A note on a sum theorem for dimension $\mathcal{K}$-Ind

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A B S T R A C T

Main results are:
1. Let $Y$ be a closed subspace of a hereditarily normal $X$ such that $\mathcal{K}$-Ind$Y \leq n$ and $\mathcal{K}$-Ind$(X \setminus Y) \leq n$. Then $\mathcal{K}$-Ind$X \leq n$.
2. Let $X$ be a perfectly normal space. Then a finite sum theorem for dimension $\mathcal{K}$-Ind holds in $X$ if and only if $\mathcal{K}$-Ind is monotonic in $X$.

We denote by $\mathcal{K}$ a non-empty set of finite complete simplicial complexes.

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1. Introduction

In [4] inductive dimension function $\mathcal{K}$-Ind, where $\mathcal{K}$ is a non-empty set of finite simplicial complexes, was introduced (look at Definition 2.4). This dimension is an extension of the classical inductive dimension Ind, since $\{0, 1\}$-Ind$X = \text{Ind}X$ for every normal space. Generally,

$\mathcal{K}$-Ind$X \leq \text{Ind}X$.

In [4] it was proved that

$\mathcal{K}$-Ind$X = \text{Ind}X$

if and only if $\mathcal{K}$ contains a disconnected complex $K$.

One of the main questions concerning dimension $\mathcal{K}$-Ind is the following one:

Let a perfectly normal space $X$ be the union of its closed subspaces $X_i$, $i = 1, 2, \ldots$. Is it true that

$\mathcal{K}$-Ind$X = \sup\{\mathcal{K}$-Ind$X_i: i = 1, 2, \ldots\}$?

The answer is unknown even if $X = X_1 \cup X_2$.

Here we prove (Theorem 3.4) that the finite sum theorem for dimension $\mathcal{K}$-Ind holds for subspaces of a perfectly normal space $X$ if and only if dimension $\mathcal{K}$-Ind is monotonic in subspaces of $X$. The proof is based on the following

Finite Dowker theorem (Theorem 3.1). Let $Y$ be a closed subspace of a hereditarily normal space $X$ such that $\mathcal{K}$-Ind$Y \leq n$, $\mathcal{K}$-Ind$(X \setminus Y) \leq n$. Then $\mathcal{K}$-Ind$X \leq n$.

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Let us recall that C.H. Dowker proved (look at [1,2]) the following

**Theorem D.** Let a hereditarily normal space \( X \) be the union of its subspaces \( X_i \), \( i = 1, 2, \ldots \), such that \( \text{Ind} \, X_i \leq n \), \( i = 1, 2, \ldots \), and \( \bigcup \{X_i: i = 1, 2, \ldots, k\} \) is closed for \( k = 1, 2, \ldots \). Then \( \text{Ind} \, X \leq n \).

Theorem 3.1 implies a finite version of Theorem D for dimension \( K \)-Ind.

### 2. Preliminaries

**2.1.** In what follows \( K \) stands for a non-empty set of finite complete simplicial complexes \( K \), which we call complexes. For a complex \( K \) by \( v(K) \) we denote the set of all its vertices. A simplicial complex, which is the nerve of a finite family \( \alpha = \{A_1, \ldots, A_s\} \) of sets, is denoted by \( N(\alpha) \).

By a space we mean a topological normal \( T_1 \)-space. For a space \( X \) by \( \exp X \) we denote the set of all closed subsets of \( X \). By \( \text{Fin}_s(\exp X) \) we denote the set of all finite sequences \( \Phi = (F_1, \ldots, F_m) \), \( F_j \in \exp X \), \( j = 1, \ldots, m \).

**Definition 2.2.** ([3]) Let \( X \) be a space, \( K \) be a complex, and \( \Phi = (F_1, \ldots, F_m) \in \text{Fin}_s(\exp X) \). A sequence \( u = (U_1, \ldots, U_s) \), \( s \geq m \), of open subsets of \( X \) is said to be a \( K \)-neighbourhood of \( \Phi \) if \( F_j \subseteq U_j \), \( j = 1, \ldots, m \), and there is an embedding \( N(u) \subseteq K \). One can number vertices \( a_j \in v(K) \) so that the embedding \( N(u) \subseteq K \) is defined by the correspondence \( U_j \rightarrow a_j \).

**Definition 2.3.** ([3]) A set \( P \subseteq X \) is called a \( K \)-partition of \( \Phi = (F_1, \ldots, F_m) \) (notation: \( P \in \text{Part}(\Phi, K) \)) if \( P = X \setminus \bigcup u \), where \( u \) is a \( K \)-neighbourhood of \( \Phi \).

