On the Kneser property for reaction–diffusion systems on unbounded domains

Francisco Morillas\textsuperscript{a}, José Valero\textsuperscript{b,*}

\textsuperscript{a} Universitat de València, Departament d’Economia Aplicada, Facultat d’Economia, Campus dels Tarongers s/n, 46022 València, Spain
\textsuperscript{b} Universidad Miguel Hernández, Centro de Investigación Operativa, Avda. Universidad s/n, Elche (Alicante), 03202, Spain

**Abstract**

We prove the Kneser property (i.e. the connectedness and compactness of the attainability set at any time) for reaction–diffusion systems on unbounded domains in which we do not know whether the property of uniqueness of the Cauchy problem holds or not. Using this property we obtain that the global attractor of such systems is connected. Finally, these results are applied to the complex Ginzburg–Landau equation.

© 2009 Elsevier B.V. All rights reserved.

1. Introduction

In this paper we study the Kneser property (i.e. the connectedness and compactness of the attainability set at any time) for reaction–diffusion systems in unbounded domains. In [18] it is studied the asymptotic behaviour of weak solutions for a rather general reaction–diffusion system on unbounded domains (in fact, for the whole $\mathbb{R}^N$) for which it is not known whether we have uniqueness of the Cauchy problem or not. In particular, it is shown the existence of a global compact invariant attractor, extending in this way previous results for attractors of reaction–diffusion and other equations in which uniqueness holds (see e.g. [1,4,7,19,22,23]). However, the question about the connectivity of this attractor was not solved there. For this aim it is necessary to use the Kneser property.

These results are applied to the complex Ginzburg–Landau equation, a well-known equation of the Mathematical–Physics used, for example, in the theory of superconductivity or in chemical turbulence.

We note that for obtaining the property of uniqueness of the Cauchy problem in reaction–diffusion systems we need in general a monotonicity assumption (see [6]) or to consider the system on a invariant region (see [21]). However, it is...
quite natural that these approaches can fail in the applications, so that it is important to study the Kneser property for such systems.

Also, observe that the absence of uniqueness in our equations force us to work with a set of solutions rather than with a solution. Hence, in order to study the asymptotic behavior of solutions we use the theory of multivalued semiflows. This approach was used in the theory of global attractors at first in [3] and then developed in other papers (see [2,15,16]). A rather similar approach, which avoids the theory of multivalued maps, is the theory of generalized semiflows [5]. Also, other approach to this question is the theory of trajectory attractors (see [6,20]).

In previous papers similar results are proved for reaction–diffusion systems on bounded domains (see [10,11]) or for scalar reaction–diffusion equations on bounded domains (see [8,9]). Also, in [12] the Kneser property is proved for a scalar reaction–diffusion on unbounded domains in which the nonlinear term is equal to $|u|^\frac{1}{2}$.

In our problem, technically, some additional difficulties appear (with respect to the case of bounded domains), so that we have to make an additional assumption which is not used for bounded domains. Nevertheless, this assumption is satisfied by important systems, as the complex Ginzburg–Landau equation.

The paper is organized as follows. In Section 2 we give the setting of the problem. In Section 3 the main result of the paper, that is, the Kneser property for general reaction–diffusion systems, is established. In Section 4 this property is applied to prove the connectivity of the global attractor. Finally, in Section 5 these results are applied to the complex Ginzburg–Landau equation.

2. Setting of the problem

We consider the following reaction–diffusion system:

\begin{align}
  u_t &= a \Delta u - f(x, u), \quad x \in \mathbb{R}^N, \ t > 0, \\
  u(0) &= u_0 \in [L^2(\mathbb{R}^N)]^d.
\end{align}

where $u$ is an unknown vector function, that is, $u(x, t) = (u^1, \ldots, u^d), x \in \mathbb{R}^N, t > 0, f(x, u) = (f^1, \ldots, f^d)$, and $u_t = \frac{\partial u}{\partial t}$. We assume the following conditions:

(H1) The real $d \times d$ matrix $a$ has a positive symmetric part $\frac{1}{2}(a + a^*) \geq AI$, where $A > 0$.

(H2) $f = f_0 + f_1, \ f_0(x, u) = (f_{01}, \ldots, f_{0d}), f_1(x, u) = (f_{11}, \ldots, f_{1d})$, where $f_j, j = 0, 1$, satisfy:

\begin{align}
  x \mapsto f_j(x, u) & \text{ is measurable for all } u \in \mathbb{R}^d, \tag{3} \\
  u \mapsto f_j(x, u) & \text{ is continuous for a.a. } x \in \mathbb{R}^N. \tag{4}
\end{align}

(H3) There exist positive functions $C_0(x), C_1(x) \in L^1(\mathbb{R}^N)$ and constants $\alpha, \beta > 0, p_i \geq 2$ verifying

\begin{align}
  (f_0(x, u), u) & \geq \alpha |u|^2 - C_0(x), \tag{5} \\
  (f_1(x, u), u) & \geq \beta \sum_{i=1}^d |u|^{|p_i|} - C_1(x). \tag{6}
\end{align}

(H4) There exist positive functions $C_2(x) \in L^2(\mathbb{R}^N), C_3(x) \in L^1(\mathbb{R}^N)$, and constants $\gamma, \eta > 0$ verifying

\begin{align}
  |f_0(x, u)| & \leq C_2(x) + \eta |u|, \tag{7} \\
  \sum_{i=1}^d |f_i^j(x, u)|^{p_i^{-1}} & \leq C_3(x) + \gamma \sum_{i=1}^d |u_i|^{p_i}. \tag{8}
\end{align}

(H5) For a.a. $x$ the functions $u \mapsto f_j(x, u), j = 0, 1$, are continuously differentiable and for any $N > 0$ there exist $D_j(N), k = 0, 1$, such that for all $u$ satisfying $|u| \leq N$ and a.a. $x$ we have

\begin{align}
  (f_{0u}(x, u)w_1, w_2) & \geq -D_0(N)|w_1||w_2|, \tag{9} \\
  (f_{1u}(x, u)w_1, w_2) & \geq -D_1(N)|w_1||w_2|, \quad \forall w_i \in \mathbb{R}^d, \tag{10}
\end{align}

where $f_{ju}$ is the Jacobian matrix of $f_j$.

Remark 1. Conditions (9)–(10) are satisfied, for example, if the partial derivatives of $u \mapsto f_j(x, u)$ are bounded on any ball of $\mathbb{R}^d$ uniformly with respect to $x$. This is true for the particular case $f_j(x, u) = h_j(u) + g_j(x)$.
In the sequel, we shall use the notation $H = [L^2(\mathbb{R}^N)]^d$, $V = [H^1(\mathbb{R}^N)]^d$ and $V' = [H^{-1}(\mathbb{R}^N)]^d$, together with the respective norms $\| \cdot \|$, $\| \cdot \|_V$ and $\| \cdot \|_V'$. By $\| \|_r, | \cdot |_r$, $(\cdot, \cdot)_H$, $(\cdot, \cdot)$ we denote the usual norm in $L'(\mathbb{R}^N)$, the norm in $\mathbb{R}^d$ (or $\mathbb{R}^N$), the scalar product in $H$ and the usual scalar product in $\mathbb{R}^d$ (or $\mathbb{R}^N$), respectively, so that $(u, v)_H = \sum_{i=1}^d \int_{\mathbb{R}^N} u_i v_i \, dx = \int_{\mathbb{R}^N} (u, v) \, dx$. For simplicity, for any $u, v \in V$ we shall use also the following notation:

$$\| \nabla u \|^2 = \sum_{i=1}^d \| \nabla u^i \|^2 = \sum_{i=1}^d \sum_{j=1}^N \left| \frac{\partial u^i}{\partial x_j} \right|^2,$$

$$(\nabla u, \nabla v) = \sum_{i=1}^d (\nabla u^i, \nabla v^i) = \sum_{i=1}^d \sum_{j=1}^N \frac{\partial u^i}{\partial x_j} \frac{\partial v^i}{\partial x_j}, \quad (\nabla u, \nabla v)_H = \int_{\mathbb{R}^N} (\nabla u, \nabla v) \, dx.$$

