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Solvability of a Three-Point Nonlinear Boundary Value Problem for a Second Order Ordinary Differential Equation

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1. INTRODUCTION

Let $f\colon [0, 1] \times \mathbf{R}^2 \to \mathbf{R}$ be a given function satisfying Caratheodory's conditions, $e : [0, 1] \to \mathbf{R}$ be a function in $L^1(0, 1)$, and let $\eta \in (0, 1)$ be given. This paper is devoted to studying the following second-order three-point nonlinear boundary value problem:

$$u'' = f(x, u(x), u'(x)) - e(x), \qquad 0 < x < 1, \tag{1.1}$$

$$u(0) = 0, \qquad u(\eta) = u(1). \tag{1.2}$$

In case $f(x, u, u') = p_0(x) + p_1(x)\, u + p_2(x)\, u'$ with $p_k : [0, 1] \to \mathbf{R}$ locally integrable, the boundary value problem (1.1)–(1.2) was studied by Kiguradze and Lomtatidze [1]. The purpose of this paper is to obtain existence and uniqueness theorems for the boundary value problem (1.1)–(1.2) under natural conditions on f using degree-theoretic arguments. We note that if u is a solution of (1.1)–(1.2) then there exists at least one $\zeta \in (\eta, 1)$, such that $u'(\zeta) = 0$. Accordingly, the boundary value problem

$$u'' = f(x, u(x), u'(x)) - e(x), \qquad 0 < x < 1, \tag{1.3}$$

$$u(0) = u'(1) = 0 \tag{1.4}$$

can be considered as a limiting case of the problem (1.1)–(1.2) when $\eta = 1$. Our results for (1.1)–(1.2) give the sharpest possible results for (1.3)–(1.4) when $\eta = 1$.

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The boundary value problem (1.1)–(1.2) can be put in the form of an operator equation

$$Lu + Nu = w,$$

where $L : D(L) \subset X \to Y$ is a linear operator, $N : X \to Y$ is a nonlinear operator, and X, Y are suitable Banach spaces in duality (denoted by $(\,,\,)$). Clearly the linear operator L in (1.1) is given by

$$Lu = -u''.$$

where the boundary conditions (1.2) are used to define the domain, $D(L)$, of L. We use Wirtinger-type inequalities to obtain the necessary a priori estimates needed to use degree theoretic arguments and the Leray Schauder continuation theorem.

We use the classical spaces $C[0, 1]$, $C^k[0, 1]$, $L^k[0, 1]$, and $L^\infty[0, 1]$ of continuous, k-times continuously differentiable, measurable real-valued functions whose kth power of the absolute value is Lebesgue integrable, or measurable functions that are essentially bounded on [0,1]. We use the Sobolev space $H^1(0, 1)$ defined by

$$H^1(0, 1) = \{u : [0, 1] \to \mathbf{R} \mid u \text{ absolutely continuous on } [0, 1] \text{ and } u' \in L^2[0, 1]\}$$

with the inner product defined by

$$(u, v)_{H^1} = \int_0^1 u'(x)\, v'(x)\, dx + \left(\int_0^1 u(x)\, dx\right)\left(\int_0^1 v(x)\, dx\right),$$

and the corresponding norm by $|\cdot|_{H^1}$. We also use the Sobolev space $W^{2,1}(0, 1)$ defined by

$$W^{2,1}(0, 1) = \{u : [0, 1] \to \mathbf{R} \mid u, u' \text{ absolutely continuous on } [0, 1]\}$$

with norm

$$|u|_{W^{2,1}} = \sum_{j=0}^{2} \int_0^1 |u^{(j)}(x)|\, dx.$$

2. Existence Theorems

Let $f : [0, 1] \times \mathbf{R}^2 \to \mathbf{R}$ be a function satisfying Caratheodory's conditions, namely,

(i) for each $(u, v) \in \mathbf{R}^2$, the function $x \in [0, 1] \to f(x, u, v) \in \mathbf{R}$ is measurable on $[0, 1]$;