If a \( K \)-partition of \( \Phi \) exists, then \( N(\Phi) \subseteq K \). Put

\[
\text{Exp}_K(X) = \{\Phi \in \text{Fin}_s(\exp X): N(\Phi) \subseteq K\}. \tag{2.1}
\]

**Definition 2.4.** ([4]) To every space \( X \) one assigns the dimension \( K \)-Ind \( X \) which is an integer \( \geq -1 \) or \( \infty \). The dimension function \( K \)-Ind is defined in the following way:

\( \text{1.} \) \( K \)-Ind \( X = -1 \iff X = \emptyset \);
\( \text{2.} \) \( K \)-Ind \( X \leq n \), where \( n = 0, 1, \ldots \), if for every \( K \in K \) and \( \Phi \in \text{Exp}_K(X) \) there exists a partition \( P \in \text{Part}(\Phi, K) \) such that \( \text{K-Ind} \, P \leq n - 1 \);
\( \text{3.} \) \( K \)-Ind \( X = \infty \), if \( K \)-Ind \( X > n \) for all \( n \geq -1 \).

If the set \( K \) contains only one complex \( K \), we write \( K = K \) and \( K \)-Ind \( X = K \)-Ind \( X \).

**Theorem 2.5.** For every space \( X \), \( \{0, 1\} \)-Ind \( X = \text{Ind} \, X \). \( \square \)

**Theorem 2.6.** ([4]) If \( Y \) is a closed subspace of a space \( X \), then \( K \)-Ind \( Y \leq K \)-Ind \( X \). \( \square \)

**Theorem 2.7.** ([4]) If \( X = \bigoplus \{X_\alpha: a \in A\} \) is a discrete union of spaces \( X_\alpha \), then

\( K \)-Ind \( X = \sup\{K \text{-Ind} \, X_\alpha: a \in A\} \). \( \square \)

**Lemma 2.8.** If \( U \) is an open \( F_\alpha \)-subspace of a space \( X \), then \( U \) is a cozero-set, i.e. there exists a continuous function \( \varphi : X \rightarrow [0, 1] \) such that \( U = \varphi^{-1}(0, 1) \). \( \square \)

**Strong swelling lemma 2.9.** Let \( \Phi = (F_1, \ldots, F_m) \in \text{Fin}_s(\exp X) \). Then there exists a family \( u = (U_1, \ldots, U_m) \) of open subsets of \( X \) such that \( F_j \subseteq U_j \), \( j = 1, \ldots, m \), and \( N(\text{Cl}(u)) = N(\Phi) \), where \( \text{Cl}(u) = (\text{Cl}(U_1), \ldots, \text{Cl}(U_m)) \). \( \square \)

**Nerve lemma 2.10.** ([4]) Let \( Y \) be subspace of a space \( X \), \( \alpha = (A_1, \ldots, A_m) \) be a sequence of subsets of \( X \), and \( \beta = (B_1, \ldots, B_m) \) be a sequence of subsets of \( Y \) such that \( N(\alpha), N(\beta) \subseteq K \) and \( A_j \cap Y \subseteq B_j \), \( j = 1, \ldots, m \). Let \( C_j = A_j \cup B_j \) and \( \gamma = (C_1, \ldots, C_m) \). Then \( N(\gamma) \subseteq K \). \( \square \)

### 3. Main results

Let \( K \) be a non-empty set of complexes and let \( X \) be a hereditarily normal space.

**Theorem 3.1.** Let \( Y \) be a closed subspace of a hereditarily normal space \( X \) such that \( K \)-Ind \( Y \leq n \), \( K \)-Ind \( (X \setminus Y) \leq n \). Then \( K \)-Ind \( X \leq n \).
Proof. We shall apply induction with respect to $n$. For $n = -1$ the theorem is obvious. Assume that the corresponding statements hold for dimensions less than $n \geq 0$ and consider a hereditarily normal space $X$ satisfying the assumption of our theorem. Let $K \in \mathcal{K}$, $\Phi = (F_1, \ldots, F_m) \in \text{Exp}_k(X)$, and $\Phi|_Y = (F_1 \cap Y, \ldots, F_m \cap Y)$. Then $\Phi|_Y \in \text{Exp}_k(Y)$. Since $\mathcal{K}$-$\text{Ind} Y \leq n$, there is a family $u = (U_1, \ldots, U_m)$ of open subsets of $Y$ such that

$$ F_j \cap Y \subset U_j, \quad j = 1, \ldots, m; \quad (3.1) $$
$$ N(u) \subset K; \quad (3.2) $$
$$ \mathcal{K}$-$\text{Ind}(Y \setminus U_1 \cup \cdots \cup U_m) \leq n - 1. \quad (3.3) $$