For $p = (p_1, \ldots, p_d)$ we define the space

$$L^p(\mathbb{R}^N) = L^{p_1}(\mathbb{R}^N) \times \cdots \times L^{p_d}(\mathbb{R}^N).$$

In a similar way we define $L^q(\mathbb{R}^N)$ for $q = (q_1, \ldots, q_d)$, where $\frac{1}{p_i} + \frac{1}{q_i} = 1$. Conditions (7)–(8) imply that for any $u \in L^p(0, T; L^p(\mathbb{R}^N)) \cap L^2(0, T; H)$ we have

$$\int_0^T \int_{\mathbb{R}^N} \left| f_0(x, u(t, x)) \right|^2 \, dx \, dt \leq K_0 \left( T + \| u \|^2_{L^2(0, T; H)} \right),$$

$$\int_0^T \int_{\mathbb{R}^N} \sum_{i=1}^d \left| f_i(x, u(t, x)) \right|^{q_i} \, dx \, dt \leq K_1 \left( T + \sum_{i=1}^d \| u^i \|_{L^{q_i}(0, T; L^{q_i}(\mathbb{R}^N))} \right).$$

First we shall give the definition of a weak solution.

**Definition 2.** The function $u(t, x), t \in [0, T], x \in \mathbb{R}^N$, is said to be a weak solution of (1)–(2) on $[0, T]$ if $u \in L^2(0, T; V) \cap L^p(0, T; L^p(\mathbb{R}^N)) \cap L^\infty(0, T; H)$ and $u$ satisfies Eq. (1) in the distribution sense, that is,

$$- \int_0^T (u, v_t)_H dt = \int_0^T (au, \Delta v)_H dt + \int_0^T \int_{\mathbb{R}^N} (f(x, u), v) \, dx \, dt = 0,$$

for all $v \in [C_0^\infty((0, T) \times \mathbb{R}^N)]^d$, and $u(0) = u_0$.

It follows from this definition and (11)–(12) that the time derivative $\frac{du}{dt}$ of any weak solution $u$ belongs to the space $L^2(0, T; V') + L^2(0, T; H) + L^p(0, T; L^p(\mathbb{R}^N)) \subset L^q(0, T; Y) = L^{q_1}(0, T; V') + \cdots + L^{q_d}(0, T; V'_1 + L^{q_d}(\mathbb{R}^N))$, where $Y = V' + L^q(\mathbb{R}^N)$. Since $u \in L^2(0, T; V) \subset L^q(0, T; Y)$, $u \subset C([0, T, Y)$, and then the inclusion $u \subset L^\infty(0, T; H)$ implies that $t \mapsto u(t, \cdot)$ is weakly continuous with values in the space $H$ (see [21, Lemma 1.4, p. 263] or [14]).

It is an immediate consequence that for any $v \in L^2(0, T; V) \cap L^p(0, T; L^p(\mathbb{R}^N))$ and any weak solution $u$ one has

$$\int_0^t \left( \frac{du}{dt}, v \right)_Y dt + \int_0^T \int_{\mathbb{R}^N} (a \nabla u, \nabla v)_H \, dx \, dt + \int_0^T \int_{\mathbb{R}^N} (f(x, u), v) \, dx \, dt = 0,$$

where $(\cdot, \cdot)_Y$ denotes pairing in the space $Y$. Since (14) implies (13), this is an equivalent definition of weak solution.

In fact, $u(t, \cdot)$ is continuous on $[0, T]$ with respect to the strong topology of the space $H$, $\| u(t, \cdot) \|$ is absolutely continuous and $\frac{d}{dt} \| u(t) \|^2 = 2 \langle \frac{du}{dt}, u \rangle_Y$ for a.a. $t \in (0, T)$ (see [18, Lemma 3]). Hence, it is standard to prove using (5)–(6) and the properties of the matrix $a$ that every weak solution $u$ of (1) satisfies for all $t \geq s$, $s, t \in [0, T]$, the following estimate

$$\| u(t) \|^2 + 2A \int_s^t \| u(\tau) \|^2 \, d\tau + 2\alpha \int_s^t \| u(\tau) \|^2 \, d\tau + 2\beta \sum_{i=1}^d \int_s^t \| u^i(\tau) \|_{L^{p_i}(\mathbb{R}^N)}^2 \, d\tau \leq \| u(s) \|^2 + 2M(t - s),$$

for some $M > 0$.

Our aim in this paper is to show that the attainability set of solutions at any moment of time $t$, that is, the set of values attained by the solution at time $t$, is connected and compact in the space $H$. This property is called the Kneser property. After that, the Kneser property will be used to prove that the global attractor of (1) is connected.
3. The Kneser property

In order to prove the Kneser property we shall need suitable approximating functions of the nonlinear terms \( f_i \).

We define a sequence of smooth functions \( \psi_k : \mathbb{R}^+ \to [0, 1] \) satisfying

\[
\psi_k(s) = \begin{cases} 
1, & \text{if } 0 \leq s \leq k, \\
0 \leq \psi_k(s) \leq 1, & \text{if } k \leq s \leq k + 1, \\
0, & \text{if } s \geq k + 1,
\end{cases}
\]

and the following approximating functions:

\[
f_{0k}(x, u) := f_0(x, \psi_{2k}(|u|)u) + (1 - \psi_k(|u|))g_k^0(u),
f_{1k}(x, u) := f_1(x, \psi_{2k}(|u|)u) + (1 - \psi_k(|u|))g_k^1(u),
\]

where \( k \geq 1, g_k^j(u) = 2\eta u^j, g_k^0(u) = (2\gamma + 1)|u|^{p_i - 1}u^i \).

Let \( f_k = f_{0k} + f_{1k} \). Then for any \( k \in \mathbb{R}^1, f_k(x, \cdot) \in C^1(\mathbb{R}^d; \mathbb{R}^d) \) and

\[
\sup_{|u| \leq A} |f_k(x, u) - f(x, u)| \to 0, \quad \text{as } k \to \infty, \quad \text{for any } A > 0.
\]

**Lemma 3.** Assume (H1)–(H5). For any \( k \) the functions \( f_{0k}, f_{1k} \) satisfy conditions (3)–(8) with functions and constants not depending on \( k \). Moreover, there exists \( D_{0k}, D_{1k} \) such that

\[
\begin{align*}
(f_{0ku}(x, u)w, w) &\geq -D_{0k}|w|^2, \\
(f_{1ku}(x, u)w, w) &\geq -D_{1k}|w|^2, \quad \forall w, u \in \mathbb{R}^d.
\end{align*}
\]

**Proof.** Conditions (3)–(4) are obvious. We check first (5) and (6). Indeed, for \( f_{0k} \) we have the following cases:

1. If \( |u| \leq 2k \), then

\[
(f_{0k}(x, u), u) = (f_0(x, u), u) + 2\eta(1 - \psi_k(|u|))(u, u) \\
\geq (f_0(x, u), u) \geq \alpha|u|^2 - C_0(x).
\]

2. If \( |u| \geq 2k \), then using (7) we get

\[
(f_{0k}(x, u), u) = (f_0(x, \psi_{2k}(|u|)u), u) + 2\eta(1 - \psi_k(|u|))(u, u) \\
\geq -C_2(x)|u| - \eta|u|^2 + 2\eta|u|^2 \geq \eta|u|^2 - \frac{C_2^2(x)}{2\eta},
\]

as \( k + 1 \leq 2k \), so that \( (1 - \psi_k(|u|)) = 1 \).

