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Note

On the structure of the adjacency matrix of the line digraph of a regular digraph

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Abstract

We show that the adjacency matrix M of the line digraph of a d-regular digraph D on n vertices can be written as $M = AB$, where the matrix A is the Kronecker product of the all-ones matrix of dimension d with the identity matrix of dimension n and the matrix B is the direct sum of the adjacency matrices of the factors in a dicycle factorization of D.

Keywords: Line digraph; Adjacency matrix; de Bruijn digraph

0. Introduction

Line digraphs of regular digraphs and their generalizations are important in the design of point-to-point interconnection networks for parallel computers and distributed systems. For instance, de Bruijn digraphs and Reddy–Pradhan–Kuhl digraphs, which are important topologies for interconnection networks, are all examples of line digraphs of regular digraphs (see, e.g., [1,2,5]). In this note, we describe a special regularity property of the adjacency matrix of the line digraph of a regular digraph. Before stating formally our main result, we recall the necessary graph-theoretic terminology.

A (*finite*) *directed graph*, for short *digraph*, consists of a non-empty finite set of elements called *vertices* and a (possibly empty) finite set of ordered pairs of vertices called *arcs*. The digraphs considered here are without multiple arcs. We denote by $D = (V, A)$ a digraph with vertex-set $V(D)$ and arc-set $A(D)$. A *labeling* of the vertices of a digraph D is a function $l : V(D) \longrightarrow L$, where L is a set of labels. Chosen a bijective labeling, the *adjacency matrix* of a digraph D with n vertices, denoted by $M(D)$, is the $n \times n$ $(0, 1)$-matrix with ijth element defined by $M_{i,j}(D) = 1$ if $(v_i, v_j) \in A(D)$ and $M_{i,j}(D) = 0$, otherwise. For any vertex $v_i \in V(D)$ of a digraph D, let $d_D^-(v_i) := |\{v_j : (v_j, v_i) \in A(D)\}|$ and $d_D^+(v_i) := |\{v_j : (v_i, v_j) \in A(D)\}|$. A digraph D is said to be *d-regular* if, for every vertex $v_i \in V(D)$, $d_D^-(v_i) = d_D^+(v_i) = d$. A digraph H is a *subdigraph* of a digraph D if $V(H) \subseteq V(D)$ and $A(H) \subseteq A(D)$. A subdigraph H of a digraph D is said to be a *spanning subdigraph* of D, or equivalently, a *factor* of D, if $V(H) = V(D)$. A *decomposition* of a digraph D is a set $\{H_1, H_2, \ldots, H_k\}$ of subdigraphs of D whose arc-sets are exactly the classes of a partition of $A(D)$. A *factorization* of a digraph D, if there exists one, is a decomposition of D into factors. A *dicycle factor* H of a digraph D is a spanning subdigraph of D such that $M(H)$ is a permutation matrix. The *disjoint union* of

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digraphs $D_1, D_2, \ldots, D_k$,is the digraph with vertex-set $\biguplus_{i=1}^{k} V(D_i)$, and arc-set $\biguplus_{i=1}^{k} A(D_i)$. Then a dicycle factor H of a digraph D is a spanning subdigraph of D and it is the disjoint union of dicycles. A *dicycle factorization* is a factorization into dicycle factors. The *line digraph* of a digraph D, denoted by $\overrightarrow{L}D$, is defined as follows: the vertex-set of $\overrightarrow{L}D$ is $A(D)$; for $v_h, v_i, v_j, v_k \in V(D)$, $((v_h, v_i), (v_j, v_k)) \in A(\overrightarrow{L}D)$ if and only if $v_i = v_j$. Kronecker product and direct sum of matrices M and N are, respectively, denoted by $M \otimes N$ and $M \oplus N$. The identity matrix and the all-ones matrix of size n are, respectively, denoted by I_n and J_n. In the next section, we prove the following theorem:

Theorem. *Let D be a d-regular digraph on n vertices and let* $\{H_1, H_2, \ldots, H_d\}$ *be a dicycle factorization of D. Then there is a labeling of* $V(\overrightarrow{L}D)$ *such that*

$$M(\overrightarrow{L}D) = (J_d \otimes I_n) \bigoplus_{i=1}^{d} M(H_i).$$

1. Proof of the Theorem

The proof of the theorem is based on two simple observations and a result proved by Hasunuma and Shibata [4] (see also Kawai et al. [6]).

Lemma 1. *Let D be a d-regular digraph. Then D has a dicycle factorization. In particular, if* $\{H_1, H_2, \ldots, H_d\}$ *is a dicycle factorization of D then* $M(H_1), M(H_2), \ldots, M(H_d)$ *are permutation matrices such that*

$$M(D) = \sum_{i=1}^{d} M(H_i).$$

Two digraphs D and D' are said to be *isomorphic* if there is a permutation matrix P such that $P \cdot M(D) \cdot P^{-1} = M(D')$. If D and D' are isomorphic we then write $D \cong D'$. An *n-dicycle*, denoted by $\overrightarrow{C}_n$, is a digraph with vertex-set $\{v_1, v_2, \ldots, v_n\}$ and arc-set $\{(v_1, v_2), \ldots, (v_{n-1}, v_n), (v_n, v_1)\}$. A *d-spiked n-dicycle* is the digraph obtained from $\overrightarrow{C}_n$ as follows: for every vertex $v_i \in V(\overrightarrow{C}_n)$, we add d new vertices $w_1, w_2, \ldots, w_d$; we connect $v_i \in (\overrightarrow{C}_n)$ to the vertices $w_1, w_2, \ldots, w_d$, obtaining the arcs $(v_i, w_1), (v_i, w_2), \ldots, (v_i, w_d)$.

