Determination of mode II fracture toughness for U-shaped notches using Brazilian disc specimen

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A B S T R A C T

A brittle fracture criterion is proposed for predicting fracture toughness of U-shaped notches under pure mode II loading. The criterion, called UMTS, is developed based on the maximum tangential stress (MTS) criterion. The UMTS criterion can be generally used for determining the mode II fracture toughness of U-notched components as well as the fracture initiation angle in U-shaped notches under pure mode II loading. To verify the validity of the proposed criterion, a set of experiments were carried out on the U-notched Brazilian disc (UNBD) specimens made of PMMA and also soda-lime glass. It is shown that there is a good agreement between the results of the UMTS criterion and the experimental results both for fracture toughness and for the fracture initiation angle under pure mode II conditions.

1. Introduction

Different types of notches particularly U-shaped notches are frequently used in the machine components or structural elements made of brittle or quasi-brittle materials like polymethyl-methacrylate (PMMA), ceramics, rock materials, graphite, etc. These notches can dramatically decrease the load bearing capacity of components due to the concentration of stress at the vicinity of their tips. Therefore, an appropriate failure criterion is critically required to predict the fracture initiation in the notched components. Brittle fracture is one of the major failure modes in the U-notched components made of brittle materials.

Several criteria have been proposed in the past and validated via experimental results for mode II and mixed-mode brittle fracture in bodies containing a sharp crack. One of the frequently used criteria is the maximum tangential stress (MTS) criterion proposed originally by Erdogan and Sih (1963). Later Smith et al. (2001) modified the MTS criterion by taking into account the effect of T-stress in the stress field around the crack tip and hence proposed the generalized MTS (GMTS) criterion. Although mixed-mode fracture has been studied for cracked components by many researchers, so far few criteria have been proposed for investigating brittle fracture in engineering components containing V or U-notches. For mode I loading conditions and in the presence of sharp V-shaped notches, several fracture criteria have been suggested by Novozhilov (1969), Seweryn (1994), Leguillon (2001, 2002), Gomez and Elices (2003a,b), Strandberg (2002), Yosibash et al. (2004), Lazzarin and Zambardi (2001) and Zhang and Li (2008). However, there are very few mixed-mode I/II brittle fracture criteria for components containing sharp V-notches (see Dunn et al. (1997a,1997), Seweryn and Lukaszewicz (2002), Yosibash et al. (2006), Priel et al. (2007) and Chen and Ozaki (2008)). Several mode I failure criteria have also been proposed and evaluated experimentally for rounded-tip V-notches and in the special case for U-notches (see Leguillon and Yosibash, 2003; Gomez and Elices, 2004, 2005 and Gomez et al., 2006). The number of failure criteria suggested in the past for predicting mixed-mode brittle fracture in U-notches is again very few. For example, Gomez et al. (2007) applied the concept of the local strain energy density to estimate fracture loads for U-notched components made of PMMA. Berto et al. (2007) used two distinct concepts (the local strain energy density (SED) and the cohesive zone model) in order to investigate mixed mode brittle failure in U-notches. The cohesive zone model has also been applied by Gomez et al. (2009) to evaluate the experimental results of mixed mode fracture in U-notched components. In another study, Livieri (2008) used the J-integral for predicting static failures in sharp V-notches and rounded U-notches.

In comparison with extensive studies conducted in the past on mode I brittle fracture, much fewer papers deal with pure mode II fracture in cracked or notched bodies. For example, in a series of recent papers Ayatollahi and his coworkers (2005, 2006, 2007 and 2009) have investigated pure mode II fracture in different materials like rocks, soda-lime glass and PMMA but only for sharp...
crack problems. However, for components containing V and U-shaped notches and loaded under pure mode II, no failure criterion has been previously suggested for predicting brittle fracture.

In the present paper, a brittle fracture criterion, called UMTS, is proposed based on the maximum tangential stress (MTS) criterion for predicting fracture toughness and fracture initiation angle in U-shaped notches under pure mode II loading. The validity of the theoretical results is verified by a set of experiments performed on an improved Brazilian disc specimen. A good agreement is shown to exist between the results of the UMTS criterion and the experimental results for various notch radii.

2. Mixed-mode stress distribution around U-shaped notches

There are two common types for rounded-tip notches: U-shape notches and V-shape notches. The U-notch is a special case for rounded tip V-notches in which the notch edges are parallel. When the notch tip radius becomes zero, the U-notch turns into a sharp crack. The notched and cracked specimens can be subjected to three different types of in-plane loading, often called pure mode I, pure mode II and mixed mode I-II loading. Under pure mode I loading, any two respective points along the notch faces open relative to the notch bi-sector line without any sliding. In this case, the shear stress along the bi-sector line is zero. In pure mode II, the two respective points along the notch faces slide relative to the notch bi-sector line without any opening and the tangential stress along the bi-sector line is zero. Any combination of mode I and mode II deformation is called mixed mode loading.

The stress field for sharp V-shaped notches was originally suggested by Williams (1952) as an exact solution. For rounded-tip V-shaped notches under mixed-mode loading, an expression was later developed by Filippi et al. (2002) to describe the stress distribution around the notch tip. This expression was an approximate solution because it satisfies the boundary conditions only in a finite number of points along the notch edge and not on the whole edge. Filippi et al. (2002) obtained the stress distribution using a conformal mapping in an auxiliary system of curvilinear coordinates “u and v” that are related to the Cartesian coordinates “x and y” as \((x + iy) = (u + iv)^{1/q}\). The power \(q\) is a real positive coefficient ranging from 1 (for a flat edge) to 2 (for a crack).

