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A central closure construction for certain algebra extensions. Applications to Hopf actions[☆]

Christian Lomp

*Centro de Matemática da Universidade do Porto, Faculdade de Ciências, Rua Campo Alegre 687,
4169-007 Porto, Portugal*

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Abstract

Algebra extensions $A \subseteq B$ where A is a left B -module such that the B -action extends the multiplication in A are ubiquitous. We encounter examples of such extensions in the study of group actions, group gradings or more general Hopf actions as well as in the study of the bimodule structure of an algebra. In this paper we are extending R. Wisbauer's method of constructing the central closure of a semiprime algebra using its multiplication algebra to those kinds of algebra extensions. More precisely if A is a k -algebra and B some subalgebra of $\text{End}(A)$ that contains the multiplication algebra of A , then the self-injective hull \widehat{A} of A as B -module becomes a k -algebra provided A does not contain any nilpotent B -stable ideals. We show that under certain assumptions \widehat{A} can be identified with a subalgebra of the Martindale quotient ring of A . This construction is then applied to Hopf module algebras.

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E-mail address: clomp@fc.up.pt (C. Lomp).

1. Introduction

Let k be a commutative ring with unit. All k -algebras in this paper are considered to be associative with unit. Unadorned tensor products are taken over k and $\text{End}(-)$ resp. $\text{Hom}(-, -)$ refer to k -linear maps. Let A be a k -algebra. For any $a \in A$, denote by L_a the k -linear map $L_a \in \text{End}(A)$ with $L_a(x) = ax$ for all $x \in A$. Denote by $R_a \in \text{End}(A)$ the k -linear map $R_a(x) = xa$ for all $x \in A$. The k -subalgebra of $\text{End}(A)$ generated by the maps L_a is denoted by $L(A)$. The *multiplication algebra* $M(A)$ of A is the k -subalgebra of $\text{End}(A)$ generated by all maps L_a and R_a ; i.e.

$$M(A) := \langle \{L_a, R_a \mid a \in A\} \rangle \subseteq \text{End}(A).$$

The k -algebra A is a cyclic left $M(A)$ -module whose submodules are precisely the two-sided ideals of A .

Definition 1.1. We say that an extension $A \subseteq B$ of k -algebras is an *extension with additional module structure* φ , if there exists a ring homomorphism

$$\varphi : B \rightarrow \text{End}(A)$$

such that $\varphi(a) = L_a$ for all $a \in A$. We denote the left B -module action on A by \cdot , i.e.

$$b \cdot a := \varphi(b)(a)$$

for all $a \in A$ and $b \in B$.

Obviously, A becomes a cyclic left B -module. We will call a left ideal B -stable if it is a B -submodule of A . Let us denote by α the left B -linear map $\alpha : B \rightarrow A$ mapping an element b of B to $b \cdot 1_A$.

Example 1.2. Let $B := A^e := A \otimes A^{op}$ be the enveloping algebra of A and define a ring homomorphism φ from A^e to $\text{End}(A)$ by $\varphi(a \otimes b) := L_a \circ R_b$. Identifying A with $A \otimes 1 \subseteq A^e$, we get that $A \subseteq A^e$ is an extension with additional module structure φ . Note that $\text{Im}(\varphi) = M(A)$.

Example 1.3. Let G be a group acting as (k -linear) automorphisms on A , i.e. there exists a group homomorphism $\eta : G \rightarrow \text{Aut}_k(A)$. We will use the notation $a^g := \eta(g)(a)$ for all $a \in A$ and $g \in G$. Define the skew-group ring $A\#G$ whose underlying A -submodule is the free left A -module with basis $\{\bar{g} \mid g \in G\}$ and whose multiplication is given by $(a\#\bar{g})(b\#\bar{h}) = ab^g\bar{gh}$. We might consider A as a subring of $A\#G$ by the map $A \rightarrow A\#G$ sending $a \mapsto a\#\bar{e}$ for all $a \in A$ where e is the neutral element of G . An action of $A\#G$ on A is given by the ring homomorphism $\varphi : A\#G \rightarrow \text{End}(A)$ with $\varphi(a\#\bar{g}) := L_a \circ \eta(g)$. Then $A \subseteq A\#G$ is an extension with additional module structure φ .

Example 1.4. Let $\delta \in \text{Der}_k(A)$ be a k -linear derivation of A . Consider the ring of differential operators $B = A[X; \delta]$, i.e. as an A -module, B is equal to $A[X]$ but the multiplication is constrained by $Xa - aX = \delta(a)$. Define a ring homomorphism

$\varphi : A[X; \delta] \rightarrow \text{End}(A)$ by $\varphi(aX^n) := L_a \circ \delta^n$. Then $A \subseteq A[X; \delta]$ is an extension with additional module structure φ .

Example 1.5. Let H be a k -Hopf algebra acting on A . Denote the action of an element $h \in H$ on A by $\lambda_h \in \text{End}(A)$. The smash product $A\#H$ of A and H is the A -module $A \otimes H$ with multiplication given by $(a\#h)(b\#g) := \sum_{(h)} a(h_1 \cdot b)\#h_2g$ where $\Delta(h) = \sum_{(h)} h_1 \otimes h_2$ is the comultiplication of h . Define $\varphi : A\#H \rightarrow \text{End}(A)$ by $\varphi(a\#h) := L_a \circ \lambda_h$. Then $A \subseteq A\#H$ is an extension with additional module structure φ .

For a group G we might choose $H = k[G]$ and recover Example 1.3. For the trivial Lie algebra $\mathfrak{g} = k$ and its enveloping algebra $H = U(\mathfrak{g}) = k[X]$ we recover Example 1.4.

If $A \subseteq B$ is an extension with additional module structure then $A \cap \text{Ann}_B(A) = 0$. Hence

$$A \subseteq B/\text{Ann}_B(A) \simeq \text{Im}(\varphi) \subseteq \text{End}(A)$$

is again an extension with additional module structure. Thus we might replace B by its image in $\text{End}(A)$ and restrict ourselves to extensions of A inside $\text{End}(A)$; where we identify A with $L(A)$.

Example 1.6. Let C be a k -bialgebra and A a right C -comodule algebra with comodule structure $\rho : A \rightarrow A \otimes C$. For any $f \in C^*$ define an action on A by $f \cdot a := (1 \otimes f)\rho(a)$, for any $a \in A$. If we write $\rho(a) = \sum_{(a)} a_0 \otimes a_1$ then $f \cdot a = \sum_{(a)} a_0 f(a_1)$. This defines an action of C^* on A , i.e. we get a ring homomorphism $\phi : C^* \rightarrow \text{End}(A)$. Let B be the subalgebra of $\text{End}(A)$ generated by $L(A)$ and $\text{Im}(\phi)$ then $A \subseteq B$ is an extension with additional module structure. All left ideals of A which are right C -comodules are B -stable. On the other hand, if C is a free k -module, then the B -stable left ideals of A are precisely the left ideals which are right C -comodules.

As an application one might consider G -graded algebras A (where G is a monoid) as $k[G]$ -comodule algebras. In order to study the G -graded left ideals of A one studies the B -stable left ideals of A where B is the subalgebra of $\text{End}(A)$ generated by $L(A)$ and $(k[G])^*$.

If we want to investigate two-sided ideals that are stable under a given action we have to restrict to extensions $A \subseteq B$ with additional module structure φ such that $M(A) \subseteq \text{Im}(\varphi)$. In some cases this happens automatically. For instance let A be a k -algebra with involution $*$. Let B be the subalgebra of $\text{End}(A)$ generated by A and $*$. Since for any $a \in A$:

$$R_a = * \circ L_a^* \circ *$$

we get $M(A) \subseteq B$. This means (as it is well-known) that any left ideal of A which is stable under $*$ is a two-sided ideal. Note that B can be seen as the factor ring of the skew-group ring $A^e\#G$ where $G = \{id, g\}$ is the cyclic group of order two and $g \in \text{Aut}(A^e)$ is given by

$$(a \otimes b)^g := b^* \otimes a^*.$$

In this case we have that $A \subseteq A^e\#G$ is an extension with additional module structure.