On the other hand, for \( f_{1k} \) in a similar way we have:

1. If \( |u| \leq 2k \), then

\[
(f_{1k}(x, u), u) \geq (f_1(x, u), u) \geq \beta \sum_{i=1}^d |u_l|^{p_i} - C_1(x).
\]

2. If \( |u| \geq 2k \), then using Young’s inequality \( ab \leq \frac{a^p}{p} + \frac{(p - 1)b^{p - 1}}{p - 1} \) and (8) we obtain

\[
(f_{1k}(x, u), u) = (f_1(x, \psi_{2k}(|u|)u), u) + (2\gamma + 1)(1 - \psi_k(|u|)) \sum_{i=1}^d |u_l|^{p_i} \\
\geq -\left( \sum_{i=1}^d \frac{|u_l|^{p_i}}{p_i} + \sum_{i=1}^d \frac{(p_i - 1)|f_1(x, \psi_{2k}(|u|)u)|^{\frac{p_i}{p_i - 1}}}{p_i} \right) + (2\gamma + 1) \sum_{i=1}^d |u_l|^{p_i} \\
\geq \sum_{i=1}^d \left( 2\gamma + 1 - \frac{1}{p_i} \right)|u_l|^{p_i} - C_3(x) - \gamma \sum_{i=1}^d |u_l|^{p_i} \\
\geq \gamma \sum_{i=1}^d |u_l|^{p_i} - C_3(x).
\]
Consider now (7) and (8). Indeed,
\[ |f_{0k}(t, u)| = |f_0(t, \psi_{2k}(|u|)u) + (1 - \psi_k(|u|))g_0(u)| \leq |f_0(t, \psi_{2k}(|u|)u)| + 2\eta|u| \leq C_2(x) + 3\eta|u|, \]
and
\[ \sum_{i=1}^{d} |f_{1k}(x, u)|^{\frac{p_i}{p_i - 1}} = \sum_{i=1}^{d} \left( |f_{1i}(x, \psi_{2k}(|u|)u) + (1 - \psi_k(|u|))g_1^i(u)|^{\frac{p_i}{p_i - 1}} \right) \leq K \sum_{i=1}^{d} \left( |f_{1i}(x, u)|^{\frac{p_i}{p_i - 1}} + |g_1^i(u)|^{\frac{p_i}{p_i - 1}} \right) \leq KC_3(x) + K(3\gamma + 1) \sum_{i=1}^{d} |u|^\frac{p_i}{p_i - 1}. \]

Taking \( \hat{C}_2(x) := C_2(x), \hat{C}_3(x) := KC_3(x), \hat{\eta} = 3\eta \) and \( \hat{\gamma} = K(3\gamma + 1) \) we obtain estimates (7) and (8).

It remains to prove (17)–(18). We note that the partial derivatives \( \frac{\partial}{\partial u} \psi_k(|u|) \) are uniformly bounded on \( \mathbb{R}^d \). Denote \( \psi_{ku}(|u|) = (\frac{\partial}{\partial u_1} \psi_k(|u|), \ldots, \frac{\partial}{\partial u_d} \psi_k(|u|)) \). Hence,
\[
\begin{align*}
f_{0k}(x, u) &= f_0u(x, \psi_{2k}(|u|)u)\psi_{2ku}(|u|) + \psi_{2k}(|u|)f_0u(x, \psi_{2k}(|u|)u) \\
&\quad - g_0(u)\psi_{ku}(|u|) + (1 - \psi_k(|u|))I,
\end{align*}
\[
\begin{align*}
f_{1k}(x, u) &= f_1u(x, \psi_{2k}(|u|)u)\psi_{2ku}(|u|) + \psi_{2k}(|u|)f_1u(x, \psi_{2k}(|u|)u) \\
&\quad - g_1(u)\psi_{ku}(|u|) + (1 - \psi_k(|u|))I.
\end{align*}
\]
where \( u\psi_{ku}(|u|) \) and \( g_j(u)\psi_{ku}(|u|) \) are \( d \times d \) matrices (a product of a column vector with a row vector), \( I \) is the identity matrix and \( I_u \) is a diagonal matrix such that \( (I_u)_{ii} = (p_i - 1)|u|^{p_i - 2} \). By (9) we have:
\[
\begin{align*}
\langle \psi_{2k}(|u|)f_0u(x, \psi_{2k}(|u|)u), w, w \rangle &\geq -D_0(2k + 1)|w|^2, \\
\langle f_0u(x, \psi_{2k}(|u|)u)\psi_{2ku}(|u|)w, w \rangle &\geq -D_0(2k + 1)C\psi_{2k}(2k + 1)|w|^2, \\
-\langle g_0(u)\psi_{ku}(|u|)w, w \rangle &\geq -2\eta(k + 1)C\psi_k|w|^2, \\
\langle (1 - \psi_k(|u|))Iw, w \rangle &\geq 0.
\end{align*}
\]
Then (17) holds and in the same way we obtain (18). \( \Box \)

**Remark 4.** In the case of bounded domains we avoided condition (H5) by using a convolution of the functions \( f_k. \) However, due to some technical difficulties we could not use this method in the case of unbounded domains.

For arbitrary \( u_0 \in H \) and \( T > 0 \) we define the following set
\[ D_T(u_0) = \{ u(\cdot) \in C^0(\bar{\Omega}): u(\cdot) \text{ is a weak solution of (1), } u(0) = u_0 \}. \]
For any \( t \in [0, T] \) we define the corresponding attainability set as
\[ K_t(u_0) = \{ u(t) \in C^0(\bar{\Omega}): u(\cdot) \in D_T(u_0) \}. \]
Our aim is to prove the connectedness of the set \( K_t(u_0) \subset H \) for any \( t \in [0, T] \). We note that the compactness of \( K_t(u_0) \) in \( H \) was proved in [18, Lemma 12].

Let us consider the following problem
\[
\begin{cases}
\frac{\partial u}{\partial t} - a\Delta u + f_k(x, u) = 0, & (t, x) \in (\gamma, T) \times \mathbb{R}^N, \\
u(\gamma, x) = u^\gamma(x),
\end{cases}
\]
where \( \gamma \in [0, T] \). In view of Lemma 3 for all \( k \geq 1 \) the function \( f_k \) satisfies (3)–(8), so that by [18, Theorem 5] for any \( u_{\gamma}^k \in [L^2(\mathbb{R}^N)]^d \) problem (19) has at least one weak solution \( u_{\gamma}^k(\cdot) \) defined on \( (\gamma, T] \). Using (17)–(18) and the properties of the matrix \( a \) it is standard to check that for the difference \( w(t) \) of two solutions we have
\[
\|w(t)\| \leq \|w(\gamma)\|e^{(D_{0k} + D_{1k})(t - \gamma)},
\]
so that the solution is unique.
We need some preliminary estimates.
Lemma 5. Let $B$ be a bounded set of $[L^2(\mathbb{R}^N)]^d$. Then there exists $R = R(B, T)$ (not depending neither on $\gamma$ or $k$) such that
\[
\|u^k_\gamma(t)\| \leq R, \quad \forall t \in [\gamma, T), \quad \|u^k_\gamma(\cdot)\|_{L^p(\gamma,T;L^p(\mathbb{R}^N))} \leq R,
\]
for any $u^\gamma \in B$, where $u^k_\gamma(\cdot)$ is the unique solution to (19) with $u^k_\gamma(\gamma) = u^\gamma$.