Lemma 2. *Let D be a d-spiked n-dicycle. Then* $D \cong \overrightarrow{L}D$.

Let D be a digraph and let H be a spanning subdigraph of D. The *growth* of D derived by H is the digraph denoted by $\Upsilon_D(H)$ and defined as follows: for every pair of vertices $v_i, v_j \in V(D)$, if $(v_i, v_j) \in A(H)$ then $(v_i, v_j) \in A(\Upsilon_D(H))$; for every vertex $v_i \in V(D)$, we add new vertices $w_1, w_2, \ldots, w_l$, where $l = d_D^+(v_i) - d_H^+(v_i)$; we connect $v_i \in V(D)$ to the vertices $w_1, w_2, \ldots, w_l$, obtaining the arcs $(v_i, w_1), (v_i, w_2), \ldots, (v_i, w_l)$.

Lemma 3 (*Hasunuma and Shibata [4]*). *If* $\{H_1, H_2, \ldots, H_k\}$ *is a decomposition of a digraph D then*

$$\{\overrightarrow{L}\Upsilon_D(H_1), \overrightarrow{L}\Upsilon_D(H_2), \ldots, \overrightarrow{L}\Upsilon_D(H_k)\}$$

is a decomposition of a digraph $D' \cong \overrightarrow{L}D$.

Proof of the Theorem. Let D be a d-regular digraph on n vertices $v_1, v_2, \ldots, v_n$. Let $\{H_1, H_2, \ldots, H_d\}$ be a dicycle factorization of D. The vertices of $H_j \in \{H_1, H_2, \ldots, H_d\}$ are denoted as $(H_j, v_1), (H_j, v_2), \ldots, (H_j, v_n)$. Let us construct $\Upsilon_D(H_j)$. For every vertex $(H_j, v_i) \in V(H_j)$, we add $d-1$ new vertices to H_j. We label these new vertices by pairs of the form (H_l, v_m), for all $l \neq j$ and v_m such that $(v_i, v_m) \in A(H_l)$. In addition, $((H_j, v_i), (H_l, v_m)) \in A(\Upsilon_D(H_j))$. The digraph $\Upsilon_D(H_j)$ has $n \cdot d$ vertices. If we label the row number $(j-1)n + i$ of $M(\Upsilon_D(H_j))$ by the vertex (H_j, v_i), the adjacency matrix of $\Upsilon_D(H_j)$ is the $(d \cdot n) \times (d \cdot n)$ block-matrix

$$M(\Upsilon_D(H_j)) = \begin{pmatrix} \mathbf{0} \\ X_j \\ \mathbf{0} \end{pmatrix},$$

where

$$X_j = (M(H_1)\ \ M(H_2)\ \ \cdots\ \ M(H_j)\ \ \cdots\ \ M(H_{d-1})\ \ M(H_d)).$$

Notice that $M(H_j)$ is the jjth block of $M(\Upsilon_D(H_j))$. Thus, we have

$$N = \sum_{i=j}^{d} M(\Upsilon_D(H_i)) = \begin{pmatrix} M(H_1) & M(H_2) & \cdots & M(H_d) \\ M(H_1) & M(H_2) & \cdots & M(H_d) \\ \vdots & \vdots & \ddots & \vdots \\ M(H_1) & M(H_2) & \cdots & M(H_d) \end{pmatrix}$$

$$= (J_d \otimes I_n) \bigoplus_{i=j}^{d} M(H_j).$$

Observe that, for every $1 \leqslant j \leqslant d$, $\Upsilon_D(H_j)$ is the disjoint union of the d-spiked cycles corresponding to the orbits of the permutation associated to H_j. It follows from Lemma 2 that, for every $1 \leqslant j \leqslant d$,

$$\Upsilon_D(H_j) \cong \overrightarrow{L}\, \Upsilon_D(H_j).$$

Then, for the chosen labeling,

$$M(\Upsilon_D(H_j)) = M(\overrightarrow{L}\, \Upsilon_D(H_j))$$

and

$$N = \sum_{j=1}^{d} M(\Upsilon_D(H_j)) = \sum_{j=1}^{d} M(\overrightarrow{L}\, \Upsilon_D(H_j)).$$

Now, by Lemma 3, $N = M(\overrightarrow{L}\, D)$. $\square$

Remark. The graph operation transforming a digraph D in its line digraph can be naturally iterated: $\overrightarrow{L}^k D := \overrightarrow{L}\, \overrightarrow{L}^{k-1} D$. Let Σ be an alphabet of cardinality d and let Σ^k be the set of all the words of length k over Σ. The *d-ary k-dimensional de Bruijn digraph*, denoted by $B(d,k)$, is defined as follows: the vertex-set of $B(d,k)$ is $V(B(d,k)) = \Sigma^k$; for every pair of vertices v_i, v_j, we have $(v_i, v_j) \in A(B(d,k))$ if and only if the last $k-1$ letters of v_i are the same as the first $k-1$ letters of v_j. Let K_d^+ be the complete digraph on d vertices with a loop at each vertex. Fiol et al. [3] proved that $B(d,k) \cong \overrightarrow{L}^{k-1} K_d^+$. This result, together with the theorem, gives

$$M(B(d,2)) \cong (J_d \otimes I_d) \bigoplus_{i=1}^{d} M(H_i),$$

where $\{H_1, H_2, \ldots, H_d\}$ is any dicycle factorization of K_d^+.

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