Put $P = Y \setminus U_1 \cup \cdots \cup U_m$ and $Z = X \setminus P$. The family $u$ is an open cover of a normal space $Y \setminus P$. Hence there exist closed subsets $A_j$ of a space $Y \setminus P$ such that

$$ F_j \cap Y \subset A_j \subset U_j; \quad (3.4) $$
$$ A_1 \cup \cdots \cup A_m = Y \setminus P. \quad (3.5) $$

Put $\alpha = (A_1, \ldots, A_m)$. From (3.2) and (3.4) it follows that

$$ N(\alpha) \subset K. \quad (3.6) $$

Since $Y \setminus P$ is closed in $Z$, the sets $B_j = A_j \cup F_j$, $j = 1, \ldots, m$, are closed in $Z$. Put $\beta = (B_1, \ldots, B_m)$. The condition $\Phi \in \text{Exp}_k(X)$ is equivalent to

$$ N(\Phi) \subset K. \quad (3.7) $$

From (3.6), (3.7) and the Nerve lemma (Lemma 2.10) it follows that

$$ N(\beta) \subset K. \quad (3.8) $$

Consequently, according to the Strong swelling lemma (Lemma 2.9) there exists a family $\nu = (V_1, \ldots, V_m)$ of open subsets of $Z$ such that

$$ B_j \subset V_j, \quad j = 1, \ldots, m; \quad (3.9) $$
$$ N(\delta) = N(\beta) \subset K, \quad (3.10) $$

where $\delta = (D_1, \ldots, D_m)$ and $D_j = \text{Cl}_Z(V_j)$.

Put $E_j = D_j \setminus Y$, $j = 1, \ldots, m$, and $\epsilon = (E_1, \ldots, E_m)$. The sets $E_j$ are closed in $X \setminus Y$ and

$$ N(\epsilon) \subset K \quad (3.11) $$

according to (3.10). But $\mathcal{K}$-$\text{Ind}(X \setminus Y) \leq n$. Consequently, according to (3.11) there exists a family $\omega = (W_1, \ldots, W_m)$ of open subsets of $X \setminus Y$ such that

$$ E_j \subset W_j, \quad j = 1, \ldots, m; \quad (3.12) $$
$$ N(\omega) \subset K; \quad (3.13) $$
$$ \mathcal{K}$-$\text{Ind} \omega \leq n - 1, \quad (3.14) $$

where

$$ Q = X \setminus Y \cup W_1 \cup \cdots \cup W_m. \quad (3.15) $$

Put $G_j = V_j \cup W_j$, $j = 1, \ldots, m$, and $\gamma = (G_1, \ldots, G_m)$.

From (3.10), (3.13), and the Nerve lemma (Lemma 2.10) it follows that

$$ N(\gamma) \subset K. \quad (3.16) $$

Condition (3.9) implies that

$$ F_j \subset G_j, \quad j = 1, \ldots, m. \quad (3.17) $$

Consequently, $\gamma$ is a $K$-neighbourhood of $\Phi$ in $X$. Then the set

$$ R = X \setminus G_1 \cup \cdots \cup G_m \quad (3.18) $$

is a $K$-partition of $\Phi$ in $X$. We claim that

$$ R = P \cup Q. \quad (3.19) $$
To check (3.19) it suffices to prove that
\[ R \cap Y = P; \]  
\[ R \cap (X \setminus Y) = Q. \]  
(3.20)

From definition of \( G_j \) it follows that
\[ G_j \cap Y = V_j \cap Y. \]  
(3.21)

Hence
\[ (G_1 \cup \cdots \cup G_m) \cap Y = (V_1 \cup \cdots \cup V_m) \cap Y. \]  
Condition (3.22) is equivalent to
\[ Y \setminus G_1 \cup \cdots \cup G_m = Y \setminus V_1 \cup \cdots \cup V_m. \]  
(3.23)

From (3.18) it follows that
\[ Y \setminus G_1 \cup \cdots \cup G_m = R \cap Y. \]  
(3.24)

From (3.24) and (3.25) we get (3.20).

On the other hand, \( V_j \subset Z \) implies that \((V_1 \cup \cdots \cup V_m) \cap P = \emptyset. \) Consequently, \( P \subset Y \setminus V_1 \cup \cdots \cup V_m \subset (\text{in view of (3.9)}) \subset Y \setminus B_1 \cup \cdots \cup B_m \subset (\text{because of } B_j = A_j \cup F_j \subset Y \setminus A_1 \cup \cdots \cup A_m = (\text{in accordance with (3.5)}) = P. \)

Hence
\[ P = Y \setminus V_1 \cup \cdots \cup V_m. \]  
(3.25)

Conditions (3.12) and (3.26) yield
\[ V_j \setminus Y \subset W_j, \quad j = 1, \ldots, m. \]  
(3.26)

Consequently, the definition of \( G_j \) implies that
\[ G_j \cap (X \setminus Y) = W_j. \]  
(3.27)

