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Nonlinear Evolution of the Kelvin–Helmholtz Instability of Supersonic Tangential Velocity Discontinuities

S. Roy Choudhury

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A nonlinear stability analysis using a multiple-scales perturbation procedure is performed for the instability of two layers of immiscible, inviscid, arbitrarily compressible fluids in relative motion. Such configurations are of relevance in a variety of astrophysical and space configurations. For modes of *all* wavenumbers on, or in the stable neighborhood of, the linear neutral curve, the nonlinear evolution of the amplitude of the linear fields on the slow first-order scales is shown to be governed by a complicated nonlinear Klein–Gordon equation. Both the spatially dependent and space-independent versions of this equation are considered to obtain the regimes of physical parameter space where the linearly unstable solutions either evolve to final permanent envelope wave patterns resembling the ensembles of interacting vortices observed empirically, or are disrupted via nonlinear modulation instability.

1. INTRODUCTION

The Kelvin–Helmholtz instability caused by tangential velocity shear in homogeneous plasmas is of interest in investigating a variety of space, astrophysical, and geophysical situations involving sheared plasma flows. Configurations, where it is relevant, include the interface between the solar wind and the magnetosphere (Sen, 1965; Southwood, 1968; Southwood, 1974; Bridge *et al.*, 1979; Ness *et al.*, 1981; Pu and Kivelson, 1983; Bull, 1984), coronal streamers moving through the solar wind, the boundaries between adjacent sectors in the solar wind (Parker, 1963; Sturrock and Hartle, 1966; Jokipii and Davis, 1969), the structure of the tails of comets (Dobrowolny and D'Angelo, 1972; Ershkovich *et al.*, 1972; Ershkovich and Chernikov, 1973; Brandt and Mendis, 1979), and the

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boundaries of the jets propagating from the nuclei of extragalactic double radio sources into their lobes (Turland and Scheuer, 1976; Blandford and Pringle, 1976; Begelman *et al.*, 1984).

Early investigations of the Kelvin−Helmholtz instability were concerned with the instability caused by a tangential velocity discontinuity or jump (or vortex sheet) in incompressible and compressible fluids and plasmas (Landau, 1944; Fejer, 1964; Sen, 1964; Miles, 1957; Gerwin, 1968).

The unmagnetized vortex sheet is found to be unstable at all wavenumbers for modes sufficiently transverse to the zero-order flow, or for modes along a flow with Mach number less than $2\sqrt{2}$. In the presence of a magnetic field parallel to the flow the instability of the incompressible vortex sheet is complete stabilized unless the velocity discontinuity exceeds twice the Alfvén speed. A magnetic field transverse to the flow has no effect on the instability.

Lerche (1966) emphasized the importance of considering the finite thickness of the shear layer. The linear Kelvin−Helmholtz instability of shear layers (a region of finite width over which the velocity change occurs) for flows with a subsonic velocity change was considered by Chandrasekhar (1981). An incompressible shear layer having a "hyperbolic tangent" profile was considered by Michalke (1964). He found a criterion $kL < 2$ for instability, with k being so that short wavelength modes were stabilized for the finite width velocity shear. The stability characteristics of finite width unmagnetized shear layers have been considered by several authors (Blumen, 1970; Blumen *et al.*, 1975; Ray, 1982; Miura and Pritchett, 1982; Roy Choudhury and Lovelace, 1986). The finite width shear layers exhibit unstable traveling wave modes satisfying radiation boundary conditions. These modes are absent for the unmagnetized vortex sheet, and present for the magnetized vortex sheet in a very small range of Mach numbers. The presence of the traveling wave modes means that the finite width layer is unstable at all Mach numbers. In addition, standing wave solutions analogous to the "warping" modes which occur for the vortex sheet are also present at long wavelengths and small values of the Mach number. Magnetized shear layers described by the MHD formalism have been considered for a linear velocity profile layer (Roy Choudhury, 1986; Ray and Ershkovich, 1983; Roy Choudhury and Lovelace, 1984) with both standing and traveling wave solutions, and for a hyperbolic tangent velocity profile (Miura and Pritchett, 1982) for only standing wave modes. A magnetic field parallel to the flow is found to stabilize both classes of modes. Computer simulation studies of the Kelvin−Helmholtz instabilities of planar, magnetized shear layers (Nepveu, 1980; Tajima and Leboeuf, 1980; Miura and Pritchett, 1982; Pritchett and Coroniti, 1984; Miura, 1984) and of cylindrical axisymmetric jets (Norman *et al.*, 1982) have also been carried out. Velocity shear of zero and finite thickness has also been considered in anisotropic plasmas (Talwar, 1965; Roy Choudhury and Patel, 1985). The results are analogous to the MHD case with larger instability growth rates.

The nonlinear development of the incompressible Kelvin–Helmholtz instability has been studied previously by Drazin (1970), Nayfeh and Saric (1971, 1972), and, in comprehensive fashion, by Weissman (1979).

The comprehensive nonlinear treatment of Weissman, as well as other early work on weakly nonlinear evolution of the amplitudes of linear fields in thermal convection (Newell and Whitehead 1969), plane Poiseuille flow (Stewartson and Stuart, 1971), a buckling problem in elasticity (Lange and Newell, 1971), and baroclinic flow (Pedlosky, 1970, 1972) have been organized into a comprehensive framework by Gibbon and McGuinness (1981) (also see Dodd *et al.*, 1982). The nonlinear analysis divides into two distinct categories. For the first category of ''dissipative'' instabilities (for example, cases where viscosity, diffusion, or other damping effects play a major role), the linear dispersion relation is a complex function of the frequency ω and wavenumber k. The central ($\omega_i = 0$) curve in the (k, μ) plane for μ some parameter of interest in the particular problem (e.g., the Mach number in the case of the Kelvin–Helmholtz instability) has a minimum at a critical wavenumber $k = k_c$, where the onset of linear instability first occurs. The weakly nonlinear evolution of the amplitude A (of the linear fields) occurs on slow second-order time and first-order space scales T_2 and $\bar{Z}$ and is governed by the canonical Ginzburg–Landau (Newell–Whitehead) equation,

$$\frac{\partial A}{\partial T_2} = \pm\alpha_1 A - \beta_1 \frac{\partial^2 A}{\partial \bar{X}^2} + \gamma_1 A|A|^2,$$