Proof. We note that, as shown in [18, p. 116], using (5)–(8) for $f_{\alpha k}$, $f_{1k}$ one can easily obtain that the functions $u^k_\gamma$ satisfy the estimate
\[
\|u^k_\gamma(t)\|^2 + 2A \int_0^t \|u^k_\gamma(\tau)\|^2 d\tau + 2\alpha \int_0^t \|u^k_\gamma(\tau)\|^2 d\tau + 2\beta \sum_{i=1}^d \int_0^t \|u^k_\gamma(\tau)\|_{L^p(\mathbb{R}^N)}^p d\tau \leq \|u^\gamma\|^2 + 2M(t - \gamma),
\]
for some constant $M > 0$. Hence, (21) follows. \hfill \Box

Lemma 6. Let $K$ be a compact set of $[L^2(\mathbb{R}^N)]^d$. For any $\varepsilon > 0$ there exists $N(\varepsilon, K, T) > 0$ (not depending neither on $\gamma$ or $k$) such that
\[
\int_{|x| \geq n} |u(x,t)|^2 \, dx \leq \varepsilon, \quad \forall t \in [0, T],
\]
\[
\int_{|x| \geq n} |u^k_\gamma(x,t)|^2 \, dx \leq \varepsilon, \quad \forall \gamma \in [0, T], \quad \forall \gamma \in [\gamma, T],
\]
if $n \geq N$, for any $u^\gamma$, $u_0 \in K$, where $u^k_\gamma(\cdot)$ is the unique solution to (19) with $u^k_\gamma(\gamma) = u^\gamma$ and $u(\cdot)$ is an arbitrary solution to (1)–(2) with $u(0) = u_0$.

Proof. We define a smooth function verifying
\[
\theta(t) = \begin{cases} 
0, & 0 \leq t \leq 1, \\
0 \leq \theta(s) \leq 1, & 1 \leq r \leq 2, \\
1, & r \geq 2.
\end{cases}
\]

It is proved in [18, p. 123] that for any $R > 0$ and $\varepsilon > 0$ there exists $N_1(\varepsilon, R)$ such that any weak solution of (1) with $u_0$ verifying $\|u_0\| \leq R$ satisfies
\[
\int_{\mathbb{R}^N} \theta \left( \frac{|x|^2}{n^2} \right) |u(x,t)|^2 \, dx \leq \left( \int_{\mathbb{R}^N} \theta \left( \frac{|x|^2}{n^2} \right) |u_0(x)|^2 \, dx \right) e^{-2\alpha t} + \varepsilon, \quad \text{if } n \geq N_1, \text{ for } t \in [0, T].
\]

Then we can find $N_2(\varepsilon, K)$ such that for any $u_0 \in K$,
\[
\int_{|x| \geq \sqrt{2}n} |u_0(x,t)|^2 \, dx \leq \int_{\mathbb{R}^N} \theta \left( \frac{|x|^2}{n^2} \right) |u_0(x,t)|^2 \, dx \leq 2\varepsilon, \quad \text{if } n \geq N_2, \text{ for } t \in [0, T].
\]

In [18] it is assumed that $\sqrt{\theta}$ is also smooth, but in fact this is not necessary (see [17]).

We shall obtain a similar estimate for $u^k_\gamma$ (denoted further just by $u^k$).

Since $u^k \in L^2(0,T;V) \cap L^p(0,T;L^p(\mathbb{R}^N))$ and $\frac{du^k}{dt} \in L^4(0,T;L^4(\mathbb{R}^N)) + L^2(0,T;V')$, the function $\|\sqrt{\theta}u^k\|^2 = \int_{\mathbb{R}^N} \theta \left( \frac{|x|^2}{n^2} \right) |u^k(x,t)|^2 \, dx$ is absolutely continuous and $\frac{d}{dt} \|\sqrt{\theta}u^k\|^2 = 2(\frac{du^k}{dt}, \theta u^k)y$ (see [17, Lemma 32]). Put $\rho_n(x) = \sqrt{\theta \left( \frac{|x|^2}{n^2} \right)}$. Then multiplying (19) by $\rho^2_n u^k$ we have
\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^N} \theta \left( \frac{|x|^2}{n^2} \right) |u^k|^2 \, dx = \int_{\mathbb{R}^N} \frac{du^k}{dt} \cdot \rho^2_n u^k \, dx = (a\Delta u^k, \rho^2_n u^k)_y - \int_{\mathbb{R}^N} \theta \left( \frac{|x|^2}{n^2} \right) (f_k(x,u^k), u^k) \, dx,
\]
for a.e. $t$. Using that $\rho^2_n u^k \in [H^1(\mathbb{R}^N)]^d$ and $\frac{d(\rho^2_n u^k)}{dx} = \frac{\rho^2_n u^k}{\rho_n x} (\rho^2_n u^k)_x$ (see [17, Lemma 31]) we obtain
\[
(a\Delta u^k, \rho^2_n u^k)_y = -\sum_{i=1}^d \int_{\mathbb{R}^N} \left( (\nabla (au^k)_i, x) \theta \left( \frac{|x|^2}{n^2} \right) u^k_i \right) \, dx - (\rho^2_n a \nabla u^k, \nabla u^k)_H.
\]
By $|\theta'(s)| \leq C$, $\theta'(\frac{\gamma^2}{n^2}) = 0$, for $|x| < n$ and $|x| > \sqrt{2}n$, (21) and the properties of the matrix $a$ we obtain

$$\left\{ \alpha \Delta u_k, \rho_0^2 u_k \right\} \leq \frac{C_1}{n} \int_{|x| \leq \sqrt{2}n} |\nabla u_k||u_k|^2 \, dx - A \int_{\mathbb{R}^N} \rho_0^2 |\nabla u_k|^2 \, dx \leq \frac{C_2}{n} (1 + \|u_k\|_V^2) \leq \epsilon' (1 + \|u_k\|_V^2).$$

(27)

if $n \geq N_3 (\epsilon', K, T)$. For the second term in (25) conditions (5)–(6) imply

$$-\int_{\mathbb{R}^N} \frac{\theta(\frac{|x|^2}{n^2})}{n^2} (f_k(x, u_k), u_k) \, dx \leq -\alpha \int_{\mathbb{R}^N} \frac{\theta(\frac{|x|^2}{n^2})}{n^2} |u_k|^2 \, dx + \int_{\mathbb{R}^N} \theta(\frac{|x|^2}{n^2}) C_0(x) \, dx

- \beta \sum_{i=1}^{d} \int_{\mathbb{R}^N} \theta(\frac{|x|^2}{n^2}) |u_k|^i \, dx + \int_{\mathbb{R}^N} \theta(\frac{|x|^2}{n^2}) C_1(x) \, dx