(ii) for a.e. $x \in [0, 1]$, the function $(u, v) \in \mathbf{R}^2 \to f(x, u, v) \in \mathbf{R}$ is continuous on $\mathbf{R}^2$; and

(iii) for each $r > 0$, there exists $\alpha_r(x) \in L^1[0, 1]$ such that $|f(x, u, v)| \leqslant \alpha_r(x)$ for a.e. $x \in [0, 1]$ and all $(u, v) \in \mathbf{R}^2$ with $\sqrt{u^2 + v^2} \leqslant r$.

THEOREM 1. *Let $f: [0, 1] \times \mathbf{R}^2 \to \mathbf{R}$ satisfy Caratheodory's conditions. Assume that*

(i) *there exist $a, b \in \mathbf{R}$ and an $\alpha(x) \in L^1[0, 1]$ such that*

$$f(x, u, v)\, u \geqslant au^2 + b\,|u|\,|v| + \alpha(x)\,|u|, \tag{2.1}$$

for a.e. $x \in [0, 1]$ and all $(u, v) \in \mathbf{R}^2$;

(ii) *there exist functions $p(x)$, $q(x) \in L^2[0, 1]$ and a function $r(x) \in L^1[0, 1]$ such that*

$$|f(x, u, v)| \leqslant p(x)\,|u| + q(x)\,|v| + r(x), \tag{2.2}$$

for a.e. x in $[0, 1]$ and all $(u, v) \in \mathbf{R}^2$.

Then for every given function $e(x) \in L^1[0, 1]$, the boundary value problem (1.1)–(1.2) has at least one solution in $C^1[0, 1]$ provided

$$\frac{4}{\pi^2}\,|a| + \frac{2}{\pi}\,|b| + \sqrt{\eta}\left(\frac{2}{\pi}\,\|p\|_{L^2[\eta, 1]} + \|q\|_{L^2[\eta, 1]}\right) < 1. \tag{2.3}$$

Proof. Let X denote the Banach space $C^1[0, 1]$ and Y denote the Banach space $L^1[0, 1]$ with their usual norms. Also for $u \in X$, $v \in Y$, let

$$(u, v) = \int_0^1 u(x)\, v(x)\, dx,$$

denote the duality pairing between X and Y. We define a linear mapping $L: D(L) \subset X \to Y$ by setting

$$D(L) = \{u \in W^{2,1}(0, 1) \mid u(0) = 0,\ u(\eta) = u(1)\},$$

and for $u \in D(L)$,

$$Lu = -u''.$$

We also define a nonlinear mapping $N: X \to Y$ by setting

$$(Nu)(x) = f(x, u(x), u'(x)), \qquad x \in [0, 1].$$

We note that N is a bounded continuous mapping from X into Y. Next, it is easy to see that the linear mapping $L : D(L) \subset X \to Y$, is a one-to-one mapping. Also the linear mapping $K : Y \to X$, defined for $y \in Y$ by

$$(Ky)(x) = -\int_0^x (x-t)\, y(t)\, dt + x \int_0^\eta y(t)\, dt + \frac{x}{1-\eta} \int_\eta^1 (1-t)\, y(t)\, dt$$

is such that for $y \in Y$, $Ky \in D(L)$ and $LKy = y$; and for $u \in D(L)$, $KLu = u$. Furthermore, it follows easily using the Arzela–Ascoli Theorem that K maps a bounded subset of Y into a relatively compact subset of X. Hence $KN : X \to X$ is a compact mapping.

We, next, note that $u \in C^1[0, 1]$ is a solution of the boundary value problem (1.1)–(1.2) if and only if u is a solution of the operator equation

$$Lu + Nu = e.$$

Now, the operator equation $Lu + Nu = e$ is equivalent to the equation

$$u + KNu = Ke.$$

We apply the Leray–Schauder continuation theorem (see, e.g., [2, Corollary IV.7]) to obtain the existence of a solution for $u + KNu = Ke$ or equivalently to the boundary value problem (1.1)–(1.2).