The elastic stresses shown in Fig. 1 can be written for mixed-mode conditions as:

\[
\begin{bmatrix}
\sigma_{uu} \\
\sigma_{uv} \\
\sigma_{vv}
\end{bmatrix}
= \frac{K_I^{V,\rho}}{\sqrt{2\pi r^{1-2q}}} \begin{bmatrix}
f_I(\theta) \\
f_{II}(\theta) \\
f_{III}(\theta)
\end{bmatrix}
+ \frac{K_{II}^{V,\rho}}{\sqrt{2\pi r^{1-2q}}} \begin{bmatrix}
f_{II}(\theta) \\
f_{II}(\theta) \\
f_{II}(\theta)
\end{bmatrix}
\]

(1)

where \(r\) and \(\theta\) are the polar coordinates shown in Fig. 1 and \(K_I^{V,\rho}\) and \(K_{II}^{V,\rho}\) are the mode I and mode II notch stress intensity factors (NSIFs), respectively. The parameter \(q\) is the distance between the origin of polar coordinate system and the notch tip which can be computed from Eq. (2). The functions \(f_I(\theta)\) and \(f_{II}(\theta)\) have been given in Appendix A. The eigenvalues \(\lambda_I\) and \(\mu_I\) which depend on the notch opening angle have been reported in Filippi et al. (2002).

The following expression is valid according to a relation that exists between the Cartesian and the curvilinear coordinate systems (Filippi et al., 2002):

\[
r_0 = \frac{q - 1}{q} \rho, \quad q = \frac{2\pi - 2\alpha}{\pi}
\]

(2)

In Eq. (2), the parameter \(2\alpha\) is the notch opening angle and \(\rho\) is the notch tip radius. The expressions for notch stress intensity factors (NSIFs) are (Lazzarin and Filippi, 2006):

\[
K_I^{V,\rho} = \sqrt{2\pi r} \left( \sigma_{uu(r,0)} \right)^{1-2q} \left( 1 + \omega_1 \left( \frac{r}{\rho} \right)^{2q-2} \right)
\]

(3)

\[
K_{II}^{V,\rho} = \sqrt{2\pi r} \left( \sigma_{uv(r,0)} \right)^{1-2q} \left( 1 + \omega_2 \left( \frac{r}{\rho} \right)^{2q-2} \right)
\]

(4)

where \(\sigma_{uu}\) and \(\sigma_{uv}\) are the tangential and the in-plane shear stresses, respectively. The auxiliary parameters \(\omega_1\) and \(\omega_2\) are defined in Appendix A. When the notch opening angle \(2\alpha\) is zero, then \(q = 2\) and the rounded-tip V-notch becomes a U-notch. Therefore, one can write Eq. (2) as:

\[
r_0 = \frac{\rho}{2}
\]

(5)

By substituting Eq. (5) into Eq. (1), an expression for the tangential stress in U-shaped notches can be derived as:

\[
\sigma_{uu}(r,0) = \frac{1}{2\sqrt{2\pi r}} \left( \frac{3}{2} + \frac{\rho}{r} \right) \left( \frac{3}{2} \cos \frac{\theta}{2} + \frac{3}{2} \cos \frac{3\theta}{2} \right)
\]

\[
+ \frac{\sqrt{2\pi r}}{2} \left( \frac{3}{2} + \frac{\rho}{r} \right) \left( \frac{3}{2} \sin \frac{\theta}{2} + \frac{3}{2} \sin \frac{3\theta}{2} \right)
\]

(6)

Eq. (6) coincides with that of Creager and Paris (1967) which has been proposed to describe the stress field ahead of a rounded-tip crack.

Using Eq. (5) and considering that \(\omega_1\) and \(\omega_2\) for U-notches are 1 and -1, respectively (Livieri, 2008), Eqs. (3) and (4) can be finally rewritten for U-notches as:

\[
K_I^{U} = \sqrt{\frac{2\pi r}{\rho}} \sigma_{uu}(\frac{\rho}{2}, 0)
\]

(7)

\[
K_{II}^{U} = \lim_{r \to 0} \sqrt{2\pi r} \left( \sigma_{uu(r,0)} \right) \left( 1 - \frac{2}{3q} \right)
\]

(8)

Note that the parameter \(r\) in Eq. (8) cannot be directly substituted by \(\rho/2\) because the denominator becomes zero. Therefore, the limit of expression very close to the notch tip is used to calculate \(K_{II}^{U}\).

In the next section, the tangential stress presented by Eq. (6) is used to develop the mode II UMTS criterion.
3. Brittle fracture criterion

The maximum tangential stress (MTS) criterion is a well known criterion frequently used for investigating brittle fracture in cracked components under mixed-mode I/II loading. For cracked bodies, the MTS criterion suggests that fracture takes place along the direction of maximum tangential stress \( \theta_0 \). Brittle fracture also initiates when the tangential stress at a critical distance \( r_c \) from the sharp crack and along \( \theta_0 \) attains its critical value \((\sigma_{\omega(c)})_c\). In the present paper, the same principles of MTS criterion are extended from cracked bodies to U-notched domains and then a criterion called UMTS is developed for predicting fracture toughness and also fracture initiation angle of U-notches under pure mode II deformation. Under mode II loading conditions, \( K_{II}^U \) becomes zero and Eq. (6) can be rewritten as:

\[
\sigma_{\omega}(r, \theta) = \frac{K_{II}^U}{2\sqrt{2\pi r}} \left[ \frac{3}{2} - \frac{\rho}{r} \right] \sin \frac{\theta}{2} + \frac{3}{2} \sin \frac{3\theta}{2}
\]

Considering the principles of the MTS criterion, one can use the following equation in order to obtain the fracture initiation angle:

\[
\frac{\partial \sigma_{\omega}(r, \theta)}{\partial \theta} = 0
\]

Substituting Eq. (9) into Eq. (10) gives:

\[
\left( \frac{3}{4} - \frac{\rho}{2r_c} \right) \cos \theta_0 + \frac{9}{2} \cos \frac{3\theta_0}{2} = 0
\]

where the fracture angle \( \theta_0 \) for mode II loading can be denoted by \( \theta_0 \).