\leq -\alpha \int_{\mathbb{R}^N} \frac{\theta(\frac{|x|^2}{n^2})}{n^2} |u_k|^2 \, dx + 2\epsilon',

(28)

if $n \geq N_4 (\epsilon')$. Denoting $Y_{k,n}(t) = \int_{\mathbb{R}^N} \theta(\frac{|x|^2}{n^2}) |u_k(x, t)|^2 \, dx$ and using (25)–(28) we get

$$\frac{1}{2} \frac{d}{dt} Y_{k,n}(t) + \alpha Y_{k,n}(t) \leq 3 \epsilon' + \epsilon' \|u_k\|^2_V,

if $n \geq N_5 = \max \{ N_3, N_4 \}$. Applying Gronwall’s lemma and using (22) we obtain

$$Y_{k,n}(t) \leq Y_{k,n}(\gamma) e^{-2\alpha(t-\gamma)} + \frac{3}{\alpha} \epsilon' + \epsilon' \int_{\gamma}^{t} e^{-2\alpha(t-s)} \|u_k(s)\|^2_V \, ds \leq Y_{k,n}(\gamma) + \frac{3}{\alpha} \epsilon' + \epsilon' C_3.

We choose $\epsilon'$ such that $\frac{3}{\alpha} \epsilon' + \epsilon' C_3 \leq \epsilon$. As $Y_{k,n}(\gamma) = \int_{\mathbb{R}^N} \theta(\frac{|x|^2}{n^2}) |u_k^*(x)|^2 \, dx$, it follows that

$$\int_{|x| \geq \sqrt{2}n} \frac{\theta(\frac{|x|^2}{n^2})}{n^2} |u_k^*(x)|^2 \, dx \leq \int_{\mathbb{R}^N} \theta(\frac{|x|^2}{n^2}) |u_k(x, t)|^2 \, dx \leq \epsilon, \quad \text{for } t \in [\gamma, T],

(29)

if $n \geq N_6 (\epsilon', K, T) = \max \{ N_2, N_5 \}$. Then (23) follows from (24) and (29).  \qquad \square

Theorem 7. Let (H1)–(H5) hold. Then the set $K_t(u_0)$ is connected in $H$ for any $t \in [0, T]$.

Proof. The case $t = 0$ is obvious. Suppose that for some $t^* \in (0, T]$ the set $K_{t^*}(u_0)$ is not connected. Then there exist two compact sets $A_1, A_2 \subset H$ such that $A_1 \cup A_2 = K_{t^*}(u_0)$, $A_1 \cap A_2 = \emptyset$. Let $u_1(\cdot), u_2(\cdot) \in D_T(u_0)$ be such that $u_1(t^*) \in U_1, u_2(t^*) \in U_2$, where $U_1, U_2$ are disjoint open neighborhoods of $A_1, A_2$, respectively.

Let $u_i(t, \gamma), i = 1, 2$, be equal to $u_i(t)$, if $t \in [0, \gamma)$, and let $u_i(t, \gamma)$ be the unique solution of problem (19), if $t \in [\gamma, T]$. We shall prove now that the maps $u_i(t, \gamma)$ are continuous on $\gamma$ for each fixed $k \geq 1$ and $t \in [0, T]$. We shall omit the index $i$ for simplicity of notation.

Lemma 8. The maps $\gamma \mapsto u_k(t, \gamma)$ are continuous.

Proof. Let $\gamma \mapsto \gamma_0$. Consider first the case where $\gamma > \gamma_0$, i.e., $\gamma > \gamma_0$. If $t \leq \gamma_0 < \gamma$, then $u_k(t, \gamma) = u(t) = u_k(t, \gamma_0)$. We note also that $u_k(t, \gamma) = u(t)$, for all $t \in [0, \gamma_0]$. Now if $t > \gamma_0$, then we can assume that $t > \gamma$, so that $u_k(t, \gamma)$ is the solution of (19) on $[\gamma, T]$ such that $u_k(\gamma, \gamma) = u(\gamma_0)$. Further, $u(\gamma) \mapsto u(\gamma_0)$, as $\gamma \mapsto \gamma_0$, by continuity. Using (20) for $w(t) = u_k(t, \gamma) - u_k(t, \gamma_0)$ we have

$$\|u_k(t, \gamma) - u_k(t, \gamma_0)\| \leq \|u_k(\gamma_0, \gamma) - u_k(\gamma_0, \gamma_0)\| e^{D_{0k} \Delta_{1k}(t-\gamma)} = \|u(\gamma) - u_k(\gamma, \gamma_0)\| e^{D_{0k} \Delta_{1k}(t-\gamma)}

\leq \left( \|u(\gamma) - u(\gamma_0)\| + \|u(\gamma) - u(\gamma_0)\| \right) e^{D_{0k} \Delta_{1k}(t-\gamma)} \to 0, \quad \text{as } \gamma \to \gamma_0.

Let now $\gamma < \gamma_0$, i.e., $\gamma \not> \gamma_0$. If $t < \gamma_0$, then we can assume that $t < \gamma$, so that $u_k(t, \gamma) = u(t) = u_k(t_0, \gamma_0)$. We note also that $u_k(t_0, \gamma_0) = u(t_0)$, for all $t \in [0, \gamma_0]$. If $t \geq \gamma_0 > \gamma$, then $u_k(t, \gamma)$ is the solution of (19) on $[\gamma, T]$, $u_k(t, \gamma) = u(\gamma_0)$, and $u_k(t, \gamma_0)$ is the solution of (19) on $[0, \gamma_0]$ such that $u_k(0, \gamma_0) = u(\gamma_0)$. Hence,

$$\|u_k(t, \gamma) - u_k(t_0, \gamma_0)\| \leq \|u_k(\gamma_0, \gamma) - u_k(\gamma_0, \gamma_0)\| e^{D_{0k} \Delta_{1k}(t_0-\gamma)} = \|u_k(\gamma_0, \gamma) - u(\gamma_0)\| e^{D_{0k} \Delta_{1k}(t_0-\gamma)} \to 0, \quad \text{as } \gamma \to \gamma_0.