To do this, it suffices to verify that the set of all possible solutions of the family of equations

$$u'' = \lambda f(x, u(x), u'(x)) - \lambda e(x), \tag{2.4}$$

$$u(0) = 0,\ u(\eta) = u(1), \tag{2.5}$$

is, a priori, bounded in $C^1[0, 1]$ by a constant independent of $\lambda \in [0, 1]$.

We observe that for $u \in W^{2,1}(0, 1)$ with $u(0) = 0$, $u(\eta) = u(1)$ there exists a $\zeta \in (0, 1)$, $\eta < \zeta < 1$, such that $u'(\zeta) = 0$. It follows that

$$\|u\|_2 \leqslant \frac{2}{\pi} \|u'\|_2, \qquad \|u\|_\infty \leqslant \|u'\|_2, \tag{2.6}$$

$$|u(1)| = |u(\eta)| = \left| \int_0^\eta u'(t)\, dt \right| \leqslant \sqrt{\eta}\, \|u'\|_2, \tag{2.7}$$

$$|u'(1)| = \left| \int_\eta^1 u''(t)\, dt \right| \leqslant \int_\eta^1 |u''(t)|\, dt. \tag{2.8}$$

Let, now, $u(x)$ be a solution for (2.4)–(2.5) for some $\lambda \in [0, 1]$, so that

$u \in W^{2,1}(0,1)$ with $u(0)=0$, $u(\eta)=u(1)$. We multiply Eq. (2.4) by u and integrate from 0 to 1 to get

$$\begin{aligned}
0 &= -\int_0^1 u''(x)\,u(x)\,dx + \lambda \int_0^1 f(x,u(x),u'(x))\,u(x)\,dx - \lambda \int_0^1 e(x)\,u(x)\,dx \\
&= \int_0^1 (u'(x))^2\,dx - u'(1)\,u(1) + \lambda \int_0^1 f(x,u(x),u'(x))\,u(x)\,dx \\
&\quad - \lambda \int_0^1 e(x)\,u(x)\,dx \\
&\geqslant \|u'\|_2^2 - |u'(1)|\,|u(1)| + \lambda \int_0^1 \{a(u(x))^2 + b\,|u(x)|\,|u'(x)| \\
&\quad + \alpha(x)\,|u(x)|\}\,dx - \lambda\,\|e\|_1\,\|u\|_\infty \\
&\geqslant \|u'\|_2^2 - \sqrt{\eta}\,\|u'\|_2\,\|u''\|_{L^1[\eta,1]} - \left(\frac{4}{\pi^2}|a| + \frac{2}{\pi}|b|\right)\|u'\|_2^2 \\
&\quad - (\|\alpha\|_1 + \|e\|_1)\,\|u'\|_2.
\end{aligned} \tag{2.9}$$

We, next, have from (2.2) and (2.4) that

$$\begin{aligned}
\|u''\|_{L^1[\eta,1]} &\leqslant \lambda\,\|f(x,u(x),u'(x))\|_{L^1[\eta,1]} + \lambda\,\|e\|_1 \\
&\leqslant \|p(x)\,|u(x)| + q(x)\,|u'(x)| + r(x)\|_{L^1[\eta,1]} + \|e\|_1 \\
&\leqslant \|p\|_{L^2[\eta,1]}\,\|u\|_2 + \|q\|_{L^2[\eta,1]}\,\|u'\|_2 + \|r\|_1 + \|e\|_1 \\
&\leqslant \left(\frac{2}{\pi}\|p\|_{L^2[\eta,1]} + \|q\|_{L^2[\eta,1]}\right)\|u'\|_2 + \|r\|_1 + \|e\|_1.
\end{aligned} \tag{2.10}$$