Note that according to the requirements of the UMTS criterion, the parameter \( r_c \) is substituted with its critical value \( r_{cU} \), called the U-notch critical distance. Eq. (11) indicates that mode II fracture initiates along the angle \( \theta_0 \) with respect to the notch bi-sector line which depends on the notch critical distance \( r_{cU} \) and the notch radius \( r \). For a more concise presentation of the relations, the following symbols are defined here:

\[
A = 2 + \frac{\rho}{r_{cU}}
\]

\[
B = \frac{1}{2} - \frac{\rho}{r_{cU}}
\]

Using Eqs. (12) and (9) and based on the UMTS criterion, the onset of mode II fracture takes place when:

\[
(\sigma_{\omega(c)})_c = \frac{K_{II}^U}{2\sqrt{2\pi r_{cU}} \left[ B \sin \frac{\theta_0}{2} + \frac{3}{2} \sin \frac{3\theta_0}{2} \right]}
\]

where \( K_{II}^U \) is the critical value of \( K_{II}^U \) at fracture or the mode II fracture toughness of U-shaped notches. It is often preferred to determine the mode II fracture toughness in terms of the well-known mode I fracture toughness. For this purpose one can simplify Eq. (6) for pure mode I loading condition as:

\[
\sigma_{\omega}(r, \theta) = \frac{K_{II}^U}{2\sqrt{2\pi r}} \left[ \frac{3}{2} - \frac{\rho}{r} \right] \cos \frac{\theta}{2} \sin \frac{\theta}{2}
\]

Mode I fracture occurs when the mode I notch stress intensity factor \( K_{II}^U \) reaches its critical value \( (K_{II}^U)_{c} \), called mode I notch fracture toughness. For a U-notch, the parameter \( K_{II}^U \) which can be determined experimentally, depends not only on the material properties but also on the notch radius. Mode I fracture initiates along the notch bi-sector line (i.e., \( \theta_0 = 0 \)) due to the symmetry in geometry and loading conditions. At the onset of fracture, the parameters \( K_{II}^U \) and \( \sigma_{\omega(c)} \) attain their critical values \( (K_{II}^U)_{c} \) and \( (\sigma_{\omega(c)})_c \), respectively. Therefore, one can write for mode I fracture:

\[
\theta_0 = 0
\]

\[
K_{II}^U = (K_{II}^U)_{c}
\]

\[
\sigma_{\omega(c)} = (\sigma_{\omega(c)})_c
\]

Substituting Eq. (15) into Eq. (14) and using Eq. (12) we have:

\[
2\sqrt{2\pi r_{cU} (\sigma_{\omega(c)})_c} = A K_{II}^U
\]

Now, the notch fracture toughness ratio \( (\sigma_{\omega(c)})_c \) can be obtained by substituting Eq. (13) into Eq. (16):

\[
\frac{K_{II}^U}{K_{II}^U} = \frac{A}{B \sin \frac{\theta_0}{2} + \frac{3}{2} \sin \frac{3\theta_0}{2}}
\]

Ultimately, using Eq. (17) one can directly predict the mode II fracture toughness of U-shaped notches in terms of its mode I fracture toughness.

For sharp cracks, it is assumed that the material parameters \( (\sigma_{\omega(c)})_c \), and \( r_c \) are independent of mode mixity and have the same values for pure mode I and pure mode II cases. This assumption has been validated through numerous fracture tests on various brittle materials, e.g., Ayatollahi and Aliha (2005, 2006), Ayatollahi et al. (2006) and Smith et al. (2006). It should be noted that the same assumption were used here to derive Eq. (17) for U-notches.

It is also noteworthy that Eq. (17) depends on \( A \) and \( B \) which are functions of \( r_{cU} \). The notch critical distance \( r_{cU} \) can be calculated by squaring both sides of Eq. (16):

\[
(\sigma_{\omega(c)})_c^2 = \frac{1}{8\pi r_{cU} \left[ 2 + \frac{\rho}{r_{cU}} \right]} K_{II}^U \frac{4\pi^2}{r_{cU}}
\]

which can be rewritten as:

\[
4r_{cU}^2 + \rho^2 + 4\rho r_{cU} \left[ \frac{K_{II}^U}{(\sigma_{\omega(c)})_c} \right]^2 - 8\pi^2 r_{cU} = 0
\]

If the two critical parameters \( K_{II}^U \) and \( (\sigma_{\omega(c)})_c \) are known, the parameter \( r_{cU} \) can be determined by solving Eq. (19) for any value of notch radius.

According to Eq. (19), the parameter \( r_{cU} \) depends on \( \rho \) and \( K_{II}^U \). Note that the parameter \( r_{cU} \) is measured from the origin of the polar coordinate system (Fig. 1) and not from the notch tip. Therefore, the critical distance \( r_{cU} \) is a geometry dependent parameter and is not a fixed material property for U-notches. In general, Eq. (19) has three roots: two complex roots and one real root. Since \( r_{cU} \) is a real parameter, only the real root is acceptable.

The parameter \( (\sigma_{\omega(c)})_c \) is a material property and is commonly considered to be the ultimate tensile strength \( \sigma_u \) for brittle and quasi-brittle materials because the experimental investigations demonstrate that for brittle materials, fracture often occurs in the plane where the tensile stress is maximum. Also, these materials have relatively low fracture resistance to the tensile loads. The final fracture for a specimen subjected to tensile loading will occur when the molecular bonds of the material are broken. This condition can be assumed to be valid for both defected (cracked or notched) and non-defected samples. The major difference between a notched or cracked specimen and a flawless one is that for the defected sample, the stress gradient in the vicinity of the notch tip is very high and hence the local stress reaches \( \sigma_u \) under lower values of the load.

It should be noted that some other parameters such as \( \sigma_s \) (inherent stress) have also been suggested in literature for being used instead of \( (\sigma_{\omega(c)})_c \). However, for brittle materials like ceramics \( \sigma_s \) can be assumed to be equal to \( \sigma_u \). Also for quasi-brittle materials which have certain amount of nonlinear deformation, the parameter \( \sigma_u \) takes on a value which is slightly higher than \( \sigma_u \) (Susmel and Taylor, 2008).
In order to evaluate the accuracy of the UMTS criterion, it is essential to perform pure mode II fracture tests on appropriate U-notched specimens. For this reason, a modified test sample is introduced in the next section and its notch parameters (NSIFs) are computed by using the finite element method.