To finish the proof of continuity, we have to check that $\|u_k(\gamma_0, \gamma) - u(\gamma_0)\| \to 0$, as $\gamma \not> \gamma_0$.\quad \square
By the properties of the matrix $a$ for the difference $v^k(t, y) = u^k(t, y) - u(t)$ we have
\[
\frac{1}{2} \frac{d}{dt} \|v^k(t, y)\|^2 + A \|v^k(t, y)\|^2 + \int_{\mathbb{R}^N} \left( (f_{0k}(x, u^k), u^k) + (f_0(x, u), u) \right) dx \\
+ \int_{\mathbb{R}^N} \left( (f_{1k}(x, u^k), u^k) + (f_1(x, u), u) \right) dx \\
\leq \int_{\mathbb{R}^N} \left( (f_0(x, u^k), u^k) + (f_{0k}(x, u^k), u^k) \right) dx + \int_{\mathbb{R}^N} \left( (f_1(x, u^k), u^k) + (f_{1k}(x, u^k), u) \right) dx,
\] (30)
for a.a. $t \in (y', T)$. Using conditions (5)–(6) and integrating (30)–(31) over $(y', y_0)$, we obtain
\[
\left\| u^k(y_0, y_0) - u(y_0) \right\|^2 \\
\leq \left\| u(y') - u(y_0) \right\|^2 + K \left( (y_0 - y') + \|f_0(\cdot, u)\|_{L^2(y_0, y_0; H)} \left\| u^k \right\|_{L^2(y_0, y_0; H)} + \|f_{0k}(\cdot, u^k)\|_{L^2(y_0, y_0; H)} \left\| u \right\|_{L^2(y_0, y_0; H)} \\
+ \left\| f_1(\cdot, u) \right\|_{L^4(y_0, y_0; L^p(\mathbb{R}^N))} \left\| u^k \right\|_{L^p(y_0, y_0; L^p(\mathbb{R}^N))} + \left\| f_{1k}(\cdot, u^k) \right\|_{L^4(y_0, y_0; L^r(\mathbb{R}^N))} \left\| u \right\|_{L^r(y_0, y_0; L^r(\mathbb{R}^N))} \right),
\] (31)
It follows from (21) and (7)–(8) that the norms $\left\| u^k \right\|_{L^p(y_0, y_0; L^p(\mathbb{R}^N))}$, $\left\| u^k \right\|_{L^2(y_0, y_0; H)}$, $\left\| f_{0k}(\cdot, u) \right\|_{L^2(y_0, y_0; H)}$ and $\left\| f_{1k}(\cdot, u^k) \right\|_{L^4(y_0, y_0; L^p(\mathbb{R}^N))}$ are bounded by a constant that does not depend on $\gamma$. On the other hand, $u \in L^p(y', y_0; L^p(\mathbb{R}^N)) \cap L^2(y', y_0; H)$ and $f_0(\cdot, u) \in L^2(y', y_0; L^2(\mathbb{R}^N))$, $f_1(\cdot, u) \in L^4(y', y_0; L^p(\mathbb{R}^N))$ (again by (7)–(8)), so that $\left\| f_0(\cdot, u) \right\|_{L^2(y_0, y_0; H)} \leq \epsilon$, $\left\| f_1(\cdot, u) \right\|_{L^4(y_0, y_0; L^p(\mathbb{R}^N))} \leq \epsilon$, $\left\| u \right\|_{L^2(y_0, y_0; H)} \leq \epsilon$, and $\left\| u \right\|_{L^p(y_0, y_0; L^p(\mathbb{R}^N))} \leq \epsilon$, as soon as $|y' - y_0| < \delta(\epsilon)$. Therefore, $\left\| u^k(y_0, y') - u(y_0) \right\| \to 0$, as $y' \to y_0$. \qed

Now we put
\[
\gamma(\lambda) = \begin{cases} 
-\lambda, & \text{if } \lambda \in [-1, 0], \\
\lambda, & \text{if } \lambda \in [0, 1],
\end{cases}
\]
and define the function
\[
\varphi(\lambda)(t) = \begin{cases} 
\varphi_1^1(t, \gamma(\lambda)), & \text{if } \lambda \in [-1, 0], \\
\varphi_2^1(t, \gamma(\lambda)), & \text{if } \lambda \in [0, 1].
\end{cases}
\]
We have $\varphi^k(-1)(t) = u_1^k(t, T) = u_1(t), \varphi^k(1)(t) = u_2^k(t, T) = u_2(t)$. The map $\lambda \mapsto \varphi(\lambda)(t) \in H$ is continuous for any fixed $k \geq 1$, $t \in [0, T]$ (note that $\varphi_1^k(0, 0) = u_1^k(t, 0)$) and $\varphi^k(-1)(t^*) \in U_1, \varphi^k(1)(t^*) \in U_2$, so that there exists $\lambda_k \in [-1, 1]$ such that $\varphi(\lambda_k)(t^*) \notin U_1 \cup U_2$.

Denote $u^k(t) = \varphi^k(\lambda_k)(t)$. Note that for each $k \geq 1$ either $u^k(t) = u_1^k(t, \gamma(\lambda_k))$ or $u^k(t) = u_2^k(t, \gamma(\lambda_k))$. For some subsequence it is equal to one of them, say $u_1^k(t, \gamma(\lambda_k))$. Now we shall consider the function $u_1^k(t, \gamma(\lambda_k)), t \in [0, T]$. We have
\[
u^k(t) = \begin{cases} 
u_1(t), & \text{if } t \in [0, \gamma(\lambda_k)], \\
u_1^k(t, \gamma(\lambda_k)), & \text{if } t \in [\gamma(\lambda_k), T],
\end{cases}
\]
where $\gamma(\lambda_k) \to y_0 \in [0, T]$. For $j = 0, 1$ we define the functions
\[
\overline{f}_{jk}(x, v) = \begin{cases} 
\overline{f}_j(x, v), & \text{if } t \in [0, \gamma(\lambda_k)], \\
\overline{f}_j(x, v), & \text{if } t \in [\gamma(\lambda_k), T],
\end{cases}
\]
By continuity $u_1(\gamma(\lambda_k)) \to u_1(y_0), k \to \infty$.

By (22) the sequence $\{u^k(t)\}$ is bounded in $L^\infty(0, T; H) \cap L^2(0, T; V) \cap L^p(0, T; L^p(\mathbb{R}^N))$. It follows also that $\frac{du^k}{dt}$ is bounded in $L^2(0, T; L^2(\mathbb{R}^N)) + L^2(0, T; V')$. Then for some function $u = u(t, x)$ we have
\[
u^k \to u \quad \text{weakly in } L^2(0, T; V),
\]
\[
u^k \to u \quad \text{weakly in } L^p(0, T; L^p(\mathbb{R}^N)),
\]
\[
\frac{du^k}{dt} \to \frac{du}{dt} \quad \text{weakly in } L^2(0, T; V') + L^2(0, T; L^2(\mathbb{R}^N)),
\]
\[
u^k \to u \quad \text{weakly star in } L^\infty(0, T; H).
\] (32)
Let $\Omega_n = \{x \in \mathbb{R}^N : |x| < n\}$ and $L_n v = v|_{\Omega_n}$, that is, $L_n v$ is the restriction of $v$ to $\Omega_n$. We can easily see that $L_n u^k$ converges to $Lu$ weakly in $L^2(0, T; H^1(\Omega_n))^d$ and $L^p(0, T; L^q(\Omega_n))$, and weakly star in $L^{\infty}(0, T; [L^2(\Omega_n)]^d).$ Moreover, $L_n u^k$ satisfies (13) for any $k \geq n$, $v \in [C^0_0((0, T) \times \Omega_n)]^d$ (but replacing $\int_{\Omega_n} f$ by $f_{1k}$). Hence, $\frac{dL_n u^k}{dt}$ is bounded in $L^2(0, T; [H^{-1}(\Omega_n)]^d) + L^q(0, T; L^q(\Omega_n))$, which is continuously embedded in $L^q(0, T; L^2(\Omega_n))$ for $s = (s_1, \ldots, s_d)$ satisfying $s_i = \max\{1, N(\frac{1}{q} - \frac{1}{2})\}$, where $H^{-1}(\Omega_n) = H^{-s_1}(\Omega_n) \times \cdots \times H^{-s_2}(\Omega_n)$, so that
\[
\frac{dL_n u^k}{dt} \to \frac{dL_n u}{dt} \text{ weakly in } L^2(0, T; H^{-s}(\Omega_n)).
\]
(33)