Using (2.10) in (2.9) we get

$$\begin{aligned}
0 \geqslant \|u'\|_2^2 &- \sqrt{\eta}\left(\frac{2}{\pi}\|p\|_{L^2[\eta,1]} + \|q\|_{L^2[\eta,1]}\right)\|u'\|_2^2 \\
&- \left(\frac{4}{\pi^2}|a| + \frac{2}{\pi}|b|\right)\|u'\|_2^2 \\
&- (\|\alpha\|_1 + (1+\sqrt{\eta})\,\|e\|_1 + \sqrt{\eta}\,\|r\|_1)\,\|u'\|_2.
\end{aligned}$$

It follows that

$$\|u'\|_2 \leqslant \frac{\|\alpha\|_1 + (1+\sqrt{\eta})\,\|e\|_1 + \sqrt{\eta}\,\|r\|_1}{1 - ((4/\pi^2)\,|a| + (2/\pi)\,|b| + \sqrt{\eta}\,((2/\pi)\,\|p\|_{L^2[\eta,1]} + \|q\|_{L^2[\eta,1]}))}. \tag{2.11}$$

Using again (2.2), (2.4) along with (2.11) we see that there exists a constant C, independent of $\lambda \in [0, 1]$ such that

$$\|u''\|_1 \leqslant C. \tag{2.12}$$

It then follows from (2.6), (2.11), (2.12) that there exists a constant, again denoted by C, independent of $\lambda \in [0, 1]$ such that

$$\|u\|_{W^{2,1}} \leqslant C. \tag{2.13}$$

Finally, since $W^{2,1}(0, 1) \subset C^1[0, 1]$ compactly, we have from (2.13) that there exists a constant, still denoted by C, independent of $\lambda \in [0, 1]$, such that

$$\|u\|_{C^1[0,1]} \leqslant C.$$

This completes the proof of the theorem. ∎

Remark 1. The a, b in condition (2.1) are interesting only when $a < 0$ and $b < 0$; for if any one or both of a or b is non-negative then we can replace the same by 0. So in assumption (2.3), $|a|$, $|b|$ appear only when a, b are negative in (2.1), otherwise they are replaced by zero.

Remark 2. We can do without assumption (2.1) in Theorem 1, if we observe that

$$f(x, u, v)\, u \geqslant -|f(x, u, v)|\, |u|$$

and then use (2.2) to get

$$\begin{aligned} f(x, u, v)\, u &\geqslant -|f(x, u, v)|\, |u| \\ &\geqslant -|p(x)|\, |u|^2 - |q(x)|\, |u|\, |v| - |r(x)|\, |u|. \end{aligned}$$

It is then easy to see from the proof of Theorem 1 that the boundary value problem (1.1)–(1.2) has at least one solution in $C^1[0, 1]$ provided

$$\frac{2}{\pi} \|p\|_2 + \|q\|_2 + \sqrt{\eta} \left(\frac{2}{\pi} \|p\|_{L^2[\eta,1]} + \|q\|_{L^2[\eta,1]} \right) < 1.$$

THEOREM 2. *Let $f\colon [0, 1] \times \mathbf{R}^2 \to \mathbf{R}$ satisfy Caratheodory's conditions. Assume that thre exist functions $p(x)$, $q(x)$, $r(x)$ in $L^1[0, 1]$ such that*

$$|f(x, u, v)| \leqslant p(x)\, |u| + q(x)\, |v| + r(x), \tag{2.14}$$

for a.e. $x \in [0, 1]$ and all $(u, v) \in \mathbf{R}^2$.

Then for every given $e(x) \in L^1[0, 1]$ the boundary value problem (1.1)–(1.2) has at least one solution in $C^1[0, 1]$ provided

$$\|q\|_1 \leqslant 1, \tag{2.15}$$

and

$$\begin{aligned}\sqrt{\eta}\, \|p\|_{L^1[\eta,1]} (1 - \|q\|_1) + \|p\|_1 (1 - U \|q\|_{L^1[\eta,1]}) \\ < (1 - \|q\|_1)(1 - \|q\|_{L^1[\eta,1]}).\end{aligned} \tag{2.16}$$

Proof. As in the proof of Theorem 1, it suffices to verify that the set of all possible solutions of the family of equations

$$u'' = \lambda f(x, u(x), u'(x)) - \lambda e(x), \tag{2.17}$$

$$u(0) = 0, \qquad u(\eta) = u(1), \tag{2.18}$$

is, a priori, bounded in $C^1[0, 1]$ by a constant independent of $\lambda \in [0, 1]$.