$$\bar{Z} = Z_1 - c_g T_1,$$

with $c_g = d\omega/dk$ being the group velocity of the linearly unstable solutions. The nonlinear evolution of A under this equation has been comprehensively discussed by Lange and Newell (1974) and Newell (1974). By contrast, the second category of instabilities is the ''dispersive'' type and occurs when no dissipation is present. For this case, the linear $\omega - k$ dispersion relation is real with roots occurring in complex conjugate pairs

$$\omega = \omega_r \pm i\omega_i.$$

For this case, the weakly nonlinear evolution of the linear amplitude A always occurs on slow first order time (instead of second order time as for the ''dissipative'' instabilities) and space scales T_1 and Z_1, being governed by either the canonical "AB equations"

$$\left(\frac{\partial}{\partial T_1} + c_1 \frac{\partial}{\partial Z_1}\right)\left(\frac{\partial}{\partial T_1} + c_2 \frac{\partial}{\partial Z_1}\right) A = \pm \alpha A - \beta A B,$$

$$\left(\frac{\partial}{\partial T_1} + c_2 \frac{\partial}{\partial Z_1}\right) B = \left(\frac{\partial}{\partial T_1} + c_1 \frac{\partial}{\partial Z_1}\right) |A|^2,$$

or a canonical nonlinear Klein–Gordon equation

$$\left(\frac{\partial}{\partial T_1} + c_1 \frac{\partial}{\partial Z_1}\right)\left(\frac{\partial}{\partial T_1} + c_2 \frac{\partial}{\partial Z_1}\right) A = \pm \alpha A - \beta A |A|^2.$$

Gibbon and McGuinness (1981) give several physical examples leading to one or the other of these equations. The nonlinear analysis of the incompressible Kelvin–Helmholtz instability by Weissman led to the nonlinear Klein–Gordon evolution equation above. The "AB equations" are completely integrable by the Inverse Scattering (or Spectral) Transform (Ablowitz and Segur, 1981; Dodd *et al.*, 1982) and have been considered by Gibbon *et al.* (1979) and Gibbon and McGuinness (1981). The nonlinear Klein–Gordon equation, however, is not completely integrable, and its solutions have been discussed by Weissman (1979), Murakami (1986), and Parkes (1991).

In this paper, we consider the weakly nonlinear evolution of the supersonic (or strongly compressible) Kelvin–Helmholtz instability. Given the broad relevance of this instability in a variety of astrophysical and space settings, such an analysis is necessary to understand the large-scale behavior of these flows. As a first step, we will consider a spatially homogeneous equilibrium here. Clearly, it will ultimately be necessary to extend this to more realistic spatially non-uniform equilibrium flow configurations to correlate the results to simulations and observations on such flows.

The rest of this paper is organized as follows. Section 2 develops the asymptotic expansion for the basic equations and derives the governing generic problem at all orders of the perturbation expansion. Section 3 reviews and extends the results for the linear problem. The second-order equations are solved in Section 4. In particular, the second-order zeroth-harmonic yields the equations for the slow evolution of the equilibrium quantities on the second-order scales, the second-harmonic equations yield the particular solutions at this order, while the suppression of the secular first-harmonic terms yields a relation between the slow first-order spatial and temporal evolutions of the linear amplitudes. Section 5 considers the third-order equation, with the zeroth harmonic or D.C. yielding the second-order corrections (or "ponderomotive" forces in the plasma physics terminology) to the equilibrium quantities, and the first harmonic then yielding the overall equation for the evolution of the linear amplitude on the slow first-order scales. Section 6 discusses this amplitude equation to delineate the parameter regimes where the nonlinear evolution does or does not lead to a final stable or permanent wave structure or nonlinear pattern.

2. BASIC EQUATIONS

The fluid-dynamical equations for a compressible, inviscid neutral fluid with an adiabatic equation of state are

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) &= 0, \\ \rho \frac{d\mathbf{v}}{dt} &= -\nabla p, \\ \frac{d}{dt}(p\rho^{-\gamma}) &= 0, \end{aligned} \tag{1}$$

with $d/dt \equiv (\partial/\partial t + \mathbf{v} \cdot \nabla)$. The equilibrium we consider has a constant density $\bar{\rho}$, pressure $\bar{p}$, temperature $\bar{T}$, and flow velocity

$$\mathbf{v}_0 = \begin{cases} v_{zm}\hat{z}, & x > 0 \\ -v_{zm}\hat{z}, & x < 0 \end{cases} \tag{2}$$

with a tangential velocity discontinuity at $x = 0$.

Each of the physical variables is expanded in a perturbation series

$$\begin{aligned} \rho &= \bar{\rho} + \sum_{i=1} \varepsilon^i \rho^{(i)}, \\ \mathbf{v} &= \mathbf{v}_0 + \sum \varepsilon^i \left(v_x^{(i)}\hat{x} + v_y^{(i)}\hat{y} + v_z^{(i)}\hat{z} \right), \\ p &= \bar{p} + \sum \varepsilon^i p^{(i)}. \end{aligned} \tag{3}$$

Introducing slow time and spatial scales T_i and Z_i such that

$$\begin{aligned} T_1 &= \varepsilon t, \qquad Z_1 = \varepsilon z, \\ T_2 &= \varepsilon^2 t, \qquad Z_2 = \varepsilon^2 z, \end{aligned} \tag{4a}$$

the temporal and spatial derivatives become (Dodd *et al.* 1982; Nayfeh 1973)

$$\begin{aligned} \frac{\partial}{\partial t} &= \partial_t + \varepsilon \partial_{T_1} + \partial^2 \partial_{T_2}, \\ \frac{\partial}{\partial z} &= \partial_z + \varepsilon \partial_{Z_1} + \partial^2 \partial_{Z_2}. \end{aligned} \tag{4b}$$