By the compactness lemma [13] we obtain (up to a subsequence) that
\[
L_n u^k \to L_n u \text{ strongly in } L^2(0, T; [L^2(\Omega_n)]^d),
\]
\[
L_n u^k(t) \to L_n u(t) \text{ in } H \text{ for a.a. } t \in (0, T),
\]
\[
L_n u^k(t, x) \to L_n u(t, x) \text{ for a.a. } (t, x) \in (0, T) \times \Omega_n, \text{ for all } n \geq 1.
\]

Hence, $\tilde{f}_{1k}(x, L_n u^k(t, x)) \to \tilde{f}_1(t, L_n u(t, x))$, for a.a. $(t, x) \in (0, T) \times \Omega_n$, and then the boundedness of $\tilde{f}_{1k}(x, L_n u^k)$ in $L^q(0, T; L^q(\Omega_n))$ implies that $f_1(x, L_n u^k)$ converges to $f_1(x, L_n u)$ weakly in $L^q(0, T; L^q(\Omega_n))$ for $n \geq 1 [13].$ In the same way $\tilde{f}_{0k}(x, L_n u^k)$ converges to $f_0(x, L_n u)$ weakly in $L^2(0, T; [L^2(\Omega_n)]^d)$.

Also, we note that (34) and Lemma 6 imply (up to a subsequence) that
\[
u^k \to \nu \text{ strongly in } L^2(0, T; [L^2(\mathbb{R}^N)]^d),
\]
\[
u^k(t) \to \nu(t) \text{ in } H \text{ for a.a. } t \in (0, T).
\]
(35)

Moreover, $L_n u^k(t) \to L_n u(t)$ weakly in $H$ for all $t \in [0, T]$ and $n \geq 1.$ Indeed, as $\frac{dL_n u^k}{dt}$ is a bounded sequence of the space $L^2(0, T; H^{-s}(\Omega_n))$, we have that $L_n u^k(t) : [0, T] \to H^{-s}(\Omega_n)$ is an equicontinuous family of functions. By (21) for each fixed $r \in [0, T]$ the sequence $L_n u^k(t)$ is bounded in $[L^2(\Omega_n)]^d$, so that the compact embedding $L^2(\Omega_n) \subset H^{-s}(\Omega_n)$ implies that $L_n u^k(t)$ is a precompact sequence in $C([0, T], H^{-s}(\Omega_n)).$ Hence, since $L_n u^k \to L_n u$ weakly in $L^2(0, T; H^{-s}(\Omega_n))$, we have $L_n u^k \to L_n u$ in $C([0, T], H^{-s}(\Omega_n)).$ The boundedness of $L_n u^k(t)$ in $H^{-s}(\Omega_n)$ implies then by a standard argument that $L_n u^k(t) \to L_n u(t)$ weakly in $[L^2(\Omega_n)]^d$ for all $r$.

Then it follows easily that $u^k(t) \to u(t)$ weakly in $[L^2(\mathbb{R}^N)]^d$ for any $t \in [0, T].$

Also, we deduce $u(0) = u_0$. As $L_n u^k$ satisfies (13) for any $k \geq n$, $v \in [C^0_0((0, T) \times \Omega_n)]^d$ (but replacing $\int_{\Omega_n} f$ by $f_{1k}$ and $f$ by $\tilde{f}_k$), passing to the limit we obtain that $u$ is a weak solution.

Finally, we shall prove the following:

Lemma 9. We have:
\[
u^k(t^*) \to \nu(t^*) \text{ strongly in } H.
\]

Proof. From (15) and (22) we have
\[
\|u^k(t)\|^2 \leq \|u^k(s)\|^2 + 2M(t - s), \quad \|u(t)\|^2 \leq \|u(s)\|^2 + 2M(t - s),
\]
(36)
for all $t \geq s$, $t, s \in [0, T]$, where the constant $M > 0$ does not depend on $k$. From (36) the functions $J_k(t) = \|u^k(t)\|^2 - 2Mt$, $J(t) = \|u(t)\|^2 - 2Mt$ are continuous and non-increasing on $[0, T].$

We state that $\limsup J_k(t^*) \leq J(t^*)$.

We know from (34) that $J_k(t) \to J(t)$, for a.a. $t \in (0, T).$ Let $t_m$ be a sequence such that $0 < t_m < t^*$, $t_m \to t^*$, as $m \to \infty$, and $J_k(t_m) \to J(t_m)$, as $k \to \infty$, for any fixed $m$. Hence, using the continuity of $f$ and the monotonicity of $J_k$, $J$ we have that for any $\varepsilon > 0$ there exist $m(\varepsilon)$ and $\varepsilon(\varepsilon, m)$ such that
\[
J_k(t^*) - J(t^*) = J_k(t^*) - J_k(t_m) + J_k(t_m) - J(t_m) + J(t_m) - J(t^*)
\]
\[
\leq |J_k(t_m) - J(t_m)| + |J(t_m) - J(t^*)| \leq 2\varepsilon,
\]
if $k \geq K$. Hence,
\[
\limsup J_k(t^*) = \limsup \|u^k(t^*)\|^2 - 2M t^* \leq \|u(t^*)\|^2 - 2M t^*.
\]
Therefore, $\limsup \|u^k(t^*)\| \leq \|u(t^*)\|$. Since $u^k(t^*) \to u(t^*)$ weakly in $H$, we have $\liminf \|u^k(t^*)\| \geq \|u(t^*)\|$. Thus, $u^k(t^*) \to u(t^*)$ in $H$. \square
4. Connectedness of the global attractor

In [18, Theorem 1] it was proved the existence of a global compact invariant minimal attractor for system (1) under conditions (H1)–(H4). Our aim now is to obtain that this attractor is also connected in $H = [L^2(\mathbb{R}^N)]^D$ if we add condition (H5).

We recall briefly the main definitions of the theory of global attractors for multivalued semiflows. Denote by $P(H) (B(H))$ the set of all non-empty (non-empty bounded) subsets of the complete metric space $H$. We recall that the multivalued map $G: \mathbb{R}^+ \times H \rightarrow P(H)$ is said to be a multivalued semiflow if:

1. $G(0, \cdot) = Id$ (the identity map);
2. $G(t + s, u) \subset G(t, G(s, u))$, for all $u \in H, t, s \in \mathbb{R}^+$.

It is called a strict multivalued semiflow if $G(t + s, u) = G(t, G(s, u))$.

The set $A$ is said to be a global attractor if it attracts any bounded set $B \in B(H)$, that is,

$$\text{dist}(G(t, B), A) \rightarrow 0, \quad \text{as } t \rightarrow +\infty,$$

where $\text{dist}(C, D) = \sup_{c \in C} \inf_{d \in D} \rho(c, d)$ ($\rho$ is the metric in $H$), and $A \subset G(t, A)$, for all $t \geq 0$. It is invariant if, moreover, $A = G(t, A)$, for all $t \geq 0$.

A trajectory of $G$ is a function $u(\cdot): \mathbb{R}^+ \rightarrow X$ such that $u(t + \tau) \in G(t, u(\tau))$, $\forall t, \tau \in \mathbb{R}^+$. The semiflow $G$ is called time-continuous if

$$G(t, u_0) = \bigcup \{u(t): u(\cdot) \text{ is a trajectory and } u(\cdot) \in C(\mathbb{R}^+; X), \ u(0) = u_0\}, \quad \forall u_0 \in X.$$ 

The map $G(t, \cdot): H \rightarrow P(H)$ is called upper semicontinuous if for any $u \in H$ and any open set $O$ containing $G(t, \cdot)$ there exists $\delta > 0$ such that $G(t, y) \subset O$, for all $y$ such that $\rho(y, u) < \delta$.