4. Finite element analysis

In this work, the U-notched Brazilian disc (UNBD) specimens were used for an experimental investigation of mode II fracture in U-shaped notches. As shown schematically in Fig. 2, the UNBD specimen is a disc of circular shape that contains a central U-notch. When the loading angle $\beta$, i.e. the angle between the notch bi-sector and the loading direction, changes the ratio of the mode I and mode II NSIFs varies.

If the direction of compressive applied load is along the notch bi-sector line ($\beta = 0$), the U-notch is subjected to pure mode I loading. For a non-zero angle between the direction of applied load and the notch bi-sector, the notch is subjected to mixed mode I/II loading. A gradual increase of the loading angle results in a reduction in mode I effects and an elevation in mode II effects. Finally, one can find a specific loading angle $\beta_B$ for which the specimen undergoes pure mode II deformation. This angle was found in this research by a series of finite element analyses as elaborated below.

A plane stress FE model with a total number of 52800 Quad 8 elements was created for simulating the specimen. The FE analyses were performed using the commercial code ABAQUS. A large number of elements were used near the notch tip due to its high stress gradient. Fig. 3 shows a sample FE grid pattern used for simulating a UNBD specimen with a notch radius $\rho$ equal to 4 mm.

In order to compute the notch stress intensity factors ($K_I^U$ and $K_{II}^U$), it is necessary to create an appropriate finite element model of the specimens that are considered for performing the fracture tests. Then, the values of the tangential and the shear stresses ($\sigma_{t0}$ and $\sigma_{s0}$) along the notch bi-sector line could be obtained from the FE results. After calculating the values of $\sigma_{t0}$ and $\sigma_{s0}$ corresponding to the applied load, one can substitute these values into Eqs. (7) and (8) in order to compute the notch stress intensity factors (NSIFs).

In order to obtain the pure mode II loading angle $\beta_B$, the angle $\beta$ was gradually increased from zero and the value of tangential stress $\sigma_{t0}(r_0, 0)$ at the notch tip was calculated for each loading angle, under an arbitrary compressive load. As the loading angle increased, the value of $\sigma_{t0}(r_0, 0)$ decreased until it was equal to zero. According to Eq. (7), when $\sigma_{t0}(r_0, 0) = 0$ the mode I NSIF ($K_I^U$) is zero, and hence the U-notch is subjected to pure mode II deformation. Therefore, the mode II loading angle $\beta_B$ is the angle for which $\sigma_{t0}(r_0, 0) = 0$. A set of finite element analyses were performed for a disc of diameter 80 mm containing a central notch of U-shape with different notch radii as $\rho = 0.5, 1, 2, 4$ mm. Similar to the specimens which were later used for the experiments, the overall notch length in the FE analyses was equal to one-half the disc diameter. The mode II loading angle $\beta_B$ was then determined from finite element results for each notch radius. Fig. 4 shows the variations of $\beta_B$ versus the notch radius $\rho$. It is seen from this figure that the mode II loading angle is not significantly dependent on the notch radius since the angle $\beta_B$ is very close to 25 (deg.) for all the notch radii. It should be noted that the results shown in Fig. 4 are related to a UNBD specimen for which the overall notch length is equal to one-half of the disc diameter. Similar analyses can also be performed for other notch length ratios as well.

5. Experiments

Several specimens have been used in the past for experimental fracture investigations in cracked and notched components such as the single-edge notched tension (SENT) and the double-edge notched tension (DENT) specimens. The single edge notched rectangular plates are also favorite test samples for fracture testing under three-point bend (TPB) and four-point bend (FPB) loading. The Brazilian disc specimen (BD) is another specimen which has been used frequently for mixed-mode fracture testing but only in cracked domains.
e.g. Awaji and Kato (1998), Awaji and Sato (1978), Atkinson et al. (1982), Shetty et al. (1987) and Chang et al. (2002). The U-notched Brazilian disc (UNBD) specimen used in our experiments is a modified version of the centrally cracked BD specimen in which the center crack is replaced by a central slit with two U-shape notch tips.

To evaluate the validity of the UMTS criterion, a set of fracture tests were performed at room temperature on the U-notched Brazilian disc (UNBD) specimens made of PMMA and soda-lime glass. PMMA is an amorphous glassy polymer that often fails by brittle fracture at room temperature. Soda-lime glass is also a favorite material for the experimental fracture study of brittle materials (Awaji and Kato, 1998; Abrams et al., 2003; Awaji et al., 1999).

The material properties of the tested PMMA and soda-lime glass at room temperature are:

<table>
<thead>
<tr>
<th>Material</th>
<th>Ultimate Tensile Strength ($\sigma_u$)</th>
<th>Fracture Toughness ($K_{IC}$)</th>
<th>Elastic Modulus ($E$)</th>
<th>Poisson's Ratio ($\nu$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>PMMA</td>
<td>70.5 MPa</td>
<td>1.96 MPa$\sqrt{m}$</td>
<td>2963 MPa</td>
<td>0.38</td>
</tr>
<tr>
<td>Soda-lime glass</td>
<td>14 MPa</td>
<td>0.6 MPa$\sqrt{m}$</td>
<td>72 GPa</td>
<td>0.22</td>
</tr>
</tbody>
</table>

The fracture toughness ($K_{IC}$), the ultimate tensile strength ($\sigma_u$), the elastic modulus ($E$) and the Poisson’s ratio ($\nu$) for PMMA and soda-lime glass were determined either experimentally using the standard test techniques or directly from the information given by the manufacturing company.

For all the specimens, the disc diameter and the overall notch length were 80 mm and 40 mm, respectively. Also, the disc thickness for PMMA and soda-lime glass specimens were 10 mm and 6 mm, respectively. The central slit was generated in the centre of the disc using a water-jet technique. In fracture testing of the UNBD specimens, a compressive load was applied diameterally to the specimen by a universal tension-compression machine. Fig. 5 shows a sample UNBD specimen inside the test machine subjected to pure mode I loading.

