

$$\|aP + bQ\| \leq |a| \|P\| + |b| \|Q\|, \quad \|PQ\| \leq \|P\| \|Q\|$$

for all $a, b \in \mathbb{C}$ and $P, Q \in \mathbb{C}^{n \times n}$; see [20, Section 5.6].

For every $n \times n$ matrix unit E_{ij} , we fix $F_{ij} \in \mathbb{C}^{n \times n}$ such that

$$E_{ij} + F_{ij}^T A + AF_{ij} \in \mathcal{D}(\mathbb{C}) \tag{47}$$

(F_{ij} exists by (46)); we take $F_{ij} = 0_n$ if $E_{ij} \in \mathcal{D}(\mathbb{C})$. Write

$$a := \|A\|, \quad f := \sum_{i,j} \|F_{ij}\|. \tag{48}$$

For each $n \times n$ matrix E , we construct a sequence

$$M_1 := E, M_2, M_3, \dots \tag{49}$$

of $n \times n$ matrices as follows: if $M_k = [m_{ij}^{(k)}]$ has been constructed, then M_{k+1} is defined by

$$A + M_{k+1} := (I_n + C_k)^T (A + M_k) (I_n + C_k) \tag{50}$$

in which

$$C_k := \sum_{i,j} m_{ij}^{(k)} F_{ij}. \tag{51}$$

In this section, we prove the following theorem.

Theorem 8.1. *Given $A \in \mathbb{C}^{n \times n}$ and a $(0, *)$ matrix \mathcal{D} of the same size that satisfy (46). Fix $\varepsilon \in \mathbb{R}$ such that*

$$0 < \varepsilon < \frac{1}{\max\{f(a + 1)(f + 2), 3\}} \quad (\text{see (48)}) \tag{52}$$

and define the neighborhood

$$U := \{E \in \mathbb{C}^{n \times n} \mid \|E\| < \varepsilon^5\}$$

of 0_n . Then for each matrix $E \in U$, the infinite product

$$S(E) := (I_n + C_1)(I_n + C_2)(I_n + C_3) \cdots \quad (\text{see (49)}) \tag{53}$$

is convergent,

$$A + D := S(E)^T (A + E) S(E) \in A + \mathcal{D}(\mathbb{C}) \tag{54}$$

and

$$\|S(E) - I_n\| < -1 + (1 + \varepsilon)(1 + \varepsilon^3)(1 + \varepsilon^5) \cdots, \quad \|D\| \leq \varepsilon^3. \tag{55}$$

The matrix $S(E)$ is a function of the entries of E ; replacing them by unknowns x_{ij} , we obtain a matrix $S(X)$ that satisfies the conditions (i)–(iii) of Definition 3.2.

The proof of Theorem 8.1 is based on two lemmas.

Lemma 8.1. *Let $\varepsilon \in \mathbb{R}$, $0 < \varepsilon < 1/3$, and let the sequence of real numbers*

$$\delta_1, \tau_1, \delta_2, \tau_2, \delta_3, \tau_3, \dots \tag{56}$$

be defined by induction:

$$\delta_1 = \tau_1 = \varepsilon^5, \quad \delta_{i+1} = \varepsilon^{-1}\delta_i\tau_i, \quad \tau_{i+1} = \tau_i + \varepsilon^{-1}\delta_i.$$

Then

$$0 < \delta_i < \varepsilon^{2i}, \quad 0 < \tau_i < \varepsilon^3 \quad \text{for all } i = 1, 2, \dots \tag{57}$$

Proof. Reasoning by induction, we assume that the inequalities (57) hold for $i = 1, \dots, k$. Then they hold for $i = k + 1$ since

$$\delta_{k+1} = \varepsilon^{-1}\delta_k\tau_k < \varepsilon^{-1}\varepsilon^{2k}\varepsilon^3 = \varepsilon^{2(k+1)}$$

and

$$\begin{aligned} \tau_{k+1} &= \tau_k + \varepsilon^{-1}\delta_k = \tau_{k-1} + \varepsilon^{-1}\delta_{k-1} + \varepsilon^{-1}\delta_k = \dots = \tau_1 + \varepsilon^{-1}(\delta_1 + \delta_2 + \dots + \delta_k) \\ &< \varepsilon^5 + \varepsilon^{-1}(\varepsilon^5 + \varepsilon^{-1}\varepsilon^5\varepsilon^5 + \varepsilon^6 + \varepsilon^8 + \varepsilon^{10} + \dots) \\ &= \varepsilon^5 + \varepsilon^4 + \varepsilon^8 + \varepsilon^5(1 + \varepsilon^2 + \varepsilon^4 + \dots) = \varepsilon^5 + \varepsilon^4 + \varepsilon^8 + \varepsilon^5/(1 - \varepsilon^2) \\ &< \varepsilon^5 + \varepsilon^4 + \varepsilon^8 + 2\varepsilon^5 < \varepsilon^4 + \varepsilon^8 + \varepsilon^4 < 3\varepsilon^4 < \varepsilon^3. \quad \square \end{aligned}$$

Lemma 8.2. Let $\varepsilon \in \mathbb{R}$ satisfy (52) and let $k \in \mathbb{N}$. Assume that the matrix $M_k = [m_{ij}^{(k)}]$ from (49) satisfies

$$\|M_k\|_{\mathcal{D}} < \delta_k, \quad \|M_k\| < \tau_k \quad (\text{see (56)}). \tag{58}$$