Let $A \subseteq B$ be an arbitrary extension with additional module structure. Recall the B -linear map $\alpha : B \rightarrow A$ with $(b)\alpha := b \cdot 1_A$ for all $b \in B$. Note that $b \cdot a = b \cdot (a \cdot 1_A) = (ba) \cdot 1_A =$

$(ba)\alpha$ for any $a \in A$ and $b \in B$. In particular $a' \cdot a = a'a$ holds for all $a, a' \in A$. Thus the map $\beta : A \rightarrow B$ with $(a)\beta := a1_B$ is A -linear and lets α split as A -module homomorphism. Hence as A -modules we have $B = A1_B \oplus \text{Ker}(\alpha)$.

We define the submodule of B -invariants of a left B -module as follows:

Definition 1.7. For any left B -module M we denote

$$M^B := \{m \in M \mid \forall b \in B, a \in A : b \cdot m = (b)\alpha m\}.$$

Note that for any $m \in M^B$ and any $a \in A$ and $b \in B$ we have

$$b \cdot (am) = (ba) \cdot m = (ba)\alpha m = (b \cdot a) m.$$

The converse holds as well, i.e. if $b \cdot (am) = (b \cdot a) m$ for all $b \in B$ and $a \in A$ then $m \in M^B$.

We can easily determine some elementary properties of B -invariants.

Lemma 1.8. Let $A \subseteq B$ be an extension with additional module structure. The following properties hold:

- (1) for all $M \in B\text{-Mod}$ we have $M^B = \text{Ann}_M(\text{Ker}(\alpha))$.
- (2) $\Psi : \text{End}_B(A) \rightarrow A$ with $\Psi(f) := (1_A)f$ is an injective ring homomorphism with image $\text{Im}(\Psi) = A^B$.
- (3) $\Psi_M : \text{Hom}_B(A, M) \rightarrow M^B$ with $\Psi_M(f) := (1_A)f$ is an isomorphism of left A^B -modules.
- (4) The isomorphisms Ψ_M are natural transformations between the functors $\text{Hom}_B(A, -)$ and $(-)^B$.

Proof. (1) If $m \in M^B$, then for all $b \in \text{Ker}(\alpha)$:

$$bm = b(1_A m) = (b \cdot 1_A)m = (b)\alpha m = 0,$$

hence $M^B \subseteq \text{Ann}_M(\text{Ker}(\alpha))$. On the other hand, if $m \in \text{Ann}_M(\text{Ker}(\alpha))$, then $m \in M^B$, since from $B = A1_B \oplus \text{Ker}(\alpha)$ it follows:

$$\forall b \in B, a \in A : b(am) = (ba)m = (ba)\alpha m = [b \cdot (a)\alpha]m = (b \cdot a)m,$$

(2) and (3) Let $f, g \in \text{End}_B(A)$, then f and g are in particular left A -linear and we have

$$\Psi(f \circ g) := (1_A)(f \circ g) := ((1_A)f)g = (1_A)f(1_A)g = \Psi(f)\Psi(g).$$

Thus Ψ is a homomorphism of rings. Moreover, for all $b \in \text{Ker}(\alpha)$:

$$b \cdot \Psi(f) = (b \cdot 1_A)f = (b)\alpha \circ f = 0.$$

By (1) it follows that $\text{Im}(\Psi) \subseteq A^B$. On the other hand, for any $x \in A^B$, the right multiplication R_x is B -linear. To see this take any $a \in A$ and $b \in B$, then

$$(b \cdot a)R_x = (b \cdot a)x = (ba)\alpha x = (ba) \cdot x = b \cdot (ax) = b \cdot (a)R_x.$$

Hence $R_x \in \text{End}_B(A)$. Thus A^B becomes a subring of A and every left B -module M is also a left A^B -module with A^B -submodule M^B . It follows as above that Ψ_M is an isomorphism.

(4) If $g : M \rightarrow N$ is a homomorphism between B -modules then for any $f \in \text{Hom}_B(A, M)$ we have

$$(f)\Psi_M \circ g|_{M^B} = (1_A)f \circ g = (f \circ g)\Psi_N = (f)\text{Hom}_B(A, g) \circ \Psi_N.$$

Thus $\Psi_M \circ g|_{M^B} = \text{Hom}_B(A, g) \circ \Psi_N$, i.e. the isomorphisms Ψ_M are natural transformations between the functors $(-)^B$ and $\text{Hom}_B(A, -)$. \square

The case when the functor of taking invariants is exact is interesting in many situations.

Proposition 1.9. *Let $A \subseteq B$ be an extension with additional module structure. The following statements are equivalent:*

- (a) $(-)^B : B\text{-Mod} \rightarrow A^B\text{-Mod}$ is an exact functor.
- (b) A is a projective left B -module.
- (c) there exists an element $t \in B^B$ with $(t)\alpha = 1_A$.
- (d) there exists an idempotent $e \in B$ with $Be \simeq A$ as left B -modules and $eBe \simeq A^B$ as rings.

Proof. (a) \Leftrightarrow (b) holds by Lemma 1.8(4).

(b) \Rightarrow (c) If ${}_B A$ is projective, then α splits and there exists a B -linear map $\beta : A \rightarrow B$ with $\beta\alpha = id_A$. Set $t := (1_A)\beta$. Then $t \in B^B$ by 1.8(3) and $(t)\alpha = 1_A$.

(c) \Rightarrow (b) If there exists an element $t \in B^B$ with $(t)\alpha = 1_A$, then one defines $\beta : A \rightarrow B$ as $(a)\beta = at$. Since t is in B^B , β is B -linear and lets α split, i.e. ${}_B A$ is projective. (b) \Leftrightarrow (d) is clear. \square

Example 1.10. For $B = A^e$ we have $M^B = Z(M) := \{m \in M \mid \forall a \in A am = ma\}$ and $Z(-)$ is exact if and only if A is a separable k -algebra.

For $B = A\#G$ we have $M^B = M^G := \{m \in M \mid \forall g \in G m^g = m\}$ and $(-)^G$ is exact if and only if G is finite and A contains an element of trace one (property Proposition 1.9(d)).

For $B = A[X; \delta]$ we have $M^B = \text{Ann}_M(X) = \{m \in M \mid Xm = 0\}$. It is impossible for A to be a projective left $A[X; \delta]$ -module simply because $B^B = 1.\text{ann}_{A[X; \delta]}(X) = 0$.

For $B = A\#H$ we have $M^B = M^H := \{m \in M \mid \forall h \in H h \cdot m = \varepsilon(h)m\}$. If H is a finite dimensional Hopf algebra over a field k then $(-)^H$ is exact in $A\#H\text{-Mod}$ if and only if there exists a left integral $t \in \int_l$ and an element $a \in A$ such that $t \cdot a = 1$.

If A is an algebra with involution $*$ and B is the subalgebra generated by A and $*$ in $\text{End}(A)$, then $M^B = Z(M; *) := \{m \in Z(M) \mid m^* = m\}$. Moreover, $Z(-; *)$ is exact if and only if A admits a separable idempotent $\gamma = \sum_{i=1}^n x_i \otimes y_i$ such that $\gamma = \sum_{i=1}^n y_i^* \otimes x_i^*$.

2. Prime and semiprime B -stable ideals

For the rest of the paper we assume that $A \subseteq B$ is an extension with additional module structure φ such that $M(A) \subseteq \text{Im}(\varphi)$. Note that then every B -submodule of A is already a two-sided ideal. Moreover, the B -invariant elements M^B for a left B -module M are

A -centralizing. In particular $M^B \subseteq Z(M)$. This approach had been used in [1] to study operator algebras B acting on semiprime algebras A .