An increase in the angle $\beta$ between the loading direction and the U-notch bi-sector line gives rise to mixed-mode I/II deformation. As mentioned in the previous section, the loading angle corresponding to mode II deformation in the UNBD specimens is approximately equal to 25 (deg.). Consequently, the mode II fracture tests were performed according to this loading angle (see Fig. 6).

To study the effects of notch radius on fracture behavior of the UNBD specimens, four different values of notch radii as 0.5, 1, 2 and 4 mm were considered for PMMA specimens. Four values were also considered for the soda-lime glass specimens as $\rho = 0.6, 1.2, 3$, and 4 mm. For each material and each notch radius, three individual tests were performed under pure mode I loading condition for determining $K_{IC}^U$. Three tests were also conducted under pure mode II loading in order to determine $K_{IC}^U$. Therefore, a total number of 42 monotonic fracture tests were performed. Fig. 7 shows the UNBD specimens made of PMMA ($\rho = 4$ mm) after the mode I and mode II fracture tests. The fracture load in each test was recorded by a data logger system. In order to obtain the fracture parameters $K_{IC}^U$ and $K_{IC}^U$, the recorded fracture loads were applied on the FE model and the corresponding tangential and shear stresses ($\sigma_{th}$ and $\sigma_{rr}$) were computed along the notch bi-sector line. These two stress components were then substituted into Eqs. (7) and (8) to obtain the critical parameters $K_{IC}^U$ and $K_{IC}^U$.

6. Results

For predicting the notch fracture toughness ratio $K_{IC}^U/K_{IC}^U$ using Eq. (17), one should first determine the critical distance $r_{cr}$ which...
according to Eq. (19) depends on $K_{IIc}$, $\rho$, and $s$. The numerical values of $r_{c,II}$ calculated for different notch radii $\rho$ are given in Tables 1 and 2 for PMMA and soda-lime glass, respectively. Also given in these tables are the average of notch fracture toughness in mode I, $K_{Ic}$, the average of notch fracture toughness ratio $K_{IIc}/K_{Ic}$ and the measured values of mode II fracture angle $\theta_{II}$. Figs. 8 and 9 show the variations of the notch fracture toughness ratio $K_{IIc}/K_{Ic}$ predicted using the UMTS criterion versus the notch radius, together with the experimental results for PMMA:

### Table 1

<table>
<thead>
<tr>
<th>$\rho$ (mm)</th>
<th>$r_{c,II}$ (mm)</th>
<th>$(K_{Ic})_{av}$ (MPa m$^{1/2}$)</th>
<th>$(K_{IIc})_{av}$ (MPa m$^{1/2}$)</th>
<th>$(K_{IIc}/K_{Ic})_{av}$</th>
<th>$\theta_{II}$ (deg.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>0.61</td>
<td>2.23</td>
<td>4.28</td>
<td>1.92</td>
<td>53</td>
</tr>
<tr>
<td>1</td>
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<td>5.51</td>
<td>1.95</td>
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</tr>
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<td>2</td>
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<td>4</td>
<td>2.60</td>
<td>5.10</td>
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<td>2.22</td>
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</tbody>
</table>

### Table 2

<table>
<thead>
<tr>
<th>$\rho$ (mm)</th>
<th>$r_{c,II}$ (mm)</th>
<th>$(K_{Ic})_{av}$ (MPa m$^{1/2}$)</th>
<th>$(K_{IIc})_{av}$ (MPa m$^{1/2}$)</th>
<th>$(K_{IIc}/K_{Ic})_{av}$</th>
<th>$\theta_{II}$ (deg.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.96</td>
<td>0.71</td>
<td>1.29</td>
<td>1.82</td>
<td>65</td>
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<tr>
<td>2</td>
<td>1.89</td>
<td>1</td>
<td>1.85</td>
<td>1.85</td>
<td>68</td>
</tr>
<tr>
<td>4</td>
<td>3.93</td>
<td>1.46</td>
<td>2.92</td>
<td>2.00</td>
<td>56</td>
</tr>
</tbody>
</table>

Fig. 7. The UNBD specimens of PMMA ($\rho = 4$) mm after the fracture tests. (a) Mode I; (b) mode II.

Fig. 8. Variations of the notch fracture toughness ratio $K_{IIc}/K_{Ic}$ versus the notch radius for PMMA-predictions of UMTS criterion together with the experimental data.

Fig. 9. Variations of the notch fracture toughness ratio $K_{IIc}/K_{Ic}$ versus the notch radius for soda-lime glass-predictions of UMTS criterion together with the experimental data.
shown in Figs. 8 and 9 when 2009) and soda-lime glass (Awaji et al., 1999) have also been experimental results reported for PMMA (Ayatollahi and Aliha, \(\frac{20}{00}\)) and soda-lime glass, respectively. When the notch tip radius vanishes.

Variations of the mode II fracture initiation angle in soda-lime glass

\[ \text{Fig. 10. Variations of the mode II fracture initiation angle in PMMA specimens—predictions of UMTS criterion together with experimental results.} \]

\[ \text{Fig. 11. Variations of the mode II fracture initiation angle in soda-lime glass specimens—predictions of UMTS criterion together with experimental results.} \]

and soda-lime glass, respectively. When the notch tip radius vanishes (\(\rho = 0\)), the U-notch becomes a sharp crack for which considerable experimental data for the fracture toughness ratio have been reported in literature. In this case the UMTS criterion simplifies to the conventional MTS criterion for sharp cracks, and the fracture toughness ratio \(K_{IIc}/K_{Ic}\) is always equal to 0.866. The experimental results reported for PMMA (Ayatollahi and Aliha, 2009) and soda-lime glass (Awaji et al., 1999) have also been shown in Figs. 8 and 9 when \(\rho = 0\). These results indicate that the fracture toughness ratio decreases abruptly when the notch tip radius vanishes.