Let us recall the following criterion for the connectedness of the global attractor.

**Theorem 10.** ([16, Theorem 5 and Remark 8]) Let $G$ be a strict time-continuous multivalued semiflow with closed and connected values. Let the map $u \mapsto G(t, u)$ be upper semicontinuous for any $t \geq 0$. Assume the existence of a compact set $K$ such that

$$\text{dist}(G(t, B), K) \rightarrow 0, \quad \text{as } t \rightarrow +\infty, \forall B \in B(H).$$

(37)

If the phase space $H$ is connected, then $G$ has the global compact invariant attractor $A$, which is connected and the minimal closed set attracting each $B \in B(H)$.

We shall apply Theorem 10 to system (1).

Let $D(u_0)$ be the set of all weak solutions (defined for $t \geq 0$) such that $u(0) = u_0$. Define the map $G$ as

$$G(t, u_0) = \{u(t): u(\cdot) \in D(u_0)\}.$$
In [18] it is shown that any weak solution of (1) can be extended to a global one, i.e., it is defined for $t \in [0, +\infty)$. Hence, $G(t, u_0) = K_t(u_0)$ for any $T > t$, so that by Theorem 7 the map $G$ has connected values.

It follows also by [18, Lemma 7] that the semiflow $G$ is strict. Moreover, the translation $u(\cdot + r)$ of a solution is again a solution for any $\tau \geq 0$, so every solution is a trajectory, and then the semiflow $G$ is time-continuous.

In [18, Lemma 12] it is proved that the map $u \mapsto G(t, u)$ is upper semicontinuous for any $t \geq 0$ and has compact values. Also, in [18, Theorem 1] it is shown the existence of a global compact invariant attractor $\mathcal{A}$ for $G$. Taking $K = \mathcal{A}$ it is clear that (37) holds.

Finally, it is obvious that the space $H = \{L^2(\mathbb{R}^N)\}^d$ is connected. Hence, all conditions of Theorem 10 are satisfied. We have:

**Theorem 11.** Let (H1)-(H5) hold. Then the multivalued semiflow $G$ generated by (1) has a global compact invariant connected attractor, which is the minimal closed set attracting each $B \in \mathcal{B}(H)$.

5. Application to the complex Ginzburg–Landau equation in $\mathbb{R}^N$

Let us consider the complex Ginzburg–Landau equation

\[
\begin{align*}
\frac{\partial z}{\partial t} &= (1 + \eta i)\Delta z + Rz - (1 + ib)|z|^2 z + g(x), \\
z(x, 0) &= z_0(x),
\end{align*}
\]

(38)

where $z = z(x, t) = u(x, t) + iv(x, t)$, $(x, t) \in \mathbb{R}^N \times [0, T]$, $g(x) = g_1(x) + g_2(x)i$, $\eta, b \in \mathbb{R}$, $R < 0$. We assume that $g_j \in \{L^2(\mathbb{R}^N)\}^d$, $j = 1, 2$.

For $y = (u, v)$, $z = u + iv$, Eq. (38) can be written as the system

\[
\begin{align*}
\frac{\partial y}{\partial \tau} &= \begin{pmatrix}
1 & -\eta \\
\eta & 1
\end{pmatrix} \Delta y + \begin{pmatrix}
Ru - (|u|^2 + |v|^2)(u - bv) \\
Rv - (|u|^2 + |v|^2)(bu + v)
\end{pmatrix} + \begin{pmatrix} g_1(x) \\ g_2(x) \end{pmatrix}.
\end{align*}
\]

(39)

We have that $f(x, y) := \begin{pmatrix}
-Ru + g_1(x) \\
-Rv + g_2(x)
\end{pmatrix}$, and this function can be written as $f(x, y) = f_0(x, y) + f_1(x, y)$, where

\[
\begin{align*}
f_0(x, y) &:= \begin{pmatrix}
-Ru + g_1(x) \\
-Rv + g_2(x)
\end{pmatrix}, & f_1(x, y) &:= |y|^2 \begin{pmatrix} u - bv \\ bu + v \end{pmatrix}.
\end{align*}
\]

Conditions (H1)-(H2) are obviously satisfied. Let us check (H3)-(H5). First,

\[
\begin{align*}
(f_0(x, y), y) &= (-R)|y|^2 + g_1 u + g_2 v \geq -\frac{R}{2} |y|^2 + \frac{1}{2R} (g_1^2 + g_2^2), \\
(f_1(x, y), y) &= |y|^2 [(u - bv)u + (bu + v)v] = |y|^2 (u^2 + v^2) = |y|^4,
\end{align*}
\]

so that, as $R < 0$, (H4) holds with $\alpha = -\frac{b}{2}$, $C_0(x) = -\frac{1}{2\pi} (g_1^2(x) + g_2^2(x))$, $\beta = 1$, $C_1(x) = 0$.

Further,

\[
|f_0(x, y)|^2 \leq \left( (-Ru + g_1)^2 + (-Rv + g_2)^2 \right) \leq 2(R^2|y|^2 + |g(x)|^2),
\]

so

\[
|f_0(x, y)| \leq \sqrt{2}(R|y| + |g(x)|),
\]

and

\[
\sum_{i=1}^{2} |f_i(x, y)|^\frac{4}{3} \leq \left( |y|^2 \right)^\frac{4}{3} \left( |u - bv|^\frac{4}{3} + (bu + v)^\frac{4}{3} \right) \leq K_1 \left( |y|^\frac{4}{3} \left( |u - bv|^2 + (bu + v)^2 \right)^\frac{2}{3} \right)
\]

\[
\leq K_2 |y| \left( 1 + b^2 \right)^\frac{2}{3} (u^2 + v^2)^\frac{2}{3} = K_3 |y|^4.
\]

Hence, (H4) holds with $\eta = \sqrt{2}R$, $C_2(x) = \sqrt{2}|g(x)|$, $\gamma = K_3$, $C_3(x) = 0$.

Finally, condition (H5) is obviously satisfied (see Remark 1).

**Remark 12.** If $|b| \leq \sqrt{3}$, then conditions (17)-(18) are satisfied for Eq. (38), so that we have uniqueness of the Cauchy problem (see [6, p. 42]). However, in the general case this property can fail.

**Remark 13.** The condition $R < 0$ is not necessary in the case of bounded domains [11]. However, for unbounded domains we need these assumptions in order to obtain (H3)-(H4).
As a consequence of Theorems 7, 11 we obtain:

**Theorem 14.** Let $R < 0$ and $g_j \in L^2(\mathbb{R}^N)$, $j = 1, 2$. Then the attainability set $K_t(y_0)$ of system (39) is compact and connected in $[L^2(\mathbb{R}^N)]^2$ for all $y_0 \in [L^2(\mathbb{R}^N)]^2$. Moreover, it generates a multivalued semiflow having a global compact invariant connected attractor.

**Acknowledgements**

We wish to express our thanks to the anonymous referee for his/her useful suggestions.

This work was partially supported by Spanish Ministerio de Ciencia e Innovación, Project MTM2008-00088, the Consejería de Innovación, Ciencia y Empresa (Junta de Andalucía), grant P07-FQM-02468, and the Consejería de Cultura y Educación (Comunidad Autónoma de Murcia), grant 08667/P1/08.

**References**