Definition 2.1. A B -stable ideal I of A is called B -prime (resp. B -semiprime) if $KL \subseteq I$ (resp. $K^2 \subseteq I$) implies $K \subseteq I$ or $L \subseteq I$ (resp. $K \subseteq I$) for all B -stable ideals K and L of A . A is called B -prime (resp. B -semiprime) if 0 is a B -prime (resp. B -semiprime) B -stable ideal.

If I is a B -stable ideal of A , then there exists a ring homomorphism $\varphi' : B \rightarrow \text{End}(A/I)$ with $M(A/I) \subseteq \text{Im}(\varphi')$. Let $B/I := \varphi'(B)$, then $A/I \subseteq M(A/I) \subseteq B/I$ is an extension with additional module structure φ' . With this notation we easily prove the following Lemma.

Lemma 2.2. Let A and B as above. Let P be a B -stable ideal of A . Then P is prime (resp. semiprime) if and only if A/P is B/P -prime (resp. B/P -semiprime).

Like in the classical case we have a description of B -stable semiprime ideals:

Proposition 2.3. A B -stable ideal of A is B -semiprime if and only if it is the intersection of B -prime B -stable ideals.

Proof. \Rightarrow : Without loss of generality we may assume that A is B -semiprime. Let

$$I := \bigcap \{P \mid P \text{ is a prime } B\text{-stable ideal of } A\}.$$

Assume $I \neq 0$. Then there exists $0 \neq x_1 \in I$. Let $I_1 := B \cdot x_1$ then $0 \neq I_1 \subseteq I$. Since A is B -semiprime, we have $(I_1)^2 \neq 0$. Hence $(I_1)^2$ contains a non-zero element x_2 . Set $I_2 := B \cdot x_2$. Again $(I_2)^2 \neq 0$, i.e. we may choose a non-zero element $x_3 \in (I_2)^2$. Continuing this process we obtain a family x_1, x_2, x_3, \dots of non-zero elements and a descending chain of non-zero B -stable ideals:

$$I \supseteq I_1 \supseteq (I_1)^2 \supseteq I_2 \supseteq (I_2)^2 \supseteq \dots \supseteq (I_{m-1})^2 \supseteq I_m \supseteq \dots$$

Consider the following set of B -stable ideals:

$$\mathcal{L} := \{P \subseteq A \mid P \text{ is a } B\text{-stable ideal of } A \text{ and for all } m : I_m \not\subseteq P\}.$$

Note that \mathcal{L} is not empty since $0 \in \mathcal{L}$. Let $\{P_\lambda\}_A$ be an ascending chain of B -stable ideals in \mathcal{L} . Suppose $I_m \subseteq \bigcup_A P_\lambda$ for some $m \geq 1$, then by definition of $I_m = B \cdot x_m$ we have $x_m \in \bigcup_A P_\lambda$. Thus there exists $\mu \in A$ such that $x_m \in P_\mu$ for some $\mu \in A$ and hence $I_m \subseteq P_\mu$ —a contradiction to $P_\mu \in \mathcal{L}$. Hence $\bigcup_A P_\lambda \in \mathcal{L}$ and we can apply Zorn's Lemma. Let P be a maximal element of \mathcal{L} . We will show that P is a B -prime B -stable ideal. Suppose there are B -stable ideals K, L such that $KL \subseteq P$. Without loss of generality we may assume $P \subseteq K$ and $P \subseteq L$. If $L \neq P \neq K$, then by the maximality of P in \mathcal{L} : $K, L \notin \mathcal{L}$, i.e. there are $m, n \geq 1$ with $I_m \subseteq K$ and $I_n \subseteq L$. Without loss of generality let $n \leq m$, then

$$I_{m+1} \subseteq (I_m)^2 \subseteq I_m I_n \subseteq KL \subseteq P,$$

a contradiction to $P \in \mathcal{L}$. Hence $P = L$ or $P = K$, i.e. P is a prime B -stable ideal. But this implies $I \subseteq P$, and in particular $I_m \subseteq P$ for all m , a contradiction. Thus the intersection I of all prime B -stable ideals is equal to zero.

The converse is clear: if $I^2 = 0$ for some B -stable ideal I then $I^2 \subseteq P$ for any prime B -stable ideal P of A . Thus $I \subseteq P$ and hence $I \subseteq \bigcap P = 0$. \square

We conclude that A is B -semiprime if and only if A is subdirect product of B/I -prime algebras A/I .

Recall the following module theoretic notions: The self-injective hull \widehat{M} of a module M is the largest M -generated submodule of the injective hull $E(M)$ of M , i.e. if $E(M)$ denotes the injective hull of M , then

$$\widehat{M} = \sum_{f \in \text{Hom}(M, E(M))} \text{Im}(f) = M\text{Hom}(M, \widehat{M}).$$

The full subcategory of $R\text{-Mod}$ consisting of submodules of M -generated modules is denoted by $\sigma[M]$. The Lambek torsion theory in $\sigma[M]$ is the torsion category whose torsion class consists of all modules X such that $\text{Hom}(X, \widehat{M}) = 0$. A submodule N of a module M is called *dense* if M/N is a torsion module with respect to Lambek torsion theory, i.e. $\text{Hom}(M/N, \widehat{M}) = 0$. It is well-known that N is dense in M if and only if $\text{Hom}(L/N, M) = 0$ for all submodules $N \subset L \subset M$ (see [6, 10.8]). M is called *polyform*, if every essential submodule of M is dense. M is called *monoform*, if every non-zero submodule of M is dense. Dense submodules are also sometimes called rational (see [6, Chapter 10]).

Lemma 2.4. *Let A be B -semiprime. Then the following statements are equivalent for a B -stable ideal I of A :*

- (a) I is a dense B -submodule of A .
- (b) I is an essential B -submodule of A .
- (c) $J I \neq 0 \neq I J$ for any non-zero B -stable ideal J of A .

Proof. (a) \Rightarrow (b) dense submodules are essential (see [6, Chapter 10]);

(b) \Rightarrow (a) Let K be a B -stable ideal of A that contains I and $f \in \text{Hom}_B(K/I, A)$. Then $f(K/I)$ is a B -stable ideal of A . Thus $N := f(K/I) \cap I$ is a nilpotent B -stable ideal of A , since $N^2 \subseteq f(K/I)I = f(KI/I) = 0$. Hence $N = 0$, as A is B -semiprime and $f(K/I) = 0$, as I is an essential B -submodule. This shows $\text{Hom}_B(K/I, A) = 0$, i.e. I is dense in A .

(b) \Leftrightarrow (c) For all B -stable ideals J we have: $(J \cap I)^2 \subseteq J I \subseteq J \cap I$. Since A is B -semiprime we have $J I = 0$ if and only if $I \cap J = 0$. Hence I is an essential B -stable ideal if and only if the left annihilator of I does not contain any non-zero B -stable ideal. Analogously one concludes the same statement for the right annihilator. \square

As a corollary from the last Lemma we get

Corollary 2.5. *Let A and B be as above.*

- (1) *If A is B -semiprime, then A is a polyform B -module and A^B is reduced.*

- (2) If A is B -prime, then A is a moniform B -module and A^B is an integral domain.
 (3) A is B -prime if and only if A is B -semiprime and a uniform B -module.

Proof. (1) It follows from Lemma 2.4[(a) \Leftrightarrow (b)] that A is polyform. Let $x \in A^B$ be such that $x^2 = 0$. Then $(Ax)^2 = Ax^2 = 0$ shows that Ax is a nilpotent B -stable ideal. Thus $x = 0$, i.e. A^B is reduced.