Using (3) and (4) in (1) yields equations at $O(\varepsilon)$, $O(\varepsilon^2)$, and $O(\varepsilon^3)$. The structure of these equations may be written as

$$\frac{\partial \rho^{(i)}}{\partial t} + \bar{\rho}\left[\frac{\partial v_x^{(i)}}{\partial x} + \frac{\partial v_y^{(i)}}{\partial y} + \frac{\partial v_z^{(i)}}{\partial z}\right] + v_0(x)\frac{\partial \rho^{(i)}}{\partial z} = \alpha^{(i)}, \qquad i = 1, 2, 3 \tag{5}$$

$$\bar{\rho}\left[\frac{\partial v_x^{(i)}}{\partial t} + v_0(x)\frac{\partial v_x^{(i)}}{\partial z}\right] + \frac{\partial p^{(i)}}{\partial x} = \beta^{(i)}, \tag{6}$$

$$\bar{\rho}\left[\frac{\partial v_y^{(i)}}{\partial t} + v_0(x)\frac{\partial v_y^{(i)}}{\partial z}\right] + \frac{\partial p^{(i)}}{\partial y} = \gamma^{(i)}, \tag{7}$$

$$\bar{\rho} v_0' v_x^{(i)} + \bar{\rho}\left[\frac{\partial v_z^{(i)}}{\partial t} + v_0(x)\frac{\partial v_z^{(i)}}{\partial z}\right] + \frac{\partial p^{(i)}}{\partial z} = \delta^{(i)}, \tag{8}$$

$$\frac{\partial}{\partial t}\left[p^{(i)} - c_s^2 \rho^{(i)}\right] = \chi^{(i)} \tag{9}$$

for various source terms $\alpha^{(i)}$, $\beta^{(i)}$, $\gamma^{(i)}$, $\delta^{(i)}$, and $\chi^{(i)}$. Here, $c_s^2 \equiv \gamma \bar{p}/\bar{\rho}$ is the adiabatic (equilibrium) sound speed.

Combining (5)–(9) will yield the generic structure of the underlying problem at linear, second, and third orders (and, in fact, at all orders). Solving (9) for $\partial_t \rho^{(i)}$ yields

$$\partial_t \rho^{(i)} = \frac{\partial_t p^{(i)} - \chi^{(i)}}{c_s^2}. \tag{10}$$

Operating on (8) with

$$D \equiv \partial_t + v_0(x)\partial_z \tag{11}$$

and using (6) yields

$$D^2 v_z^{(i)} = \frac{1}{\bar{\rho}} \left\{ D \left(\delta^{(i)} - \frac{\partial p^{(i)}}{\partial z} \right) - v_0' \left(\beta^{(i)} - \frac{\partial p^{(i)}}{\partial x} \right) \right\}. \tag{12}$$

Operating on (5) with $D^2 \partial_t$ and using (6), (7), (10), and (11) yield the composite generic form of the equation for the pressure $p^{(i)}$ at all orders i:

$$\begin{aligned} L p^{(i)} &\equiv \left[\frac{1}{c_s^2} D^3 \partial_t - \partial_t D \left(\partial_x^2 + \partial_y^2 + \partial_z^2 \right) + 2 v_0' \partial_t \, \partial_z \, \partial_x \right] p^{(i)} \\ &= \Gamma^{(i)}. \end{aligned} \tag{13}$$

Here, the ith-order source term is

$$\begin{aligned} \Gamma^{(i)} \equiv{}& \frac{1}{c_s^2} D^3 \chi^{(i)} - \partial_t D \, \partial_x \beta^{(i)} - \partial_t D \, \partial_y \gamma^{(i)} - \partial_t D \, \partial_z \delta^{(i)} \\ &+ 2 v_0' \partial_t \partial_z \beta^{(i)} + D^2 \partial_t \alpha^{(i)}. \end{aligned} \tag{14}$$

Notice that the operator L does not contain a constant term, i.e., one not containing a derivative. This fact will be significant in the nonlinear analysis of the following sections.

3. LINEAR PROBLEM

In this section, we briefly recapitulate the results for the $i = 1$ or linear problem (Landau, 1944; Gerwin, 1968; Roy Choudhury and Lovelace, 1984; Ray, 1982), which will be used in the higher-order calculations. In addition, two alternative formulations for the boundary conditions at the velocity discontinuity at $x = 0$ are also considered. These will be important in deriving and interpreting the appropriate boundary conditions at the discontinuity. Equations (5)–(9) for $i = 1$ have source terms

$$\alpha^{(1)} = \beta^{(1)} = \gamma^{(1)} = \delta^{(1)} = \chi^{(1)} = 0.$$

Also, for linear solutions or normal modes of the form (Roy Choudhury and Lovelace, 1984)

$$\chi^{(1)} = \tilde{\chi}_1(x, Z_1, Z_2, T_1, T_2) \exp\left[i \left(k_y y + k_z z - \omega t \right) \right] \tag{15}$$

(with χ representing any of the field variables ρ, v_x, v_y, v_z, or p), (11) yields $D \equiv i(k_z v_0 - \omega)$. Using these, (13) and (14) for $i = 1$ reduce to the linear eigenvalue problem

$$(p^{(1)})'' - \frac{2U'}{U}(p^{(1)})' = k^2(1 - U^2)p^{(1)}. \tag{16}$$

Here, $' \equiv d/dx$, the dimensionless flow velocity is

$$U \equiv \frac{k_z v_0(x) - \omega}{kc_s} = \frac{k_z v_0(x)}{kc_s} - W, \tag{17a}$$

with

$$k \equiv \left(k_y^2 + k_z^2\right)^{1/2}, \tag{17b}$$

and the dimensionless frequency

$$W \equiv \omega/kc_s. \tag{17c}$$

Equation (16) is solved with boundary conditions for $x \to \pm\infty$, together with a matching condition at $x = 0$ between the solutions in the two half-spaces $x < 0$ and $x > 0$. We consider the conditions at $x \to \pm\infty$ first. Away from the discontinuity at $x = 0$, the solutions of (16) are $p^{(1)} =$ constant $\exp[\pm ik_- x]$ for $x < 0$, and $p^{(1)} =$ constant $\exp[\pm ik_+ x]$ for $x > 0$, where

$$k_\pm \equiv k\left[(\pm C - W)^2 - 1\right]^{1/2} = k_{r\pm} - ik_{i\pm}, \tag{18}$$

and the "reduced" Mach number C and Mach number M

$$C \equiv \frac{k_z M}{2k}, \qquad M \equiv \frac{2v_{zm}}{c_s}. \tag{19}$$

We are interested in the initial-value problem so that ω has a nonzero, positive imaginary part. The correct choice for the $\pm$ signs in the solutions for $p^{(1)}$ for $x \gtrless 0$ are determined by dual considerations (Roy Choudhury and Lovelace, 1984). We must ensure that away from the discontinuity the pressure perturbation corresponds to both a spatially damping (in x) and an outgoing wave in the co-moving frame of the fluid in each half-space. The requirement of damping follows since there is no source of energy away from $x = 0$. The Sommerfeld radiation condition applied in a frame co-moving with the fluid (Miles, 1957; Gerwin, 1968; Pearlstein and Berk, 1969) results in outgoing, spatially damping for $x \to \pm\infty$, if we choose the solutions $p^{(1)} \alpha e^{-ik_- x}$ for $x < 0$, and $p^{(1)} \alpha e^{-ik_+ x}$ for $x > 0$. Combining these with (15) and (18) yields