Then

$$\|M_{k+1}\|_{\mathcal{D}} < \delta_{k+1}, \quad \|M_{k+1}\| < \tau_{k+1} \tag{59}$$

and

$$\|C_k\| < \varepsilon^{-1}\delta_k \quad (\text{see (51)}). \tag{60}$$

Proof. By (47),

$$\sum_{i,j} m_{ij}^{(k)} E_{ij} + \sum_{i,j} m_{ij}^{(k)} F_{ij}^T A + \sum_{i,j} m_{ij}^{(k)} A F_{ij} \in \mathcal{D}(\mathbb{C}),$$

and so

$$M_k + C_k^T A + A C_k \in \mathcal{D}(\mathbb{C}). \tag{61}$$

If $(i, j) \in \mathcal{I}(\mathcal{D})$, then $E_{ij} \in \mathcal{D}(\mathbb{C})$ and $F_{ij} = 0$ by the definition of F_{ij} . If $(i, j) \notin \mathcal{I}(\mathcal{D})$, then $|m_{ij}^{(k)}| < \delta_k$ by the first inequality in (58). The inequality (60) holds because

$$\|C_k\| \leq \sum_{(i,j) \notin \mathcal{I}(\mathcal{D})} |m_{ij}^{(k)}| \|F_{ij}\| < \sum_{(i,j) \notin \mathcal{I}(\mathcal{D})} \delta_k \|F_{ij}\| = \delta_k f < \delta_k \varepsilon^{-1}.$$

By (50) and (58),

$$\begin{aligned} M_{k+1} &= M_k + C_k^T(A + M_k) + (A + M_k)C_k + C_k^T(A + M_k)C_k, \\ \|M_{k+1}\| &\leq \|M_k\| + 2\|C_k\|(\|A\| + \|M_k\|) + \|C_k\|\|A + M_k\|\|C_k\| \\ &< \tau_k + 2\delta_k f(a + \tau_k) + \delta_k f(a + \tau_k)\delta_k f \\ &= \tau_k + \delta_k f(a + \tau_k)(2 + \delta_k f) \\ &< \tau_k + \delta_k f(a + 1)(2 + f) < \tau_k + \delta_k \varepsilon^{-1} = \tau_{k+1}. \end{aligned} \tag{62}$$

By (61) and (62),

$$\begin{aligned} \|M_{k+1}\|_D &= \|C_k^T M_k + M_k C_k + C_k^T (A + M_k) C_k\| \\ &\leq 2\|C_k\| \|M_k\| + \|C_k\|^2 (\|A\| + \|M_k\|) \\ &< 2\delta_k f \tau_k + (\delta_k f)^2 (a + \tau_k) \\ &< \delta_k f \tau_k (2 + f(a + 1)) \\ &< \delta_k \tau_k f (2(a + 1) + f(a + 1)) < \delta_k \tau_k \varepsilon^{-1} = \delta_{k+1}, \end{aligned}$$

which proves (59). □

Proof of Theorem 8.1. Since $M_1 = E \in U$, $\|M_1\| < \varepsilon^5 = \delta_1 = \tau_1$. Hence, the inequalities (58) hold for $k = 1$. Reasoning by induction and using Lemma 8.2, we get

$$\|M_i\|_D < \delta_i, \quad \|M_i\| < \tau_i, \quad \|C_i\| < \varepsilon^{-1} \delta_i, \quad i = 1, 2, \dots,$$

and by (57)

$$\begin{aligned} \|C_1\| + \|C_2\| + \|C_3\| + \dots &< \varepsilon^{-1} (\delta_1 + \delta_2 + \delta_3 + \dots) \\ &< \varepsilon^{-1} (\varepsilon^2 + \varepsilon^4 + \varepsilon^6 + \dots) = \varepsilon (1 + \varepsilon^2 + \varepsilon^4 + \dots) \\ &= \varepsilon / (1 - \varepsilon^2) = 1 / (\varepsilon^{-1} - \varepsilon) < 1 / (3 - 3^{-1}). \end{aligned}$$

The infinite product (53) converges to some matrix $S(E)$ due to [33, Theorem 4] (which generalizes [28, Theorem 15.14]). By (50) and (51), the entries of each C_i are polynomials in the entries of E . Thus, the entries of each

$$S_k(E) := (I_n + C_1)(I_n + C_2) \cdots (I_n + C_k), \quad k = 1, 2, \dots$$

are polynomials in the entries of E . Since $S_k(E) \rightarrow S(E)$, the Weierstrass theorem on uniformly convergent sequences of analytic functions [28, Theorem 15.8] ensures that the entries of $S(E)$ are holomorphic functions in the entries of M .

The inclusion (54) holds since $A + M_i \rightarrow S(E)^T (A + E) S(E)$ and $\|M_i\|_D < \delta_i \rightarrow 0$ as $i \rightarrow \infty$. The inequalities (55) hold since for each $k \in \mathbb{N}$ we have

$$\begin{aligned} \|S_k(E) - I_n\| &= \|(I_n + C_1)(I_n + C_2) \cdots (I_n + C_k) - I_n\| \\ &\leq \sum_{i \leq k} \|C_i\| + \sum_{i < j \leq k} \|C_i\| \|C_j\| + \dots \\ &\leq -1 + (1 + \|C_1\|)(1 + \|C_2\|)(1 + \|C_3\|) \cdots \\ &< -1 + (1 + \varepsilon^{-1} \delta_1)(1 + \varepsilon^{-1} \delta_2)(1 + \varepsilon^{-1} \delta_3) \cdots \\ &< -1 + (1 + \varepsilon)(1 + \varepsilon^3)(1 + \varepsilon^5) \cdots \quad (\text{by (57)}) \end{aligned}$$

and

$$M_i \rightarrow D, \quad \|M_i\| \leq \tau_i < \varepsilon^3.$$

If $E = 0_n$ then all $M_i = C_i = 0_n$, and so $S(0_n) = I_n$. □

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