In addition to the fracture toughness ratio, the other important parameter in mode II fracture study of U-shaped notches is the fracture initiation angle \(\theta_{II}\). Figs. 10 and 11 show the curve of the UMTS criterion for the mode II fracture initiation angles of PMMA and soda-lime glass specimens in terms of notch radius. Also shown in these figures are the fracture angles \(\theta_{II}\) measured from the broken specimens. Like the fracture toughness ratio, the conventional MTS criterion suggested for a sharp crack (\(\rho = 0\)) predicts a constant value for the fracture initiation angle under pure mode II loading. The absolute value of this constant angle is \(\theta_{II} = 70.51\) (deg.) which is in agreement with the estimate of the UMTS criterion for a U-notch with \(\rho \neq 0\). Fig. 10 also shows the mode II fracture initiation angle obtained experimentally by Ayatollahi and Aliha (2009) from mode II fracture tests on a centrally cracked PMMA specimen. It is seen that the experimental results are in a good agreement with UMTS predictions for \(\rho = 0\) as well.

It should be noted that the signs of notch fracture toughness ratio and the fracture initiation angle are negative. However, the absolute values are shown in Tables 1 and 2, also in Figs. 8–11.

**7. Discussion**

According to Tables 1 and 2, the mode I notch fracture toughness \((K_{IIc})\) increases when the notch tip radius is increased. This is because the stress concentration at the notch tip decreases for larger notch radii requiring a larger load for fracturing the notched specimen.

It is seen from Tables 1 and 2 that the notch critical distance \(r_{c,II}\) depends on the notch radius \(\rho\) and the material properties as well. In general, \(r_{c,II}\) increases for U-notches having a larger notch radius \(\rho\). For calculating \(r_{c,II}\) more conveniently, we define here the parameter \(\gamma\) as:

\[ \gamma = \frac{K_{IIc}}{\sigma_{IIc}} \] (20c)

Substituting Eq. (20c) into Eq. (19) gives:

\[ \gamma^2(4r_{c,II}^2 + \rho^2 + 4\rho r_{c,II}) - 8\pi r_{c,II}^3 = 0 \] (20d)

The variations of the notch critical distance \(r_{c,II}\) versus the notch radius \(\rho\) is plotted in Fig. 12 for different values of \(\gamma\).

It is noteworthy that for the particular case of \(\rho = 0\) (i.e. for a sharp crack), \(r_{c,II}\) is found from Fig. 12 as 0.12 mm for PMMA and 0.29 mm for soda-lime glass. These values are in good agreements with the critical distance for cracks obtained earlier from experimental results in Kharazi (2008) for PMMA as 0.1 mm and in Ayatollahi and Aliha (2009) for soda-lime glass as 0.23 mm.

Fig. 8 indicates that for \(\rho = 0.5\) mm, the theoretically predicted notch fracture toughness ratio \(K_{IIc}/K_{Ic}\) is approximately equal to the mean experimental value. However, for other notch radii, the UMTS criterion overestimates slightly the notch fracture toughness ratio of PMMA specimens. Meanwhile, a comparison between the results shown in Figs. 8 and 9 reveals that the discrepancy between the UMTS curve and the test data for soda-lime glass is less than that of PMMA. The average discrepancies between the theoretical and the experimental results for \(K_{IIc}/K_{Ic}\) in the PMMA and soda-lime glass specimens are about 8% and 5%, respectively. The average discrepancies in predicting the fracture initiation angles are 6% and 3.5% for the same materials, respectively (see Figs. 10 and 11).

Conducting mode I fracture tests is usually more convenient and less expensive than mode II fracture tests. There are also rather extensive data in the literature about the mode I notch fracture toughness for different materials and for different notch tip radii. Thus, the UMTS criterion is a very useful method for predicting mode II notch fracture toughness without having to perform complicated and expensive mode II fracture tests on notched specimens.

According to the UMTS curves shown in Figs. 8 and 9, when the notch tip radius is larger than 1 mm, the fracture toughness ratio for each material is almost a constant and independent of the notch tip radius. Meanwhile, the U-notches which are found in practical design of engineering components often have a radius \(\rho\) larger than 1 mm. Therefore, it might be suggested that not only the mode II notch fracture toughness can be predicted directly from the mode I notch fracture toughness using the UMTS criterion but also the calculation of fracture toughness ratio for different values of \(\rho\) is not necessary when the notch tip radius is larger than
1 mm. It is also interesting that when \( \rho > 1 \) mm, the fracture toughness ratios for PMMA and the soda-lime glass, are almost identical and both are about 2. However, these findings are based only on the results obtained in the present study. For a more comprehensive conclusion, it is very constructive to investigate both theoretically and experimentally the mode II fracture behavior in U-notches for other brittle materials and also for a wider range of notch tip radii.

As mentioned earlier, the U-notch stress field (Filippi et al., 2002) used in this work is an approximate expression because it satisfies the boundary conditions only in several points on the notch edge and not on the whole edge. Also, the stress field which is developed for a parabolic notch in Curvilinear coordinate system (Filippi et al., 2002) is very similar to an ideal U-notch, and hence the curvatures of these notches do not completely match each other. In pure mode I loading conditions, fracture occurs from the notch tip along the notch bi-sector line (i.e. \( \phi_0 = 0 \)) where the radii of curvatures are exactly equal and the stress field can be considered to be accurate. With increasing the contribution of mode II loading, the location of maximum tangential stress moves along the notch edge away from the notch root. At such points, the radii of curvatures for a parabolic notch and for an ideal U-notch diverge, so that the stress field (Filippi et al., 2002) can be insufficiently accurate. Consequently, it can be said that, as the effect of mode II increases, the accuracy of the stress field at the fracture initiation point for U-notches is expected to decrease, particularly when the notch is subjected to pure mode II deformation. Therefore, improved predictions can be expected to obtain from the UMTS criterion if a more accurate mathematical expression is found for elastic stresses near a U-notch.