(2) Let A be B -prime and left I be a non-zero B -stable ideal of A . Note that $JI = 0$ implies $J = 0$ for all B -stable ideals as A is B -prime and $I \neq 0$. By Lemma 2.4, I is a dense left B -submodule of A , i.e. A is moniform. If $xy = 0$ holds for $x, y \in A^B$, then $(Ax)(Ay) = Axy = 0$. Since A is B -prime, $x = 0$ or $y = 0$, i.e. A^B is an integral domain.

(3) Follows from the definitions and Lemma 2.4. \square

3. The central closure

In the sequel we will extend Wisbauer's construction of the extended centroid and of the central closure of a semiprime algebra (see [6, Chapter 32]) to our situation of an extension $A \subseteq B$ with additional module structure. We will reduce ourselves to subalgebras B of $\text{End}(A)$ which contain the multiplication algebra $M(A)$.

Let $Q_{\max}(A)$ denote the (right) maximal quotient ring of A .

Theorem 3.1. *Let A be B -semiprime and let \widehat{A} be the self-injective hull of A as B -module. Then the following hold:*

- (1) The map $\Psi : \text{End}_B(\widehat{A}) \longrightarrow \widehat{A}^B$ with $\Psi(f) := (1_A)f$ is an isomorphism of A^B -modules and defines a ring structure on \widehat{A}^B making it a commutative, self-injective and von Neumann regular ring with subring A^B .
- (2) There is a bijection between the set of (essentially) closed B -stable ideals of A and of the set of central idempotents of \widehat{A}^B .
- (3) \widehat{A}^B is a field if and only if A is B -prime.
- (4) \widehat{A}^B is a finite product of n fields ($n \geq 1$) if and only if A has finite Goldie dimension n as left B -module.
- (5) If A^B is large in A , i.e. $A^B \cap I \neq 0$ for all non-zero B -stable ideals I of A , then $\widehat{A}^B = Q_{\max}(A^B)$ and A is non-singular as A^B -module.

Proof. (1) We know from Corollary 2.5 that A is a polyform B -module. Hence

$$\text{End}_B(\widehat{A}) = \text{Hom}_B(A, \widehat{A}) \xrightarrow{\Psi_{\widehat{A}}} \widehat{A}^B.$$

In particular $f = 0$ if and only if $(1_A)f = 0$ for all $f \in \text{End}_B(\widehat{A})$. \widehat{A}^B carries a ring structure induced by $\Psi_{\widehat{A}}$ where

$$(1) f(1)g = (1)(f \circ g)$$

for all $f, g \in \text{End}_B(\widehat{A})$. Moreover, let $I := (A)f^{-1} \cap (A)g^{-1} \cap A$. Then for all $x, y \in I$ we have

$$(xy)(f \circ g) = (x(y)f)g = (x)g(y)f = ((x)gy)f = (xy)(g \circ f),$$

i.e. $f \circ g - g \circ f \in \text{Hom}_B(\widehat{A}/I^2, \widehat{A})$. As intersection of two essential B -submodules, I is essential and no non-zero B -stable ideal annihilates I on the left (see Lemma 2.4). Thus I^2 is also an essential B -submodule of A . By Lemma 2.4 is I^2 dense. And henceforth as \widehat{A} is polyform, $f \circ g = g \circ f$, i.e. $\text{End}_B(\widehat{A}) \simeq \widehat{A}^B$ is commutative. As endomorphism ring of a self-injective polyform module, \widehat{A}^B is self-injective and von Neumann regular and contains A^B as subring (see [6, 11.2]).

(2) follows from [6, 12.7];

(3) and (4) follow from (2) and (1);

(5) By [6, 11.5(1)] $\widehat{A}^B = Q_{\max}(A^B)$. Let $a \in A$ and I an essential ideal of A^B with $aI = 0$. Set $J := (B \cdot a)^B = (B \cdot a) \cap A^B$, Then $JI = 0$ and hence $(J \cap I)^2 = 0$. As A^B is reduced and I is essential in A^B we conclude $J = 0$. But since A^B is large in A we can also conclude $(B \cdot a) = 0$, i.e. $a = 0$. Thus A is a non-singular A^B -module. \square

In the next theorem we will see that the self-injective hull \widehat{A} itself carries a ring structure.

Theorem 3.2. *Let A be B -semiprime and let \widehat{A} be the self-injective hull of A as B -module. Then*

(1) $\widehat{A} = Q_{\mathcal{D}}(A)$ is the torsion theoretic quotient module with respect to the Lambek torsion theory \mathcal{D} in $\sigma[B A]$ and

$$\widehat{A}^B \simeq \varinjlim \{ \text{Hom}_B(I, A) \mid I \text{ is an essential } B\text{-submodule of } A \}.$$

(2) The map $\phi : A \otimes_{A^B} \text{End}_B(\widehat{A}) \rightarrow \widehat{A}$ with $\psi(a \otimes f) := (a)f$ is left B -linear. Its kernel is an ideal and thus we may define a ring structure on \widehat{A} given by the following multiplication:

$$\forall a, b \in A; s, t \in \widehat{A}^B : (as) \cdot (bt) := (ab)st,$$

where A is a subring of \widehat{A} .

Let $\widehat{B} := \langle B, \widehat{A}^B \rangle \subseteq \text{End}(\widehat{A})$. Then $\widehat{A} \subseteq M(\widehat{A}) \subseteq \widehat{B}$ is again an extension with additional module structure and the following hold:

(3) \widehat{A} is \widehat{B} -semiprime and a self-injective \widehat{B} -module.

(4) \widehat{A} is a non-singular \widehat{A}^B -module.

(5) A is B -prime if and only if \widehat{A} is \widehat{B} -prime.

We call \widehat{A} the central closure of A with respect to B and \widehat{A}^B the extended centre of A with respect to B .

Proof. (1) From the fact that A is a polyform B -module it follows from [6, 9.13] that $\widehat{A} = Q_{\mathcal{D}}(A)$. From [6, 9.17] follows the description of the endomorphism ring \widehat{A}^B .

(2) By definition the self-injective hull of a module M is M -generated, i.e. the map $\phi : A \otimes \text{Hom}_B(A, \widehat{A}) \rightarrow \widehat{A}$ with $\phi(a \otimes f) := (a)f$ is an epimorphism of left B -modules,

where B acting just on the first component of the tensor product $A \otimes \text{Hom}_B (A, \widehat{A})$. Since A is polyform we have $\text{Hom}_B (A, \widehat{A}) = \text{End}_B (\widehat{A})$. Let $a_i \in A$ and $f_i \in \text{End} (\widehat{A})$ and assume that $\gamma := \sum_{i=1}^n a_i \otimes f_i$ is in the kernel of ϕ , i.e.

$$0 = \phi(\gamma) = \sum_{i=1}^n (a_i) f_i$$

For any $b \otimes g$ we have

$$\phi((b \otimes g)\gamma) = \phi\left(\sum_{i=1}^n ba_i \otimes gf_i\right) = \sum_{i=1}^n (ba_i)g \circ f_i = b\left(\sum_{i=1}^n (a_i) f_i\right)g = 0.$$

Hence, $(b \otimes g)\gamma \in \text{Ker} (\phi)$. Moreover,

$$\phi(\gamma(b \otimes g)) = \phi\left(\sum_{i=1}^n a_ib \otimes f_i g\right) = \sum_{i=1}^n (a_ib) f_i \circ g = \left(\sum_{i=1}^n (a_i) f_i\right)gb = 0.$$

Thus $\gamma(b \otimes g) \in \text{Ker} (\phi)$ shows that $\text{Ker} (\phi)$ is an ideal of $A \otimes \text{End}_B (\widehat{A})$ as claimed. This allows us to define an associative ring structure on the B -module \widehat{A} that is also compatible with that B -action and contains A as a subring.