$$p^{(1)} = A_\pm(Z_1, Z_2, T_1, T_2)e^{i\theta}e^{k_{i\pm}x} + \text{c.c.}, \qquad x \gtrless 0 \tag{20}$$

with

$$\theta = k_y y + k_z z - \omega t - k_{r\pm} x. \tag{21}$$

A is the amplitude of the pressure perturbation and c.c. denotes complex conjugate. Using (20) and the other $O(\varepsilon)$ equations yields the solutions for the other linear fields (away from $x = 0$),

$$\rho^{(1)} = \frac{A_\pm}{c_s^2} e^{i\theta} e^{k_{i\pm} x} + \text{c.c.} \tag{22}$$

$$v_x^{(1)} = \left[\frac{k_\pm A_\pm}{\bar{\rho}(k_z v_0 - \omega)} e^{i\theta} e^{k_{i\pm} x} + \text{c.c.} \right], \tag{23a}$$

$$v_y^{(1)} = -\left[\frac{k_y A_\pm e^{i\theta} e^{k_{i\pm} x}}{\bar{\rho}(k_z v_0 - \omega)} + \text{c.c.} \right], \tag{23b}$$

$$v_z^{(1)} = -\left[\left\{ \frac{ik_z(k_z v_0 - \omega) A_\pm + k_\pm v_0' A_\pm}{i(k_z v_0 - \omega)^2 \bar{\rho}} \right\} e^{i\theta} e^{k_{i\pm} x} + \text{c.c.} \right]. \tag{23c}$$

Next, considering the matching of solutions at $x = 0$, the dynamical condition at the interface (Chandrasekhar, 1981; Gerwin, 1968; Weissman, 1979) corresponds to any pressure difference across $x = 0$ being due to surface tension. In the absence of a density difference and, hence, surface tension in the relevant linear equation of motion, we have

$$p^{(1)}\Big|_{x=0_-} = p^{(1)}\Big|_{x=0_+},$$

so that (20) is consistent with the same amplitude A for $x > 0$ and $x < 0$, i.e., $A_+ = A_-$ to linear order. However, we distinguish A_+ and A_- as they evolve differently at higher orders. The kinematic condition at the interface (Chandrasekhar, 1981; Weissman, 1979) corresponds to uniqueness of the normal displacement at the interface (or no cavitation). If the position of the interface is

$$x_s = 0 + \varepsilon x_s^{(1)} e^{i\theta} e^{k_{i\pm} x} + \varepsilon^2 x_s^{(2)} + \ldots, \tag{24}$$

this condition may be written as

$$\left(\frac{\partial}{\partial t} + \mathbf{v}_0 \cdot \boldsymbol{\nabla} \right) x_s = v_x. \tag{25}$$

Using (3) and (24), the $O(\varepsilon)$ or linear part of this yields

$$x_s^{(1)} = \frac{v_x^{(1)}}{i(k_z v_0 - \omega)} \text{ continuous} \tag{26}$$

across $x = 0$ for no cavitation at $O(\varepsilon)$. Consider another alternative approach to deriving this condition. Equation (16) may be rewritten as

$$\left[\frac{\left(p^{(1)}\right)'}{U^2} \right]' = \frac{k^2(1 - U^2)\, p^{(1)}}{U^2}.$$

Integrating this from $x = 0_-$ to $x = 0_+$ (recognizing that U is continuous, while U' contains a Kronecker delta) yields

$$\left. \frac{\left(p^{(1)}\right)'}{U^2} \right|_{x=0_-} = \left. \frac{\left(p^{(1)}\right)'}{U^2} \right|_{x=0_+}. \tag{27}$$

It is straightforward to verify using (A4) that (26) and (27) are identical conditions. Using (18) and (20) in (27) (or (26)) yields the linear dispersion relation for arbitrarily compressible modes (Landau, 1944; Roy Choudhury and Lovelace, 1984),

$$W^2 = (1 + C^2) - (1 + 4C^2)^{1/2}, \tag{28}$$

where a spurious solution introduced by squaring has been omitted. We will refer to (28) subsequently in the alternative form

$$F(\omega, k_y, k_z) \equiv \frac{\omega^2}{k^2 c_s^2} - \left(1 + \frac{k_y^2 v_{zm}^2}{k^2 c_s^2}\right) + \left[1 + \frac{4k_y^2 v_{zm}^2}{k^2 c_s^2}\right]^{1/2}$$
$$= 0. \tag{29}$$

Note that (28) or (29) correspond to $W = 0$ or the neutral curve at $C = \sqrt{2}$, with purely growing solutions for $C < \sqrt{2}$ (Roy Choudhury and Lovelace, 1984). Two complex conjugate solutions of (28) for $C < \sqrt{2}$ coalesce on the neutral curve $C = \sqrt{2}$, which is typical of dispersive instabilities characterized by real dispersion relations such as (28) as discussed in Section 1. Notice that the neutral curve $C = \sqrt{2}$ for this problem (see Fig. 2 of Roy Choudhury and Lovelace (1984)) is a horizontal line in the (k, C) plane (the (B, A) plane in that paper). Thus, the onset of linear instability for this problem occurs simultaneously at all wavenumbers $k \equiv (k_y^2 + k_z)^{1/2}$, which is somewhat different from the more typical situation of instability onset first occurring at a characteristic wavenumber k_c corresponding to the minimum of the neutral curve in the (k, C) plane. The most linearly unstable modes (those driven unstable for the lowest Mach number or shear velocity) correspond to those propagating along the equilibrium flow (with $k_y = 0$), so that the least value of shear for $C = \sqrt{2}$ corresponds to Mach number $M = 2\sqrt{2}$ (see (19)).

Notice that for the linearly unstable modes of the compressible (or supersonic) Kelvin–Helmholtz instability, the kinematic boundary condition (26) has a completely different form from the corresponding condition in the incompressible limit (Weissman, 1979). In the incompressible limit, this condition is a relation between the interface displacement and the velocity potential (from which v_x, v_y, and v_z may all be obtained). Thus, the relevant variables for this case are the displacement of the interface, the velocity potential, and the pressure. By contrast, for the compressible problem the interfacial displacement is somewhat peripheral, being related only to v_x. Since no velocity potential now exists, one works with the physical fields ρ, v_x, v_y, v_z, and ρ and the kinematic condition corresponding to the uniqueness of the interface position enters as an extra condition. This difference will be significant in the subsequent nonlinear analysis.