Letting $u(x)$ to be a solution of (2.17)–(2.18) for some $\lambda \in [0, 1]$, we get, as in Theorem 1, that

$$\begin{aligned} 0 &= -\int_0^1 u''(x)\, u(x)\, dx + \lambda \int_0^1 f(x, u(x), u'(x))\, u(x)\, dx - \lambda \int_0^1 e(x)\, u(x)\, dx \\ &\geqslant \|u'\|_2^2 - |u'(1)|\, |u(1)| - \int_0^1 |f(x, u(x), u'(x))|\, |u(x)|\, dx - \|e\|_1 \|u\|_\infty \\ &\geqslant \|u\|_2^2 - |u'(1)|\, |u(1)| \\ &\quad - \int_0^1 [p(x)\, |u(x)|^2 + q(x)\, |u(x)|\, |u'(x)| + r(x) |u(x)|] - \|e\|_1 \|u\|_\infty \\ &\geqslant \|u'\|_2^2 - \sqrt{\eta}\, \|u'\|_2 \|u''\|_{L^1[\eta,1]} \\ &\quad - [\|p\|_1 \|u\|_\infty^2 + \|q\|_1 \|u\|_\infty \|u'\|_\infty + \|r\|_1 \|u\|_\infty] - \|e\|_1 \|u\|_\infty \\ &\geqslant \|u'\|_2^2 - \sqrt{\eta}\, \|u'\|_2 \|u''\|_{L^1[\eta,1]} - [\|p\|_1 \|u'\|_2^2 + \|q\|_1 \|u'\|_2 \|u''\|_1] \\ &\quad - (\|r\|_1 + \|e\|_1) \|u'\|_2. \end{aligned} \tag{2.19}$$

Now, we use Eq. (2.17) and assumption (2.14) to get

$$\begin{aligned} \|u''\|_1 &\leqslant \|p\|_1 \|u\|_\infty + \|q\|_1 \|u'\|_\infty + \|r\|_1 \\ &\leqslant \|p\|_1 \|u'\|_2 + \|q\|_1 \|u''\|_1 + \|r\|_1 \end{aligned}$$

so that, we get using (2.15) that

$$\|u''\|_1 \leqslant \frac{\|p\|_1 \|u'\|_2}{1 - \|q\|_1} + \frac{\|r\|_1}{1 - \|q\|_1}. \tag{2.20}$$

Similarly, we have

$$\|u''\|_{L^1[\eta,1]} \leqslant \frac{\|p\|_{L^1[\eta,1]} \|u'\|_2}{1-\|q\|_{L^1[\eta,1]}} + \frac{\|r\|_1}{1-\|q\|_{L^1[\eta,1]}}. \tag{2.21}$$

We, next, use (2.20), (2.21), and (2.16) in (2.19) to get

$$\begin{aligned}\{(1-\|q\|_1)(1-\|q\|_{L^1[\eta,1]}) - \sqrt{\eta}\, \|p\|_{L^1[\eta,1]} (1-\|q\|_1) \\ - \|p\|_1 (1-\|q\|_{L^1[\eta,1]})\} \|u'\|_2 \\ \leqslant \|q\|_1 \|r\|_1 (1-\|q\|_{L^1[\eta,1]}) + \sqrt{\eta}\, \|r\|_1 (1-\|q\|_1) \\ + (\|r\|_1 + \|e\|_1)(1-\|q\|_{L^1[\eta,1]})(1-\|q\|_1).\end{aligned}$$