(3) By definition \widehat{A} is a left B -module, hence there exists a ring homomorphism $\Theta: B \rightarrow \text{End} (\widehat{A})$ that is injective as \widehat{A} is a faithful B -module. Without loss of generality we might identify B with its image $\Theta(B)$. Let $\widehat{B} := \langle \Theta(B), \widehat{A}^B \rangle$. By hypothesis $M(A) \subseteq B$ implies

$$M(\widehat{A}) = M(A)\widehat{A}^B \subseteq \widehat{B}$$

and $\widehat{A} \subseteq \widehat{B}$ is an extension with additional module structure. Let I be a \widehat{B} -stable ideal of \widehat{A} , with $I^2 = 0$. In particular $(I \cap A)^2 = 0$ holds. But since I is B -stable, A is essential as B -submodule of \widehat{A} and A is B -semiprime we conclude $I = 0$. Thus \widehat{A} is \widehat{B} -semiprime.

Every \widehat{B} -endomorphism of \widehat{A} is also B -linear. On the other hand, $\text{End}_B (\widehat{A}) \simeq \widehat{A}^B \simeq \text{End}_{\widehat{B}} (\widehat{A})$ holds. As \widehat{A} was self-injective as B -module, it is also self-injective as \widehat{B} -module.

(4) follows from [6, 11.11(5)].

(5) Let I, J be non-trivial \widehat{B} -stable ideals in \widehat{A} . As $B \subseteq \widehat{B}$ these ideals are also B -stable. Since A is essential as B -submodule, $(I \cap A)$ and $(J \cap A)$ are non-trivial B -stable ideals of A and $(I \cap A)(J \cap A)$ is contained in IJ . Hence if A is B -prime, then \widehat{A} is \widehat{B} -prime.

On the other hand if \widehat{A} is \widehat{B} -prime and $IJ = 0$ for some B -stable ideals I and J of A , then $(I\widehat{A}^B)(J\widehat{A}^B) = IJ\widehat{A}^B = 0$, i.e. $I\widehat{A}^B = 0$ or $J\widehat{A}^B = 0$. And A is B -prime. \square

For $B = M(A)$ we recover Wisbauer’s construction of the central closure of A (see [5]).

4. The Martindale quotient ring

Let \mathcal{F} denote the set of ideals of A with zero left and right annihilator. The right Martindale ring of quotients of A is

$$Q(A) := \varinjlim \{\text{Hom}_{-A} (I, A) \mid I \in \mathcal{F}\}.$$

Alternatively, one might construct $Q(A)$ as follows: define an equivalence relation \sim on $\bigcup_{I \in \mathcal{F}} \text{Hom}_{-A}(I, A)$ by letting $f : I \rightarrow A$ be equivalent to $g : J \rightarrow A$ if there exists $K \in \mathcal{F}$ such that $K \subseteq I \cap J$ and $f|_K = g|_K$. Denote by $[f]$ the equivalence class of a map $f : I \rightarrow A$. Note that the equivalence class of the zero map $A \rightarrow A$ contains all maps f that vanish on some ideal in \mathcal{F} . Addition is defined by $[f] + [g] := [f + g : I \cap J \rightarrow A]$ while multiplication is set to be $[f][g] := [fg : JI \rightarrow A]$ where fg denotes the composition map $a \mapsto f(g(a))$.

In case we have an extension $A \subseteq B$ with additional module structure, we are going to construct a subring of $Q(A)$ related to all B -stable ideals in \mathcal{F} . Let \mathcal{F}_B be the set of B -stable ideals with zero left and right annihilator. We assume from now on that A is B -semiprime and that the left annihilator of an B -stable ideal is again B -stable. By Lemma 2.4 all ideals in \mathcal{F}_B are essential B -submodules of A . Consider the following construction:

$$Q_0(A) := \varinjlim \{ \text{Hom}_{-A}(I, A) \mid I \in \mathcal{F}_B \}.$$

We will refer to the elements of $Q_0(A)$ as equivalence classes in the above sense, i.e.

$$Q_0(A) = \left(\bigcup_{I \in \mathcal{F}_B} \text{Hom}_{-A}(I, A) \right) / \sim,$$

where

$$f \sim g \Leftrightarrow f|_I = g|_I \text{ for some } I \in \mathcal{F}_B.$$

With the operations $+$ and \cdot as above $Q_0(A)$ becomes a k -algebra and a subring of $Q(A)$.

Before we show that the central closure \widehat{A} can be identified with a subring of $Q_0(A)$ we show that \widehat{A}^B lies in the centre of $Q_0(A)$.

Proposition 4.1. *Let A be B -semiprime and denote by \widehat{A} the central closure of A with respect to B . Assume that for any essential B -stable ideal I of A the left annihilator $\text{l.ann}_A(I)$ and right annihilator $\text{r.ann}_A(I)$ are B -stable ideals. Define for any $f \in \text{End}_B(\widehat{A})$ the ideal $I_f := (A)f^{-1} \cap A$. Then the map*

$$\psi : \text{End}_B(\widehat{A}) \rightarrow Q_0(A) \text{ with } f \mapsto [f : I_f \rightarrow A]$$

is an injective homomorphism of k -algebras whose image lies in the centre of $Q_0(A)$ and consists of all elements $[f : I \rightarrow A]$ where f is left B -linear.

Proof. For each endomorphism $f \in \text{End}_B(\widehat{A})$ define $I_f := f^{-1}(A) \cap A$. Since pre-images of essential submodules are essential, I_f is an essential B -submodule of A . By Lemma 2.4(c) and the hypothesis I_f has zero left and right annihilator, i.e. $I_f \in \mathcal{F}_B$. We will show that ψ is a ring homomorphism. Let $f, g \in \text{End}_B(\widehat{A})$. Note that $I_f I_g \in \mathcal{F}_B$ and $I_f I_g \subseteq I_{fg}$ since for all $x \in I_f, y \in I_g$ the following holds:

$$fg(xy) = f(xg(y)) = f(x)g(y) \in A.$$

Thus

$$\begin{aligned}\psi(f)\psi(g) &= [f : I_f \rightarrow A][g : I_g \rightarrow A] = [fg : I_f I_g \rightarrow A] \\ &= [fg : I_{fg} \rightarrow A] = \psi(fg).\end{aligned}$$

This shows that ψ is a ring homomorphism.

Assume $\psi(f) = 0$ for some $f \in \text{End}_B(\widehat{A})$. Then there exists an $J \in \mathcal{F}_B$ with $J \subseteq I_f$ and $f(J) = 0$. Hence $f \in \text{Hom}_B(\widehat{A}/J, \widehat{A}) = 0$ as J is dense by Lemma 2.4. Thus $f = 0$ and ψ is injective.

Let $[f : I \rightarrow A]$ be such that f is B -linear, then f can be uniquely extended to an endomorphism $\bar{f} \in \text{End}_B(\widehat{A})$ since \widehat{A} is self-injective and polyform as B -module. By definition $\psi(\bar{f}) = [f : I \rightarrow A]$ since $I \subseteq I_{\bar{f}}$ and $f|_I = \bar{f}|_I$. Hence the image of ψ consists of all elements $[f : I \rightarrow A]$ such that f is B -linear. \square

Let $\iota : A \rightarrow Q_0(A)$ be the inclusion of A into $Q_0(A)$ given by $\iota(a) := [L_a : A \rightarrow A]$. Together with ψ we have a map $A \times \text{End}_B(\widehat{A}) \rightarrow Q_0(A)$ sending a pair (a, f) to the product $\iota(a)\psi(f)$. Since $A^B \simeq \iota(A^B) \subseteq Z(Q_0(A))$ this map is A^B -balanced and induces a k -algebra homomorphism

$$\psi^* : A \otimes_{A^B} \text{End}_B(\widehat{A}) \rightarrow Q_0(A).$$

Recall from Theorem 3.2 the k -algebra homomorphism $\phi : A \otimes_{A^B} \widehat{A}^B \rightarrow \widehat{A}$ with $\phi(a \otimes f) = af$.