Thus, the condition (3.21) is checked as well. Hence the equality (3.19) is proved. Since \( P \cap Q = \emptyset, \) we have \( Q = R \setminus P. \)

On the other hand,
\[ \mathcal{K} \text{-Ind } P \leq (\text{in view of (3.3)}) \leq n - 1; \]
\[ \mathcal{K} \text{-Ind } Q \leq (\text{because of (3.14)}) \leq n - 1. \]

Consequently, by the inductive assumption we have
\[ \mathcal{K} \text{-Ind } R \leq n - 1. \]  
(3.28)

The condition (3.29) implies that \( \mathcal{K} \text{-Ind } X \leq n. \) \( \square \)

Let us consider the following properties of a space \( X: \)

- \((\mu_n)\) For each subspace \( Y \subset X \) and every open subspace \( U \) of \( Y, \) if \( \mathcal{K} \text{-Ind } Y \leq n, \) then \( \mathcal{K} \text{-Ind } U \leq n. \)
- \((\mu^0_n)\) For each subspace \( Y \subset X \) and every open \( F_n \text{-subspace } U \) of \( Y, \) if \( \mathcal{K} \text{-Ind } Y \leq n, \) then \( \mathcal{K} \text{-Ind } U \leq n. \)
- \((\sigma_n)\) For each subspace \( Y \subset X \) and every pair \( Y_1, Y_2 \) of closed subspaces of \( Y \) such that \( Y = Y_1 \cup Y_2, \) if \( \mathcal{K} \text{-Ind } Y_i \leq n, \) then \( \mathcal{K} \text{-Ind } Y \leq n. \)

As a corollary of Theorem 3.1 we have

**Proposition 3.2.** If a hereditarily normal space \( X \) has property \((\mu_n),\) then it also has property \((\sigma_n).\)
Proof. Consider a subspace \( Y \subset X \) and a pair \( Y_1, Y_2 \) of closed subspaces of \( Y \) such that \( Y = Y_1 \cup Y_2 \) and \( K\text{-Ind} Y_i \leq n \), \( i = 1, 2 \). By virtue of \((\mu_n)\) the set \( Y \setminus Y_1 \) satisfies the inequality \( K\text{-Ind}(Y \setminus Y_1) \leq n \). Applying Theorem 3.1 to the space \( Y \) and the pair \( Y_1, Y \setminus Y_1 \) we obtain the inequality \( K\text{-Ind} Y \leq n \). \( \square \)

**Proposition 3.3.** If a hereditarily normal space \( X \) has property \((\sigma_n)\), then it also has property \((\mu_0^n)\).

**Proof.** Let \( Y \subset X \) and let \( U \) be an open \( F_\sigma \)-set in \( Y \). Then there exists a continuous function \( f : Y \to I \) such that \( U = f^{-1}((0, 1]) \) (see Lemma 2.8). The sets \( B_i = f^{-1}([1/i+1, 1/i]) \), \( i = 1, 2, \ldots \), are closed in \( Y \). By the closed subspace theorem (Theorem 2.6) we have

\[
K\text{-Ind} B_i \leq n, \quad i = 1, 2, \ldots \tag{3.30}
\]

Consider the sequences

\[
B_{2i+1} = 0, 1, 2, \ldots; \quad B_{2i+2} = 0, 1, 2, \ldots.
\]

They are discrete. Put

\[
A_1 = \bigcup \{B_{2i+1}, \ i = 0, 1, 2, \ldots\}; \quad A_2 = \bigcup \{B_{2i+2}, \ i = 0, 1, 2, \ldots\}.
\]

By the Discrete sum theorem (Theorem 2.7) and (3.30) we have

\[
K\text{-Ind} A_1 \leq n; \quad K\text{-Ind} A_2 \leq n.
\]

But \( A_1 \cup A_2 = U \). Consequently, property \((\sigma_n)\) yields \( K\text{-Ind} U \leq n \). \( \square \)

Since properties \((\mu_n)\) and \((\mu_0^n)\) are equivalent in perfectly normal spaces, from Propositions 3.2 and 3.3 we get

**Theorem 3.4.** Properties \((\mu_n)\) and \((\sigma_n)\) are equivalent in the class of perfectly normal spaces. \( \square \)

**Question 3.5.** Does a perfectly normal space \( X \) satisfy property \((\sigma_n), n = 0, 1, 2, \ldots \), for an arbitrary \( K \)?

**Remark 3.6.** The answer is “yes” if \( K \) contains a disconnected complex \( K \). In fact, in this case, \( K\text{-Ind} X = \text{Ind} X \) (look at [4]) for every normal space \( X \), and dimension \( \text{Ind} \) satisfies the countable sum theorem in the class of all perfectly normal spaces (look at [1,2]).

**References**