Hence, there exists, a constant C, independent of $\lambda \in [0, 1]$ such that

$$\|u'\|_2 \leqslant C.$$

It, now, follows as in the proof of Theorem 1, that there is a constant, still denoted by C, independent of $\lambda \in [0, 1]$ such that

$$\|u\|_{C^1[0,1]} \leqslant C.$$

This completes the proof of the theorem. ∎

Remark 3. We notice that for a $u \in W^{2,1}(0, 1)$ with

$$u(0) = 0, \qquad u(\eta) = u(1) \ (\eta \in (0, 1))$$

there exists a ζ, $\eta < \zeta < 1$, such that $u'(\zeta) = 0$. We can, accordingly, consider the boundary value problem

$$u'' = f(x, u(x), u'(x)) - e(x), \tag{2.22}$$

$$u(0) = u'(1) = 0, \tag{2.23}$$

as a degenerate case of the boundary value problem (1.1)–(1.2) when $\eta = 1$.

Accordingly, we get the following corollary as an immediate consequence of Theorem 2 when $\eta = 1$ for the boundary value problem (2.22)–(2.23).

COROLLARY 1. *Let* $f: [0, 1] \times \mathbf{R}^2 \to \mathbf{R}$ *be as in Theorem* 2. *Then for every given* $e(x) \in L^1[0, 1]$ *the boundary value problem* (2.22)–(2.23) *has at least one solution in* $C^1[0, 1]$ *provided*

$$\|p\|_1 + \|q\|_1 < 1.$$

Proof. When $\eta = 1$, condition (2.16) reduces to $\|p\|_1 < 1 - \|q\|_1$ which is equivalent to $\|p\|_1 + \|q\|_1 < 1$. ∎

THEOREM 3. *Let $f: [0, 1] \times \mathbf{R}^2 \to \mathbf{R}$ satisfy Caratheodory's conditions. Assume that $a \geqslant 0$, $b \geqslant 0$, $\alpha(x) \in L^1[0, 1]$ are such that*

$$|f(x, u, v)| \leqslant a\,|u| + b\,|v| + \alpha(x), \tag{2.24}$$

for a.e. $x \in [0, 1]$ and all $(u, v) \in \mathbf{R}^2$.

Then for every given $e(x) \in L^1[0, 1]$ the boundary value problem (1.1)–(1.2) has at least one solution u in $C^1[0, 1]$, provided

$$\left(\frac{2}{\pi} a + b\right)\left(\frac{2}{\pi} + \sqrt{\eta(1-\eta)}\right) < 1. \tag{2.25}$$

Proof. As in the proof of Theorem 1, it suffices to verify that the set of all possible solutions of the family of equations

$$u'' = \lambda f(x, u(x), u'(x)) - \lambda e(x) \tag{2.26}$$

$$u(0) = 0, \qquad u(\eta) = u(1), \tag{2.27}$$

is, a priori, bounded in $C^1[0, 1]$ by a constant independent of $\lambda \in [0, 1]$.

Letting $u(x)$ to be a solution of (2.26)–(2.27) for some $\lambda \in [0, 1]$ we get, as in Theorem 1, that

$$\begin{aligned}
0 &= -\int_0^1 u''(x)\,u(x)\,dx + \lambda \int_0^1 f(x, u(x), u'(x))\,u(x)\,dx - \lambda \int_0^1 e(x)\,u(x)\,dx \\
&\geqslant \|u'\|_2^2 - |u'(1)|\,|u(1)| - \int_0^1 (a\,|u|^2 + b\,|u|\,|u'| + \alpha(x)\,|u|)\,dx - \|e\|_1\,\|u\|_\infty \\
&\geqslant \|u'\|_2^2 - \sqrt{\eta}\,\|u'\|_2\,\|u''\|_{L^1[\eta,1]} - a\,\|u\|_2^2 - b\,\|u\|_2\,\|u'\|_2 - (\|\alpha\|_1 + \|e\|_1)\,\|u\|_\infty \\
&\geqslant \|u'\|_2^2 - \sqrt{\eta}\,\|u'\|_2\,\|u''\|_{L^1[\eta,1]} - \left(\frac{4}{\pi^2} a + \frac{2}{\pi} b\right)\|u'\|_2^2 - (\|\alpha\|_1 + \|e\|_1)\,\|u'\|_2.
\end{aligned} \tag{2.28}$$