Lemma 4.2. *For any element $\gamma \in A \otimes_{A^B} \text{End}_B(\widehat{A})$ there exists a B -stable ideal $I \in \mathcal{F}_B$ such that*

$$\psi^*(\gamma) = [L_{\phi(\gamma)} : I \rightarrow A],$$

where $L_{\phi(\gamma)}(x) = \phi(\gamma)x$ for all $x \in I$.

Proof. Write $\gamma = \sum_{i=1}^n a_i \otimes x_i$. By definition

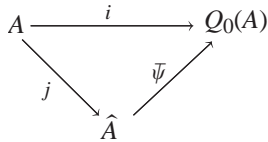
$$\begin{aligned}\psi^*(\gamma) &= \sum_{i=1}^n [L_{a_i} : A \rightarrow A][L_{x_i} : I_{x_i} \rightarrow A] \\ &= \sum_{i=1}^n [L_{a_i x_i} : I_{x_i} \rightarrow A] \\ &= \left[\sum_{i=1}^n L_{a_i x_i} : \bigcap_{i=1}^n I_{x_i} \rightarrow A \right] \\ &= [L_{\sum_{i=1}^n a_i x_i} : I \rightarrow A],\end{aligned}$$

where we set $I := \bigcap_{i=1}^n I_{x_i} \in \mathcal{F}_B$. Since $\phi(\gamma) = \sum_{i=1}^n a_i x_i$ our claim is proved. \square

In particular this implies that $\text{Ker}(\phi) \subseteq \text{Ker}(\psi^*)$, i.e. ψ^* extends to a k -algebra homomorphism $\bar{\psi} : \widehat{A} \rightarrow Q_0(A)$.

Let $j : A \rightarrow \widehat{A}$ denote the inclusion map.

Proposition 4.3. *The following diagram in the category of k -algebras commutes:*



The image $\text{Im}(\bar{\psi})$ is the subalgebra of $Q_0(A)$ generated by the image of A and all elements $[f : I \rightarrow A]$ such that f is B -linear. The kernel of $\bar{\psi}$ is equal to

$$\text{Ker}(\bar{\psi}) = \phi(\text{Ker}(\psi^*)) = \bigcup_{I \in \mathcal{F}_B} \text{l.ann}_{\widehat{A}}(I).$$

Proof. Assume that $\gamma \in \text{Ker}(\psi^*)$, then $\psi^*(\gamma) = [L_\phi(\gamma) : I \rightarrow A] = 0$ for some $I \in \mathcal{F}_B$, i.e. $\phi(\gamma)I = 0$. Thus $\phi(\gamma)$ is an element of the left annihilator in \widehat{A} of I . Hence $\text{Ker}(\bar{\psi}) \subseteq \bigcup_{I \in \mathcal{F}_B} \text{l.ann}_{\widehat{A}}(I)$. On the other hand each element z in $\text{l.ann}_{\widehat{A}}(I)$ is mapped by $\bar{\psi}$ to the zero class in $Q_0(A)$. \square

Under some conditions $\bar{\psi}$ is injective.

Theorem 4.4. *Let A be B -semiprime and \widehat{A} its central closure with respect to B . Assume that the following conditions are fulfilled:*

- (i) $\text{l.ann}_A(I)$ and $\text{r.ann}_A(I)$ are B -stable for all essential B -stable ideals I of A .
- (ii) $\text{l.ann}_{\widehat{A}}(I)$ is B -stable for all B -submodules I of \widehat{A} .

Then $\bar{\psi} : \widehat{A} \rightarrow Q_0(A)$ is an injective k -algebra homomorphism.

Proof. It is enough to show that $\phi(\text{Ker}(\bar{\psi})) = 0$ since then $\text{Ker}(\bar{\psi}) = \text{Ker}(\phi)$ holds, i.e. $\bar{\psi}$ is injective. Let $I \in \mathcal{F}_B$. Then $0 = \text{l.ann}_A(I) = A \cap \text{l.ann}_{\widehat{A}}(I)$. Since I is an essential B -submodule of A and hence of \widehat{A} , by hypothesis $\text{l.ann}_{\widehat{A}}(I)$ is B -stable. But then $\text{l.ann}_{\widehat{A}}(I) = 0$ as A is essential as B -submodule of \widehat{A} . \square

5. Applications to Hopf actions

In this section we apply our construction to Hopf actions. Let H be a Hopf algebra over k and let A be a left H -module algebra. Then there exists a left H -module structure on A given by some ring homomorphism $\varphi : H \rightarrow \text{End}(A)$. Let us denote the action of an element $h \in H$ as an endomorphism of A by λ_h , i.e. $\lambda_h(a) = h \cdot a$ for all $a \in A$. In order for A to be a left H -module algebra the H -action has to satisfy the following condition in $\text{End}(A)$

for $h \in H$ and $a \in A$:

$$\lambda_h \circ L_a = \sum_{(h)} L_{h_1 \cdot a} \circ \lambda_{h_2},$$

where $\Delta(h) = \sum_{(h)} h_1 \otimes h_2$ is the Sweedler notation.

The k -subalgebra generated by the maps L_a, R_a and λ_h for $a \in A$ and $h \in H$ is denoted by $M_H(A)$, i.e.

$$M_H(A) := \{ \{L_a, R_a, \lambda_h \mid a \in A, h \in H\} \} \subseteq \text{End}(A).$$

Instead of $M_H(A)$ -prime resp. $M_H(A)$ -semiprime one says that A is H -prime resp. H -semiprime. It is easy to verify that $A^{M_H(A)} = Z(A) \cap A^H$ holds. Let us denote $Z(A) \cap A^H$ by $Z(A)^H$. Note that in general $Z(A)$ is not closed under the action of H , but $Z(A)$ is a left H -module algebra in case H is cocommutative.

Let us first realize $M_H(A)$ as the factor of some kind of smash product. If A is commutative, then $M_H(A)$ is generated by $\{L_a, \lambda_h \mid a \in A, h \in H\}$. Hence we might identify $M_H(A)$ with the image of $A\#H \rightarrow \text{End}(A)$ mapping $a\#h$ to $L_a \circ \lambda_h$ (where $A\#H$ is the smash product as defined in Example 1.5). In this case $M_H(A)$ is isomorphic to $A\#H/\text{Ann}_{A\#H}(A)$.

In the following theorem we represent $M_H(A)$ as a factor ring of some smash product. For that reason we are going to recall a general construction taken from [2]:

Definition 5.1. Let A and B be k -algebras with multiplication maps μ_A resp. μ_B . Let $v : B \otimes A \rightarrow A \otimes B$ be a k -linear map and define:

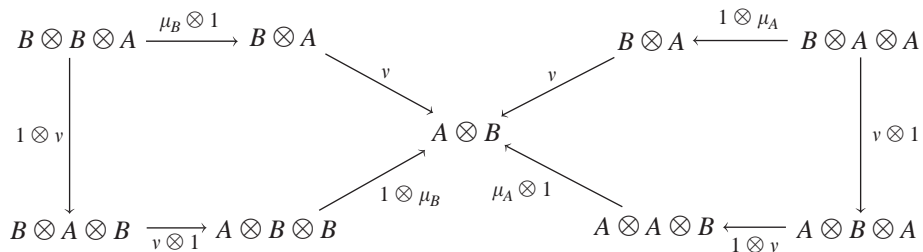
$$\mu : (A \otimes B) \otimes (A \otimes B) \xrightarrow{1 \otimes v \otimes 1} (A \otimes A) \otimes (B \otimes B) \xrightarrow{\mu_A \otimes \mu_B} A \otimes B.$$

If $A \otimes B$ becomes through μ an associative k -algebra with unit $1_A \otimes 1_B$, then we will write $A\#_v B$ and call this ring the *smash-product* or factorization structure of A and B with respect to v .