4. SECOND-ORDER SOLUTIONS

To consider the nonlinear evolution within the usual weakly nonlinear theory (Dodd *et al.*, 1982; Drazin and Reid, 1981; Ochoa and Murray, 1983), we consider the first onset of instability near $C = \sqrt{2}$. From Section 3, this corresponds to the expansion of the reduced Mach number

$$C = \sqrt{2} \mp \delta,$$

independent of wavenumber k since all wavenumbers go unstable simultaneously at $C = \sqrt{2}$. This is different from the usual Stuart-type expansion, which would be of the form (Dodd *et al.*, 1982; Gibbon and McGuinness, 1981; Drazin and Reid, 1981; Stuart, 1960)

$$C = C_{\text{crit.}} \pm \varepsilon^2 \zeta$$

$$\zeta = \left| \frac{C''(k_c)}{2} \right|, \qquad \varepsilon = (k - k_c)$$

in the more usual case of a most unstable wavenumber k_c which first goes unstable at $C = C_{\text{crit}}$ (corresponding to a minimum of the neutral curve at (k_c, C_{crit}) in the (k, C) plane).

The $O(\varepsilon^2)$ source terms $\alpha_\pm^{(2)}$, $\beta_\pm^{(2)}$, $\gamma_\pm^{(2)}$, $\delta_\pm^{(2)}$, and $\chi_\pm^{(2)}$ in the two half-spaces are computed using (20)–(23). Each of these terms has the structure

$$\delta^{(2)} = \delta_{0\pm}^{(2)} + \delta_{1\pm}^{(2)} e^{i\delta} + \delta_{2\pm}^{(2)} e^{2i\delta}, \tag{30}$$

where δ stands for α, β, γ, δ, or χ. Using these, the inhomogeneous term on the right-hand side of (13) may be computed yielding the equation

$$Lp_{\pm}^{(2)} = \Gamma_{0\pm}^{(2)} + \Gamma_{1\pm}^{(2)} e^{i\theta} + \Gamma_{2\pm}^{(2)} e^{2i\theta} + \text{c.c.}, \tag{31}$$

with $\Gamma_{1\pm}^{(2)}, \Gamma_{2\pm}^{(2)}, \Gamma_{3\pm}^{(2)}$ given by (14) with $i = 2$ and $\delta^{(2)}$ of (30).

The zeroth-harmonic term $\Gamma_0^{(2)}$ in (31) is secular and would cause $p_{\pm}^{(2)}$ to contain terms proportional to x and t, causing the expansions (3) to become non-uniform after a time of $O(\varepsilon^{-1})$ since the operator L does not contain a constant term (Dodd *et al.*, 1982; Nayfeh, 1973), as noted following (13) and (14). To suppress this secular term, one would need to set $\Gamma_0^{(2)} = 0$. We will not pursue this here since it will shortly be demonstrated that one needs to look at the individual zeroth-harmonic source terms $\alpha_0^{(2)}$, $\beta_0^{(2)}$, $\gamma_0^{(2)}$, $\delta_0^{(2)}$, and $\chi_0^{(2)}$ in (30) (to suppress such secular terms), which takes the above condition $\Gamma_0^{(2)} = 0$ into account automatically.

The first-harmonic term $\Gamma_1^{(2)} e^{i\theta}$ in (31) is also secular (since the solutions of the homogeneous equation have the form $e^{i\theta}$). Suppressing this requires (using (14), the $O(\varepsilon^2)$ sources, and simplifying)

$$\begin{aligned}
\Gamma_{1\pm}^{(2)} = & -\frac{i(k_z v_0 - \omega)^3}{c_s^2} \chi_{1\pm}^{(2)} + i\omega k_{\pm}(k_z v_0 - \omega)\beta_{1\pm}^{(2)} \\
& - i\omega k_y (k_z v_0 - \omega)\gamma_{1\pm}^{(2)} - i\omega k_z (k_z v_0 - \omega)\delta_{1\pm}^{(2)} \\
& + i\omega (k_z v_0 - \omega)^2 \alpha_{1\pm}^{(2)} \\
= & \, i\omega(k_z v_0 - \omega)\left[-\left\{ \frac{(k_{\pm}^2 + k_y^2 + k_z^2)}{(k_z v_0 - \omega)^2} + c_s^2 \right\} \left(\frac{\partial A_{\pm}}{\partial T_1} + v_0 \frac{\partial A_{\pm}}{\partial Z_1} \right) \right. \\
& \left. + \frac{2ky}{k_z v_0 - \omega} \frac{\partial A_{\pm}}{\partial Y_1} \right] = 0.
\end{aligned} \tag{32}$$

On the neutral curve $C = \sqrt{2}$ (at any wavenumber k), $\Gamma_1^{(2)} \equiv 0$. Away from the neutral curve and assuming $\partial/\partial Y_1 = 0$, Eq. (32) implies that

$$A_{\pm}(Z_1, Z_2, T_1, T_2) = A_{\pm}(Z_1 - v_0 T_1, Z_2, T_2), \qquad \text{off the neutral curve,} \tag{33}$$

with v_0 given in (2). This behavior is standard in that suppression of secularity at the second-order first harmonic requires the linear amplitude $A_{\pm}$ in (20) to depend on a linear combination of T_1 and Z_1, in this case $\bar{Z} \equiv Z_1 - V_0 T_1$, *off* the neutral curve, while leaving T_1 and Z_1 dependence of $A_{\pm}$ unconnected *on* the neutral curve (Weissman, 1979). However, it is non-standard in that the usual linear combination of T_1 and Z_1

is $Z_1 - c_g T_1$, where $c_g \equiv d\omega/dk$ is the group velocity obtained from the linear dispersion relation (28). The reason for this non-standard behavior is that, as noted in Section 3, our set of equations does not include explicit equations for the boundary conditions at the discontinuity at $x = 0$. Rather, these boundary conditions correspond to an extra no cavitation condition as discussed earlier. As a consequence, the condition (32) may not be cast into the form $-F_\omega(\partial A/\partial T_1) + F_{k_z}(\partial A/\partial Z_1) + F_{k_y}(\partial A/\partial Y_1) = 0$, as in the incompressible case (Weissman, 1979).