Next, using, (2.24), (2.26) we get

$$\begin{aligned}
\|u''\|_{L^1[\eta,1]} &\leqslant \|f(x, u(x), u'(x))\|_{L^1[\eta,1]} + \|e\|_1 \\
&\leqslant a \int_\eta^1 |u|\,dx + b \int_\eta^1 |u'|\,dx + \|\alpha\|_1 + \|e\|_1 \\
&\leqslant \sqrt{1-\eta}\left(\frac{2}{\pi} a + b\right)\|u'\|_2 + \|\alpha\|_1 + \|e\|_1.
\end{aligned} \tag{2.29}$$

Using (2.29) in (2.28) we get, in view of (2.25), that

$$\|u'\|_2 \leqslant \frac{(\|\alpha\|_1 + \|e\|_1)(1 + \sqrt{\eta})}{1 - ((2/\pi)\,a + b)(2/\pi + \sqrt{\eta(1-\eta)})}.$$

It then follows as in the proof of Theorem 1 that there exists a constant C, independent of $\lambda \in [0, 1]$ such that

$$\|u\|_{C^1[0,1]} \leqslant C.$$

Hence the theorem. ∎

COROLLARY 2. *Let $f\colon [0, 1] \times \mathbf{R}^2 \to \mathbf{R}$ be as in Theorem 3. Then for every given $e(x) \in L^1[0, 1]$ the boundary value problem (2.22)–(2.23) has at least one solution $u(x)$ in $C^1[0, 1]$ provided*

$$\frac{4}{\pi^2} a + \frac{2}{\pi} b < 1.$$

Proof. The corollary follows immediately from Theorem 3 when $\eta = 1$. ∎

Remark 4. We notice that the boundary value problem

$$u'' = -\frac{\pi^2}{4} u + \operatorname{Sin} \frac{\pi}{2} x$$

$$u(0) = 0, \qquad u'(1) = 0$$

has no solution. Indeed, the general solution to the differential equation $u'' + (\pi^2/4)\, u = \sin(\pi/2)\, x$ can be easily seen to be

$$u(x) = c_1 \cos \frac{\pi}{2} x + c_2 \sin \frac{\pi}{2} x - \frac{1}{\pi} x \cos \frac{\pi}{2} x + \frac{1}{\pi^2} \sin \frac{\pi}{2} x.$$

If one then tries to compute c_1, c_2 so that $u(0) = 0$ and $u'(1) = 0$, one is led to the equation $1/2 = 0$, an absurdity. This shows that the result of Corollary 2 is the sharpest possible. Corollary 2 has been well known. The point of Corollary 2 (as well as Corollary 1) is to show that our results and methods for the boundary value problem (1.1)–(1.2) are such that they even give best results for the boundary value problem (2.22)–(2.23).

3. UNIQUENESS THEOREMS

THEOREM 4. *Let $f\colon [0, 1] \times \mathbf{R}^2 \to \mathbf{R}$ satisfy Caratheodory's conditions. Let $a \geqslant 0$, $b \geqslant 0$ be such that*

$$|f(x, u_1, v_1) - f(x, u_2, v_2)| \leqslant a\, |u_1 - u_2| + b\, |u_1 - v_2| \tag{3.1}$$

for a.e. $x \in [0, 1]$ and all $(u_i, v_i) \in \mathbf{R}^2$, $i = 1, 2$.