Caenepeel et al. gave a characterization of smash-products.

Theorem 5.2 (Caenepeel et al. [2, Theorem 2.5]). *Let A, B and v as above. Then v defines a smash-product for A and B if and only if the following statements hold:*

- (i) $v(b \otimes 1_A) = 1_A \otimes b$ and $v(1_B \otimes a) = a \otimes 1_B$ for all $a \in A, b \in B$.
- (ii) The following diagrams commute:



Let $A^e := A \otimes A^{op}$ be the enveloping algebra of A . We want to define a smash product of A^e and H . If H is cocommutative, then A^e is a left H -module algebra and we can use the (ordinary) smash product $A^e \# H$ of a module algebra and a Hopf algebra (see Example 1.5 for definition), but in general A^e will not be an H -module algebra.

Define the map $v : H \otimes A^e \rightarrow A^e \otimes H$ by

$$v(h \otimes a \otimes b) := \sum_{(h)} (h_1 \cdot a) \otimes (h_3 \cdot b) \otimes h_2$$

for all $a, b \in A$ and $h \in H$.

Then $A^e \#_v H$ is a smash product in the above sense. To see this we have to check that the diagrams above commute.

Property (i) of Theorem 5.2 is obviously fulfilled. Let $a, b, x, y \in A$ and $h, g \in H$. Then

$$\begin{aligned} & (1 \otimes \mu_H)(v \otimes 1)(1 \otimes v)(h \otimes g \otimes (a \otimes b)) \\ &= \sum_{(g)} (1 \otimes \mu_H)(v \otimes 1)(h \otimes (g_1 \cdot a \otimes g_3 \cdot b) \otimes g_2) \\ &= \sum_{(g,h)} (1 \otimes \mu_H)((h_1 g_1 \cdot a) \otimes (h_3 g_3 \cdot b) \otimes h_2 \otimes g_2) \\ &= \sum_{(g,h)} h_1 g_1 \cdot a \otimes h_3 g_3 \cdot b \otimes h_2 g_2. \\ &= v(\mu_H \otimes 1)(h \otimes g \otimes (a \otimes b)). \end{aligned}$$

Hence $(1 \otimes \mu_H)(v \otimes 1)(1 \otimes v) = v(\mu_H \otimes 1)$ holds, i.e. the left part of the diagram in Theorem 5.2(ii) commutes.

$$\begin{aligned} & (\mu_{A^e} \otimes 1)(1 \otimes v)(v \otimes 1)(h \otimes (x \otimes y) \otimes (a \otimes b)) \\ &= \sum_{(h)} (\mu_{A^e} \otimes 1)(1 \otimes v)((h_1 \cdot x \otimes h_3 \cdot y) \otimes h_2 \otimes (a \otimes b)) \\ &= \sum_{(h)} (\mu_{A^e} \otimes 1)((h_1 \cdot x \otimes h_5 \cdot y) \otimes (h_2 \cdot a \otimes h_4 \cdot b) \otimes h_3) \\ &= \sum_{(h)} (h_1 \cdot x)(h_2 \cdot a) \otimes (h_4 \cdot b)(h_5 \cdot y) \otimes h_3 \\ &= \sum_{(h)} ((h_1 \cdot (xa)) \otimes (h_3 \cdot (by))) \otimes h_2 \\ &= v(1 \otimes \mu_{A^e})(h \otimes (x \otimes y) \otimes (a \otimes b)). \end{aligned}$$

Hence $(\mu_{A^e} \otimes 1)(1 \otimes v)(v \otimes 1) = v(1 \otimes \mu_{A^e})$, i.e. the right part of the diagram of Theorem 5.2(ii) commutes and $A^e \#_v H$ is a smash product.

One could also define a smash product on the k -algebras A^{op} and $A \# H$. It is not difficult to check that the map $\sigma : A^{op} \otimes A \# H \rightarrow A \# H \otimes A^{op}$ with

$$\sigma(b \otimes a \# h) := \sum_{(h)} a \# h_1 \otimes (S(h_2) \cdot b),$$

will define a smash product $(A\#H)\#_{\sigma}A^{op}$. Moreover, one checks that the map $\Psi: (A\#H)\#_{\sigma}A^{op} \rightarrow A^e\#_v H$ with $\Psi(a\#h \otimes x) := \sum_{(h)} (a \otimes (h_2 \cdot x))\#h_1$ is an isomorphism of k -algebras.

We can now represent $M_H(A)$ as the factor of the smash product $A^e\#_v H$.

Theorem 5.3. *The map $\Phi: A^e\#_v H \rightarrow M_H(A)$ with $\Phi((a \otimes b)\#h) := L_a \circ R_b \circ \lambda_h$ is a surjective map of k -algebras. Moreover, $A \subseteq A^e\#_v H$ is a ring extension with additional module structure Φ .*

Proof. We have the well-defined maps $L_{A^e}: A^e \rightarrow \text{End}(A)$, and $\lambda_H: H \rightarrow \text{End}(A)$. Let μ denote the multiplication in $\text{End}(A)$, then $\Phi = \mu \circ (L_{A^e} \otimes \lambda_H)$ is well-defined. By definition $\text{Im}(\Phi) = M_H(A)$, i.e. Φ is surjective. To show that Φ is a ring homomorphism note that for all $h \in H, a, b, x \in A$

$$h \cdot (axb) = \sum_{(h)} (h_1 \cdot a)(h_2 \cdot x)(h_3 \cdot b)$$

holds and therefore also

$$\lambda_h \circ L_a \circ R_b = \sum_{(h)} L_{h_1 \cdot a} \circ R_{h_3 \cdot b} \circ \lambda_{h_2}.$$

By definition of the multiplication in $A^e\#_v H$ this implies that Φ is a ring homomorphism. \square

In case H is cocommutative, we have that $A^e\#_v H$ and $A^e\#H$ coincide:

Proposition 5.4. *If H is cocommutative, then A^e is a left H -module algebra and $A^e\#_v H$ is equal to the ordinary smash product $A^e\#H$ of a module algebra and the Hopf algebra.*

Proof. A^e is always a left H -module by the diagonal module structure, i.e.

$$h \cdot (a \otimes b) := \sum_{(h)} (h_1 \cdot a) \otimes (h_2 \cdot b)$$

for all $h \in H$ and $a, b \in A$. Suppose that H is cocommutative. Let $a \otimes x, b \otimes y \in A^e$ and $h \in H$. Then

$$\begin{aligned} h \cdot ((a \otimes x)(b \otimes y)) &= h \cdot (ab \otimes yx) \\ &= \sum_{(h)} h_1 \cdot (ab) \otimes h_2 \cdot (yx) \\ &= \sum_{(h)} (h_1 \cdot a)(h_2 \cdot b) \otimes (h_3 \cdot y)(h_4 \cdot x) \end{aligned}$$

$$\begin{aligned}
 &= \sum_{(h)} (h_1 \cdot a)(h_3 \cdot b) \otimes (h_4 \cdot y)(h_2 \cdot x) \\
 &= \sum_{(h)} [(h_1 \cdot a) \otimes (h_2 \cdot x)][(h_3 \cdot b) \otimes (h_4 \cdot y)] \\
 &= \sum_{(h)} [h_1 \cdot (a \otimes x)][h_2 \cdot (b \otimes y)].
 \end{aligned}$$

Moreover, $h \cdot (1 \otimes 1) = \varepsilon(h)(1 \otimes 1)$ holds, i.e. A^e is a left H -module algebra. Moreover, for all $h \in H$ and $a, x \in A$:

$$v(h \otimes (a \otimes x)) = \sum_{(h)} (h_1 \cdot a) \otimes (h_3 \cdot x) \otimes h_2 = \sum_{(h)} [h_1 \cdot (a \otimes x)] \otimes h_2.$$

Thus the multiplication defined by v is equal to the multiplication in the ordinary smash product, i.e. $A^e \#_v H = A^e \# H$. \square

Now we are in a position to apply our previous results to Hopf module algebras.