Note that an alternative way to derive $\Gamma_{1\pm}^{(2)} = 0$ (Weissman, 1979) is by considering only the $e^{i\theta}$ or first-harmonic parts of the $O(\varepsilon^2)$ equations and combining them into a composite equation for $p_1^{(2)}$ which yields

$$0 \cdot p_1^{(2)} = \Gamma_{1\pm}^{(2)}$$

thus recovering (32).

Note that (33) demonstrates that the first-order variable $\bar{Z} \equiv Z_1 - v_0 T_1$ which controls the dependence of the linear amplitude $A_\pm$ away from the neutral curve is different for $x > 0$ where $v_0 = v_{zm}$, and $x < 0$ where $v_0 = -v_{zm}$. Thus, the amplitudes A_+ and A_- evolve differently on the slow first-order scales T_1, Z_1, or $\bar{Z} = Z_1 - v_0 T_1$. We will see this explicitly at third-order.

The solutions for the second-order fields which we consider next are only the particular solutions, as is standard for singular perturbation or asymptotic expansions (Dodd *et al.*, 1982; Nayfeh, 1973). Hence, in the usual way we need not enforce the boundary conditions at $x = 0$, i.e., continuity of pressure and no cavitation. Enforcing these would involve including the homogeneous solutions (with the no-cavitation condition obtained by integrating (13) with $i = 2$ across $x = 0$), which would combine with the linear solutions (20)–(23). These conditions would then hold up to the time $A_\pm$ start evolving on the slow first-order timescales T_1, Z_1 (or $\bar{Z}$). We follow the more usual approach (Dodd *et al.*, 1982; Nayfeh, 1973) of including only the particular solutions.

With the secular terms $\Gamma_0^{(2)}$ and $\Gamma_1^{(2)}$ suppressed in (31), a particular solution of (31) for the second-order pressure

$$p_\pm^{(2)} = p_{0\pm}^{(2)}(Z_1, Z_2, T_1, T_2) + \left(c_\pm A_\pm^2 e^{2i\theta} + \text{c.c.}\right), \tag{34}$$

with

$$c_\pm = \frac{\Gamma_{2\pm}^{(2)}}{\left[-16\omega(k_z v_0 - \omega)^3 + 16\omega(k_z v_0 - \omega)\left(k_\pm^2 + k^2\right)\right]}, \tag{35}$$

and $\Gamma_{2\pm}^{(2)}$ given in (B7).

Returning to the individual second-order equations, and writing the individual second-order fields in the form

$$\phi_{\pm}^{(2)} = \phi_{0\pm}^{(2)} + \phi_{1\pm}^{(2)} e^{i\theta} + \phi_{2\pm}^{(2)} e^{2i\theta} + \text{c.c.}, \tag{36}$$

with ϕ representing ρ, v_x, v_y, v_z, or p, a typical equation yields

$$\begin{aligned} Dv_x^{(2)} &= i(k_z v_0 - \omega) v_{x,1}^2 e^{i\theta} + 2i(k_z v_0 - \omega) v_{x,2}^{(2)} e^{2i\theta} \\ &= \frac{1}{\bar{\rho}} \left[2ik_{\pm} c_{\pm} A^2 e^{2i\theta} + \beta_0^{(2)} + \beta_1^{(2)} e^{i\theta} + \beta_2^{(2)} e^{2i\theta} \right]. \end{aligned}$$

Hence,

$$\beta_0^{(2)} = 0. \tag{37a}$$

(Alternatively, balancing $\beta_0^{(2)}$ would require secular terms proportional to t and z in $v_x^{(2)}$. Note that such terms are inadmissible since we work with physical field variables and the boundary conditions at $x \to +\infty$ are violated by such terms. Weissman (1979) used such terms in his velocity potential for the incompressible case, consistent with bounded velocities at infinity.) Similar considerations yield

$$\alpha_0^{(2)} = 0, \tag{37b}$$

$$\gamma_0^{(2)} = 0, \tag{37c}$$

$$\delta_0^{(2)} = 0, \tag{37d}$$

$$\chi_0^{(2)} = 0. \tag{37e}$$

Using (37), $\Gamma_0^{(2)}$ (from (14)) $\equiv 0$, so that the zeroth-harmonic secular term in (31) is indeed zero, as required earlier. The suppression of second-order zeroth-harmonic secular terms represented by (37) are conditions for the evolution of the equilibrium quantities $\bar{\rho}, v_0, \bar{p}$ on the even slower second-order scales T_2, X_2, Y_2, and Z_2. For instance, using (B2), (37a) is

$$\begin{aligned} \frac{\partial \bar{p}}{\partial X_2} = e^{2k_{i\pm} x} \Bigg\{ & \frac{i\omega k_{\pm} |A|^2}{\bar{\rho} c_s^2 (k_z v_0 - \omega)} - \frac{v_0 i k_z k_{\pm} |A|^2}{\bar{\rho} c_s^2 (k_z v_0 - \omega)} + \frac{ik_{\pm}^2 k_{\pm *} |A|^2}{\bar{\rho} |k_z v_0 - \omega|^2} \\ & + \frac{ik_y^2 k_{\pm} |A|^2}{\bar{\rho} |k_z v_0 - \omega|^2} - \frac{ik_z^2 k_{\pm *} (k_z v_0 - \omega) |A|^2}{\bar{\rho} |k_z v_0 - \omega|^2 (k_z v_0 - \omega)} \Bigg\}. \end{aligned} \tag{38}$$

In cases where the equilibrium configuration is continuously spatially inhomogeneous in one direction (at least one of the equilibrium quantities is a function of the variable in this direction, e.g., if v_0 is a continuous function of x over a finite-width shear layer), conditions such as (38) typically generalize to a diffusion equation in T_2 and the zeroth-order spatial variable along the direction of inhomogeneity (Kaup *et al.*, 1988).

Also, note that $p_{\pm}^{(2)}$ may become infinite if the denominator of $c_{\pm}$ in (35) is zero. This so-called "second harmonic resonance" would cause the expansions (3) to fail at $O(\varepsilon^2)$. Treating this resonance requires different expansions (Weissman, 1979; Craik *et al.*, 1992), and will not be considered here.