It is essential to highlight that our proposed criterion seems to be a frame-dependent criterion since it is based on a frame-dependent driving quantity i.e. the tangential stress. However, as elaborated in Appendix B, it can be demonstrated that the location of the frame origin has almost no effect on the value of mode II notch fracture toughness. In other words, although the tangential stress is frame-dependent, the UMTS criterion itself is virtually a frame-independent failure criterion when brittle fracture is studied for

Fig. 12. The variations of the notch critical distance \( r_c, U \) versus notch radius \( \rho \) for different values of \( \gamma \).
U-notched components under mode II loading. Accordingly, among different reasonable choices for the origin of frame, we have used the most convenient polar frame of reference which gives simpler equation for the tangential stress. On the other hand, frame-independent driving quantities and their associated failure criteria like the maximum principal stress or the minimum shear stress also involve very complicated mathematical relations. It is important to note that the use of mathematically complicated criteria is normally confined only to research papers. But, a prime objective in this research was to provide a simpler criterion which not only is virtually frame-independent but also can be conveniently used by engineers in the practical engineering design and analysis of U-notched components made of brittle materials.

8. Conclusions

1. The classical MTS criterion originally suggested for sharp cracks was extended to U-notched domains for investigating brittle fracture under pure mode II loading.

2. According to the MTS criterion, for sharp cracks the values of fracture toughness ratio \(K_{IIc}/K_{IIc}^0\) and the mode II fracture initiation angle \(\theta_{IIc}\) are constant and equal to 0.866 and -70.51 (deg.), respectively. However, for U-notched components, these parameters depend on the notch radius (\(\rho\)), mode I notch fracture toughness \(K_{IIc}^0\) and the ultimate tensile strength of the material.

3. The UMTS criterion predicts well both the notch fracture toughness ratio and the mode II fracture initiation angle for UNBD specimens made of soda-lime glass and PMMA and tested at room temperature. It can be suggested that the UMTS criterion is able to estimate mode II fracture resistance of U-notched brittle components with a good degree of accuracy.

4. The UNBD specimen is an appropriate test specimen for fracture tests of U-shaped notches particularly when the notch is subjected to pure mode II loading.

Appendix A

(1) Functions used in the stress field for rounded V-shaped notches (mode I and II) (Filippi et al., 2002):

\[
\begin{align*}
\sigma_{II}(r, \theta) &= 0 \\
\tau_{II}(r, \theta) &= \frac{1}{1 + \lambda_1 + \lambda_2} \left[ \frac{1}{1 + \lambda_1 + \lambda_2} \right] \\
\tau_{II}(r, \theta) &= \frac{1}{1 + \lambda_1 + \lambda_2} \left[ \frac{1}{1 + \lambda_1 + \lambda_2} \right] \\
\sigma_{II}(r, \theta) &= \frac{1}{1 + \lambda_1 + \lambda_2} \left[ \frac{1}{1 + \lambda_1 + \lambda_2} \right] \\
\tau_{II}(r, \theta) &= \frac{1}{1 + \lambda_1 + \lambda_2} \left[ \frac{1}{1 + \lambda_1 + \lambda_2} \right] \\
\sigma_{II}(r, \theta) &= \frac{1}{1 + \lambda_1 + \lambda_2} \left[ \frac{1}{1 + \lambda_1 + \lambda_2} \right]
\end{align*}
\]

(2) The expressions for parameters \(\omega_1\) and \(\omega_2\) (Filippi et al., 2002):

\[
\omega_1 = \frac{q}{4(q - 1)} \left[ \frac{1}{1 + \lambda_1 + \lambda_2} \right] \\
\omega_2 = \frac{1}{4(\mu_2 - 1)} \left[ \frac{1}{1 + \lambda_1 + \lambda_2} \right]
\]

The values of the parameters \(\lambda_1, \lambda_2, \mu_1, \mu_2, \lambda_0, \lambda_2, \lambda_0, \lambda_2\) are reported in Filippi et al. (2002) for various notch opening angles.

Appendix B

It is shown in this Appendix that although the tangential stress is a frame-dependent parameter, the UMTS criterion itself is virtually a frame-independent theory for mode II brittle fracture in U-notched components. Fig. B1 shows a U-notch with three appropriate and reasonable points 1, 2 and 3 for locating the reference frames.

The points 1, 2 and 3 locate at the distances \(\rho, \rho/2\) and \(\rho/4\) from the notch tip, respectively. The stress field prescribed in the paper is developed based on a frame located at point 2. This stress field is an approximate expression with a high degree of accuracy. The stress fields for two other frames (based on the points 1 and 3) can be achieved by a transformation in the horizontal direction. The tangential stress \(\sigma_{II}(r, \theta)\) for three reference frames 1, 2 and 3 in pure mode II loading conditions are respectively:

\[
\sigma_{II}(r, \theta) = \frac{K_{IIc}^0}{4\pi} \rho \sin \theta (0.5642\rho \sin \theta) (1.95)\left(0.5642\rho \sin \theta\right) (1.95)
\]

\[
\sigma_{II}(r, \theta) = \frac{K_{IIc}^0}{4\pi} \rho \sin \theta (0.5642\rho \sin \theta) (1.95)\left(0.5642\rho \sin \theta\right) (1.95)
\]

\[
\sigma_{II}(r, \theta) = \frac{K_{IIc}^0}{4\pi} \rho \sin \theta (0.5642\rho \sin \theta) (1.95)\left(0.5642\rho \sin \theta\right) (1.95)
\]

where the superscripts \(i = 1, 2, 3\) for \(\sigma_{II}(r, \theta)\) represent the frame numbers.

Fig. B1 shows three circles drawn according to the three distinct centers (points 1, 2 and 3) which their radii correspond to critical distances \(r_{cr}\). Since the length of damage zone from the notch tip is determined by using a mode I fracture test, the three circles touch each other on the notch bisector line where mode I fracture is expected to occur. As the contribution of mode II increases, the circles deviate from each other. It is clear that the radii of the circles (the critical distances) can be determined by using Equations similar to Eq. (19).
Now we define for each circle a point $P_i$ that is the point where the tangential stress in pure mode II loading conditions is a maximum. The stress analysis of notch shows that the lines drawn between the points $i$ and $P_i$ ($i = 1, 2, 3$) intersect each other at the point $P_i$. Here the main purpose is to evaluate the quantity of the tangential stresses in these points rather than their directions because the quantities are required in the UMTS criterion for predicting the notch fracture toughness. In other words, the tangential stress with respect to each frame is, in fact, a tensile stress and the fracture is expected to occur when the tangential stress attains a constant critical value.