Theorem 5.5. *Let H be a k -Hopf algebra, A be an H -semiprime left H -module algebra and let \widehat{A} be the self-injective hull of A as $A^e \#_v H$ -module. Then the following statements hold:*

- (1) A is a polyform $A^e \#_v H$ -module and a subdirect product of H -prime module algebras. Furthermore, $Z(A)^H$ is reduced.
- (2) \widehat{A} is an H -semiprime left H -module algebra with submodule algebra A . \widehat{A} is self-injective as $\widehat{A}^e \#_v H$ -module and a non-singular module over the self-injective and von Neumann regular ring $Z(\widehat{A})^H$.
- (3) If $Z(A)^H$ is large in A , then $Z(\widehat{A})^H = Q_{\max}(Z(A)^H)$ and A is non-singular as $Z(A)^H$ -module

Proof. Note that the $A^e \#_v H$ -module structure of A coincides with that of $M_H(A)$ since $M_H(A) \simeq A^e \#_v H / \text{Ann}_{A^e \#_v H}(A)$.

- (1) follows from Corollary 2.5 and Proposition 2.3.
- (2) follows from Theorem 3.2 and Theorem 3.1. Note that

$$M_H(\widehat{A}) = \langle M_H(A), Z(\widehat{A})^H \rangle = \widehat{M_H(A)} \subseteq \text{End}(\widehat{A}).$$

We still have to prove that \widehat{A} is a left H -module algebra. The H -module structure on $\widehat{A} = AZ(\widehat{A})^H$ is given by $h \cdot (as) = (h \cdot a)s$. Let $as, bt \in \widehat{A}$ and $h \in H$, then:

$$\begin{aligned}
 h \cdot [(as)(bt)] &= (h \cdot (ab))st = \sum_{(h)} (h_1 \cdot a)(h_2 \cdot b)st = \sum_{(h)} [(h_1 \cdot a)s][(h_2 \cdot b)t] \\
 &= \sum_{(h)} [h_1 \cdot (as)][h_2 \cdot (bt)].
 \end{aligned}$$

- (3) follows from Theorem 3.1. \square

From 3.1(2–4) follows also

Corollary 5.6. *Let H be a k -Hopf algebra, A be an H -semiprime left H -module algebra and let \widehat{A} be the self-injective hull of A as $A^e \#_v H$ -module. Then the following statements hold:*

- (1) *There exists a bijection between the (essentially) closed H -stable ideals of A , the central idempotents of \widehat{A} and the central idempotents of $Z(\widehat{A})^H$.*
- (2) *The following statements are equivalent:*
 - (a) *Every direct sum of non-trivial H -stable ideals in A is finite.*
 - (b) *\widehat{A} is a finite direct product of H -prime H -module algebras.*
 - (c) *$Z(\widehat{A})^H$ is a finite product of fields.*
- (3) *A is H -prime if and only if $Z(\widehat{A})^H$ is a field.*

Let G be a group and consider the group ring $H = k[G]$. Let A be a k -algebra where G acts on, then G acts also on A^e and we can form the skew-group ring $A^e \# G$. The G -central closure constructed in [6] coincides with our construction of the central closure \widehat{A} as self-injective hull of $A^e \#_v H$ since, as mentioned before, $A^e \#_v H$ coincides with the ordinary smash product of A^e and $k[G]$ which is in this case the skew-group ring of A^e and $k[G]$.

Using the results of the last section we show that our central closure embeds into the Martindale ring of quotients of a Hopf-module algebra. For a left H -module algebra A our Martindale ring of quotient Q_0 constructed with respect to \mathcal{F}_B where $B = M_H(A)$ coincides with the Martindale ring of quotients constructed by Cohen in [3].

Proposition 5.7. *Let H be a Hopf algebra over k and let A be a left H -semiprime module algebra with right Martindale ring of quotients Q_0 . Let \widehat{A} be the self-injective hull of A as $A^e \#_v H$ -module. Assume that one of the following conditions hold:*

- (i) *A is commutative or*
- (ii) *A is semiprime or*
- (iii) *H has a bijective antipode.*

Then

- (1) *$\psi : Z(\widehat{A})^H \rightarrow Z(Q_0)^H$ with $f \mapsto [f : I_f \rightarrow A]$ is an isomorphism of k -algebras where $I_f := f^{-1}(A) \cap A$.*
- (2) *$Z(Q_0)^H$ is a von Neumann regular self-injective k -algebra.*
- (3) *$\psi : A \rightarrow Q_0$ with $\gamma \mapsto [L_\gamma : I \rightarrow A]$ is an injective homomorphism of k -algebras where $I = \bigcup_{j=1}^n I_{x_j}$ for $\gamma = \sum_{j=1}^n a_j x_j$.*
- (4) *\widehat{A} is isomorphic to the subalgebra of Q_0 generated by A and $Z(Q_0)^H$.*

Proof. We just have to check that any of hypotheses (i–iii) implies that the left and right annihilator of an H -stable ideal in \mathcal{F}_H of a left H -module algebra is H -stable. First of all note that the left annihilator in A of an H -stable (left) ideal I is always H -stable. Let $a \in A$

satisfy $aI = 0$; then for all $h \in H$ and $x \in I$ one has

$$(h \cdot a)x = \sum_{(h)} h_1 \cdot (a(S(h_2) \cdot x)) = 0.$$

We still have to show that the right annihilator in A of an essential H -stable ideal is H -stable.

Case (i): If A is commutative then left and right annihilator are equal and hence H -stable.

Case (ii): Let I be an essential H -stable ideal. Since the left annihilator $\text{l.ann}_A(I)$ of I in A is H -stable, $\text{l.ann}_A(I) = 0$ follows by Lemma 2.4. If A is semiprime, then also $\text{r.ann}_A(I) = 0$ follows, i.e. the right annihilator of any essential H -stable ideal is H -stable.

Case (iii): If H has a bijective antipode, then the right annihilator in A of an H -stable ideal I is always H -stable. Let $a \in A$ satisfy $Ia = 0$; then for all $h \in H$ and $x \in I$ one has:

$$x(h \cdot a) = \sum_{(h)} h_2 \cdot ((S^{-1}(h_1) \cdot x)a) = 0.$$

Hence we can apply Theorem 4.4 and Proposition 4.1. \square

In [4] Matczuk constructs the central closure of an H -prime module algebra A directly as the subalgebra of the Martindale quotient ring Q_0 of a module algebra A , generated by A and $Z(Q_0)^H$. We see by (4) that his construction coincides with ours.

A H -semiprime left H -module algebra A is called H -centrally closed, if $\widehat{A} = A$ holds. From Theorem 3.2(4) follows that if A is an H -semiprime left H -module algebra, then \widehat{A} is H -centrally closed. Is A H -prime, then \widehat{A} is H -prime and $Z(\widehat{A})^H$ is a field. We might consider $\bar{H} := H \otimes Z(\widehat{A})^H$ as a $Z(\widehat{A})^H$ -Hopf algebra and \widehat{A} becomes a \bar{H} -prime left \bar{H} -module algebra over the field $k := Z(\widehat{A})^H$. Was H separable over k , then \bar{H} is also separable over k and hence finite dimensional and semisimple. Thus questions with respect to H -prime module algebras over separable Hopf algebras H over commutative rings can be reduced to H -prime H -centrally closed module algebras over finite dimensional semisimple Hopf algebras over fields.

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