5. THIRD-ORDER ANALYSIS AND AMPLITUDE EVOLUTION EQUATIONS

Comparing (5)–(9) for $i = 3$ with the $O(\varepsilon^3)$ sources yields $\alpha^{(3)}$, $\beta^{(3)}$, $\gamma^{(3)}$, $\delta^{(3)}$, and $\chi^{(3)}$. Computing these using (20)–(23), (34), (36) (with ϕ standing for ρ, v_x, v_y, v_z, or p) and (C1)–(C12) yields

$$\xi^{(3)} = \xi_0^{(3)} + \xi_1^{(3)} e^{i\theta} + \xi_2^{(3)} e^{2i\theta} + \xi_3^{(3)} e^{3i\theta} + \text{c.c.}, \tag{39}$$

where ξ represents α, β, γ, δ, or χ. Note that we do not use the linear and second-order solutions to explicitly expand these coefficients for $i = 2, 3$ as they will not be required in our analysis. Once these coefficients are known, the right-hand side of (13) may be evaluated for $i = 3$ yielding the equation

$$Lp^{(3)} = \Gamma_0^{(3)} + \Gamma_1^{(3)} e^{i\theta} + \Gamma_2^{(3)} e^{2i\theta} + \Gamma_3^{(3)} e^{3i\theta} + \text{c.c.}, \tag{40}$$

with $\Gamma_i^{(3)}$, $i = 0, 1$, given in terms of these by (14) with $i = 3$.

a. *Zeroth Harmonic*

Considering the structure of the third-order equations, the operator $D \equiv (\partial_t + v_0 \partial_z)$ has no constant term. Hence, as in second-order, the third-order zeroth-harmonic sources must vanish, i.e.,

$$\alpha_0^{(3)} = 0, \tag{41a}$$

$$\beta_0^{(3)} = 0, \tag{41b}$$

$$\gamma_0^{(3)} = 0, \tag{41c}$$

$$\delta_0^{(3)} = 0, \tag{41d}$$

$$\chi_0^{(3)} = 0, \tag{41e}$$

to prevent the occurrence of aperiodic z or t dependent terms in the third-order zeroth-harmonic fields $\rho_0^{(3)}$, $v_{x,0}^{(3)}$, $v_{y,0}^{(3)}$, $v_{z,0}^{(3)}$, and $p_0^{(3)}$ (such terms violate the boundary conditions on these fields and are inadmissible, unlike the case of the velocity potential in the incompressible case (Weissman, 1979)). Thus the zeroth-harmonic secular term $\Gamma_0^{(3)}$ in (40) is automatically zero. Neglecting the evolution on the slower second-order T_2 and Z_2 scales (the evolution for the present dispersive instability occurs on the faster first-order T_1, Z_1 scales as discussed in Section 1), Eqs. (41) may be recast as the matrix equation

$$\left(\underline{\underline{\delta}}\frac{\partial}{\partial T_1} + \underline{\underline{\Delta}}\frac{\partial}{\partial Z_1}\right)\underline{D} = \left(\underline{g}^{(1)}\frac{\partial}{\partial T_1} + \underline{g}^{(2)}\frac{\partial}{\partial Z_1}\right)|A|^2, \tag{42}$$

where

$$\overline{D} = \begin{pmatrix} \rho_0^{(2)} \\ v_{x,0}^{(2)} \\ v_{y,0}^{(2)} \\ v_{z,0}^{(2)} \\ p_0^2 \end{pmatrix}, \tag{43}$$

$$\underline{\underline{\delta}} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -\bar{\rho} & 0 & 0 & 0 \\ 0 & 0 & -\bar{\rho} & 0 & 0 \\ 0 & 0 & 0 & -\bar{\rho} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{44}$$

$$\underline{\underline{\Delta}} = \begin{pmatrix} -v_0 & 0 & 0 & -\bar{\rho} & 0 \\ 0 & -\bar{\rho}v_0 & 0 & 0 & 0 \\ 0 & 0 & -\bar{\rho}v_0 & 0 & 0 \\ 0 & 0 & 0 & -\bar{\rho}v_0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{45}$$

$$\bar{g}^{(1)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \tag{46}$$

$$\bar{g}^{(2)} = \begin{pmatrix} -\dfrac{2}{\bar{\rho} c_s^2 v_0} \\ 0 \\ 0 \\ \dfrac{k_{\pm}^2 + k^2}{\bar{\rho}(k_z v_0)^2} - \dfrac{1}{\bar{\rho} c_s^2} \\ v_0 \end{pmatrix}, \tag{47}$$

where we employ the fact that we are in the vicinity of the neutral curve $C = \sqrt{2}$ for the weakly nonlinear analysis, so that from (18), $k_{i\pm} \approx 0$.

Notice that the last line of the matrix equation (42) corresponding to (41e) implies that on the first-order scales the linear amplitude $A_{\pm}$ of (20)–(23) satisfies

$$A_{\pm}(Z_1, T_1) = A_{\pm}(Z_1 - v_0 T_1) = A_{\pm}(\bar{Z}) \quad \text{on and off the neutral curve.} \tag{48}$$

Here, $v_0 = v_{zm}$ for $x > 0$, and $v_0 = -v_{zm}$ for $x < 0$. Note that (48) generalizes (33), away from the linear neutral curve $C = \sqrt{2}$ where $\omega = 0$. Using (48) (or (33) in the vicinity of the neutral curve, as usually done in weakly nonlinear analysis), the second, third, and fifth equations in (42) are satisfied for any values of $\rho_0^{(2)}$, $v_{x,0}^{(2)}$, and $v_{y,0}^{(2)}$. The first and fourth equations in (42) yield (upon integrating with respect to $\bar{Z} \equiv Z_1 - v_0 T_1$) the second order zeroth-harmonic fields,

$$v_{z,0\pm}^{(2)} = \frac{2|A|^2}{\bar{\rho} c_s^2 v_0}, \tag{49a}$$

$$p_{0\pm}^{(2)} = \left[\frac{1}{\bar{\rho} c_s^2} - \frac{k_{\pm}^2 + k^2}{\bar{\rho}(k_z v_0)^2}\right]|A|^2, \tag{49b}$$

which are dimensionally consistent as A in (20) has dimensions of pressure. Since $\rho_0^{(2)}$, $v_{x,0}^{(2)}$, and $v_{y,0}^{(2)}$ are arbitrary, we choose

$$\rho_0^{(2)} = v_{x,0}^{(2)} = v_{y,0}^{(2)} = 0. \tag{49c}$$

Equations (49) yielding the second-order zeroth-harmonic fields are the so-called "ponderomotive" forces in the plasma physics terminology (Kaup *et al.*, 1988). Note that $v_{z,0}^{(2)}$ given by (49a) represents a second-order correction to the mean flow. This is absent for the incompressible Kelvin–Helmholtz flow (Weissman, 1979) as seen from (49a) for $c_s^2 \to \infty$, and is analogous to a similar term arising in the nonlinear evolution of the baroclinic instability (Pedlosky, 1972).