According to the requirements of the UMTS criterion, the first derivative of the tangential stress must be zero:

$$\frac{\partial \sigma_{\text{tan}}(r^i_{U}, \theta)}{\partial \theta} = 0 \Rightarrow \theta = \theta_{i\text{tan}} \quad (i = 1, 2, 3) \quad (B4)$$

By substituting each of Eqs. (B1–B3) into Eq. B4 and solving the corresponding equation using an appropriate mathematical code or software (for example Mathematica code in our calculations), one can find the mode II fracture initiation angle ($\theta_{i\text{tan}}$) associated with each frame of reference. If the critical distances $r^i_{U}$ for the three frames are known, one can substitute the values of $\theta_{i\text{tan}}$ and $r^i_{U}$ into Eqs. (B1–B3) and obtain the value of the tangential stress at the critical position around the notch tip for any value of the applied load (i.e. different values of $K_{II}^0$). Tables B1 and B2 summarize the critical parameters and the tangential stresses (for an arbitrary value of $K_{II}^0 = 10 \text{MPa}\sqrt{\text{mm}}$) at the critical position around the notch tip corresponding to the three different coordinates for the U-notched specimens of PMMA and soda-lime glass, respectively.

The last columns of Tables B1 and B2 show that for each notch tip radius ($\rho$), the values of the tangential stresses associated with the three frames at the critical points are almost identical. This results demonstrate that for an arbitrary value of $K_{II}^0$ (e.g. $K_{II}^0 = 10 \text{MPa}\sqrt{\text{mm}}$ in our analyses), the stress value at the critical point is actually independent of the frame position. Since the onset fracture takes place when the value of $\sigma_{\text{tan}}$ attains the critical value ($\sigma_{\text{tan}}^c$), (which is a material property), all the frames predict almost identical values for the notch fracture toughness (or for the fracture load) under mode II loading.

Although our analytical calculations demonstrate that the fracture toughness of the UNBD specimens is independent of the frame location, however, the fracture initiation angle depends on the reference frame because the direction of the maximum tangential stress at any point around the notch tip is a frame-dependent parameter. Here, a set of geometrical relations are proposed in order to convert the fracture initiation angles of the frames 1, 2 and 3 ($\theta_{i\text{tan}}^1$, $\theta_{i\text{tan}}^2$, and $\theta_{i\text{tan}}^3$) to each other. Fig. B2 shows the points 1, 2, 3 and the angles $\theta_{i\text{tan}}^i$ together with several auxiliary angles.

In the triangle $1 – 2 – P_1$, one can write:

$$\frac{\sin \alpha}{r_{\text{tan}}^{1U} + \rho/2} = \frac{\sin \beta}{\rho/2} \Rightarrow \beta = \sin^{-1} \left( \frac{\rho/2}{r_{\text{tan}}^{1U} + \rho/2} \sin \theta_{i\text{tan}}^1 \right)$$

$$\alpha = \pi - \theta_{i\text{tan}}^1$$

Thus

$$\theta_{i\text{tan}}^1 = \theta_{i\text{tan}}^0 - \beta \quad (B6)$$

Similarly for the triangle $1 – 3 – P_1$:

$$\frac{\sin \eta}{r_{\text{tan}}^{1U} + \rho/2} = \frac{\sin(\beta + \gamma)}{3\rho/4}$$

$$\eta = \pi - (\gamma + \theta_{i\text{tan}}^1)$$

Table B1

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<th>$\rho$ (mm)</th>
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<th>$\sigma_{\text{tan}}^i$ (MPa)</th>
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Table B2

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Substituting Eq. B8 into Eq. B7
\[
\sin \left( \gamma + \theta_{\text{out}} \right) = \sin \left( \beta + \gamma \right) \frac{r^2_{\text{ci}} + \rho/2}{3\rho/4}
\]  
(B9)

Note that the parameters \( r^2_{\text{ci}} \) and \( \theta_{\text{out}} \) are the critical distance and the fracture initiation angle associated with the frame 2. These values have already been reported in the paper. By solving Eq. B9, the angle \( \gamma \) can be determined. Finally, one can simply obtain the angle \( \theta_{\text{out}} \) by using Eq. B10:
\[
\theta_{\text{out}}^2 = \theta_{\text{out}}^2 + \gamma 
\]  
(B10)

Therefore, although the angles \( \theta_{\text{out}} = 1,2,3 \) depend on the frame of reference, these values can be simply converted to each other by equations similar to B6 and B10. According to the UMTS criterion, the main parameter for characterizing the onset of fracture is the magnitude of the tangential stress rather than its direction. Above, it was demonstrated that the magnitude of the tangential stresses at the three critical points \( P_i \) are almost identical, however, the directions are different. It can be therefore concluded that although the tangential stress is a frame-dependent driving parameter, the results of UMTS criterion for mode II fracture toughness is virtually independent of the frame of reference. Thus, one can arbitrarily select one of the three frames for predicting the mode II fracture toughness of U-notches while paying attention to the fact that the numerical value of fracture initiation angle depends on the frame position.

Note that from an experimental point of view, the measurement of the fracture initiation angle with respect to the frame 1 is easier than the other two frames, because the line 1 – \( P_i \) in Fig. B2 is perpendicular to the notch edge. Therefore, the angles \( \theta_{\text{out}} \) for the broken specimens were first measured experimentally in this research and then converted to the corresponding angles \( \theta_{\text{out}} \) by using Eq. B6 in order to compare the experimental values of \( \theta_{\text{out}} \) with the theoretical ones as shown in Figs. 10 and 11.

References