Then for every given $e(x) \in L^1[0, 1]$ *the boundary value problem* (1.1)–(1.2) *has a unique solution in* $C^1[0, 1]$ *provided*

$$\left(\frac{2}{\pi}a+b\right)\left(\frac{2}{\pi}+\sqrt{\eta(1-\eta)}\right)<1. \tag{3.2}$$

Proof. We note that (3.1) implies

$$|f(x, u, v)| < a\,|u| + b\,|v| + |f(x, 0, 0)|$$

for a.e. $x \in [0, 1]$ and all $(u, v) \in \mathbf{R}^2$. Hence the boundary value problem (1.1)–(1.2) has at least one solution in $C^1[0, 1]$ by Theorem 3. Let, now, u_1, u_2 be two solutions of (1.1)–(1.2) in $C^1[0, 1]$. Setting $w = u_1 - u_2$ we have

$$w'' = f(x, u_1(x), u_1'(x)) - f(x, u_2(x), u_2'(x)) \tag{3.3}$$

$$w(0) = 0, \qquad w(\eta) = w(1). \tag{3.4}$$

Multiplying Eq. (3.3) by w and integrating over $[0, 1]$ we get

$$\begin{aligned}
0 &= -\int_0^1 w''w\,dx + \int_0^1 (f(x, u_1(x), u_1'(x)) - f(x, u_2(x), u_2'(x)))\,w(x)\,dx \\
&\geqslant \|w'\|_2^2 - |w(1)|\,|w'(1)| \\
&\quad - \int_0^1 |f(x, u_1(x), u_1'(x)) - f(x, u_2(x), u_2'(x))|\,|w(x)|\,dx \\
&\geqslant \|w'\|_2^2 - \sqrt{\eta}\,\|w'\|_1\,\|w''\|_{L^1[\eta,1]} \\
&\quad - \int_0^1 [a\,|u_1 - u_2|^2 + b\,|u_1' - u_2'|\,|u_1 - u|_2]\,dx \\
&\geqslant \|w'\|_2^2 - \sqrt{\eta}\,\|w'\|_2\,\|w''\|_{L^1[\eta,1]} - \left(\frac{4}{\pi^2}a + \frac{2}{\pi}b\right)\|w'\|_2^2.
\end{aligned} \tag{3.5}$$

Next, we get from (3.1) and (3.3) as in the proof of Theorem 3, that

$$\begin{aligned}
\|w''\|_{L^1[\eta,1]} &= \|f(x, u_1(x), u_1'(x)) - f(x, u_2(x), u_2'(x))\|_{L^1[\eta,1]} \\
&\leqslant \sqrt{1-\eta}\left(\frac{2}{\pi}a + b\right)\|w'\|_2.
\end{aligned} \tag{3.6}$$

Finally (3.5), (3.6) give that

$$\left(1 - \left(\sqrt{\eta}\,\sqrt{1-\eta} + \frac{2}{\pi}\right)\left(\frac{2}{\pi}a + b\right)\right)\|w'\|_2^2 \leqslant 0,$$

so that $\|w'\|_2 = 0$. Since, now, $w(0) = 0$ we obtain using $w(x) = \int_0^x w'(t)\,dt$ that $\|w\|_\infty \leqslant \|w'\|_2 = 0$. Hence $w = 0$ a.e. in $[0, 1]$ and thus $w(x) = 0$ for every $x \in [0, 1]$ because w is continuous.

Hence the theorem. ∎

Remark 5. We note that $f\colon [0, 1] \times \mathbf{R}^2 \to \mathbf{R}$ satisfies condition (3.1) in Theorem 4, if the partial derivatives $\partial f/\partial u$, $\partial f/\partial v$ exist for a.e. $x \in [0, 1]$ and all $(u, v) \in \mathbf{R}^2$ and

$$\left|\frac{\partial f}{\partial u}(x, u, v)\right| \leqslant a, \qquad \left|\frac{\partial f}{\partial v}(x, u, v)\right| \leqslant b.$$

Remark 6. One can easily obtain uniqueness theorems analogous to Theorems 1 and 2 of Section 1.

References

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