b. *First Harmonic Terms and Evolution Equation*

The $\Gamma_1^{(3)} e^{i\theta}$ term on the right of (40) resonates with the homogeneous solution of this equation. Suppressing this term in the usual way so as to prevent the occurrence of non-uniform secular terms yields after considerable algebra,

$$\Gamma_{1\pm}^{(3)} = \frac{\omega\left(k_\pm^2 + k^2\right)}{(k_z v_0 - \omega)} \left\{ \frac{\partial^2 A}{\partial T_1^2} + \left[2v_0 - \frac{2k_z(k_z v_0 - \omega)}{k_\pm^2 + k^2} \right] \frac{\partial^2 A}{\partial T_1\, \partial Z_1} + \left[v_0^2 + \frac{\omega(k_z v_0 - \omega)}{k_\pm^2 + k^2} \right] \frac{\partial^2 A}{\partial Z_1^2} \right\} - GA + \left(\sum_i \bar{\beta}_i D_i \right) A - \bar{e} A |A|^2 = 0,$$

where $G, \bar{\beta}_i, \bar{e}$ are complicated constants and the D_i are the components of the vector (43) of second-order zeroth-harmonic fields. Using (48) and (49) (for the D_i) in the above equation yields the final evolution equation

$$\frac{d^2 A}{dT_1^2} = G_\pm A + \bar{N}_\pm |A|^2 A, \tag{50}$$

which is a nonlinear Klein−Gordon equation, as for the incompressible inviscid Kelvin−Helmholtz instability (Weissman, 1979) and a buckling problem in elasticity (Lange and Newell, 1974; Gibbon and McGuinness, 1981). The coefficients $G_\pm$ and $\bar{N}_\pm$ are very involved.

Note that, using (48), (50) may be rewritten as an equation for evolution on the slow spatial scale Z_1, i.e.,

$$v_0^2 \frac{d^2 A}{dZ_1^2} = G_\pm A + \bar{N}_\pm |A|^2 A. \tag{51}$$

In both the forms (50) and (51), the evolution of the amplitude A (of the linear fields (20)−(23)) occurs on the slow first-order scales, as expected from the discussion of Section 1 for inviscid or dispersive instabilities.

6. DISCUSSION

The evolution equation (50) or (51) governing the weakly nonlinear evolution of the amplitude A (of the linear fields of (20)–(23)) for the supersonic Kelvin–Helmholtz instability is of the form of a non-integrable nonlinear Klein–Gordon equation as for the incompressible Kelvin–Helmholtz instability (Weissman, 1979). In both these cases, we do not obtain the integrable AB equations, which are the other possible form of the canonical evolution equation for inviscid or dispersive instabilities as discussed in Section 1 (Gibbon and McGuinness, 1981; Gibbon *et al.*, 1979).

For the supersonic Kelvin–Helmholtz instability considered here, there is no critical point of wavenumber k_c (corresponding to a minimum of the neutral curve or surface in the (k, C) or (k, M) plane) at which the onset of instability first occurs. Rather, as discussed in Section 3, the onset of instability occurs simultaneously at all z wavenumber values k_z at Mach number $M = 2\sqrt{2}$. Thus, the second and third cases of the incompressible analysis (Weissman, 1979) corresponding to the critical point (the minimum of the neutral surface or curve) and other points on the neutral curve coalesce into the latter for the supersonic Kelvin–Helmholtz instability. Also, for the incompressible case, the second-order first harmonic analysis shows that the evolution equation at the critical point is characterized by T_1 and Z_1 remaining independent of each other, in contrast to the stable region slightly away from the neutral surface where these variables occur in the combination $Z_1 - c_g T_1$ with c_g being the linear dispersion relation. This remains valid at the second-order first harmonic level for the present supersonic instability as seen in (33). However, in the present supersonic case, the third-order zeroth-harmonic analysis yields the additional feature that the first-order slow time and spatial scales T_1 and Z_1 always occur in the combination $\overline{Z} = Z_1 - v_0 T_1$ as seen in (48). Thus, for the supersonic analysis, the first and third cases of Weissman (1979), corresponding to the stable region and points on the neutral curve, also coalesce. In summary, for the compressible Kelvin–Helmholtz modes, the weakly nonlinear evolution at all points (or z wavenumbers k_z) on the neutral surface as well as in the stable regime $M = 2\sqrt{2} + \delta$ and all k_z is governed by the same equation (50) (or (51)).

Another difference between the nonlinear evolution of the incompressible and compressible modes is that a second-order correction (given by (49a)) to the mean or equilibrium flow occurs for the latter case, in contrast to the former incompressible limit $c_s^2 \to \infty$ (Weissman, 1979) when the correction (49a) vanishes.

Note that the nonlinear coefficient $\overline{N}$ in (50) may be stabilizing (for $\overline{N} < 0$) or destabilizing ($\overline{N} > 0$).

Equation (50) and its solutions have been extensively discussed by Weissman (1979), which also reviews the results of earlier treatments of this equation (Pedlosky, 1970; Drazin, 1970; Nayfeh and Saric, 1972). The modulational instability (linear stability of Stokes wave or spatially uniform solutions) of the complex nonlinear Klein−Gordon equation (50) has been considered using numerical Floquet analysis by Murakami (1986) and Parkes (1991) for both uniform solutions $A = A_0 e^{i\sigma t}$ (A_0 real) corresponding to a wavetrain, as well as spatially uniform but temporarily periodic Jacobian elliptic function solutions. This Floquet analysis is relatively recent and extends and complements the more well-known treatments of (50) summarized by Weissman. Consider a general nonlinear Klein−Gordon equation

$$c_1 \frac{\partial^2 A}{\partial T_1^2} - c_2 \frac{\partial^2 A}{\partial Z_1^2} + V'(A) = 0 \tag{52}$$

which includes (50) or (51). In order to study the linear stability of a periodic nonlinear carrier wave of arbitrary strength we write

$$A(x,t) = A_0(t) + \psi(x,t),$$

where ψ is a small perturbation. Substitution into (52) and linearization gives

$$\psi_{tt} - \psi_{xx} + V''(A_0)\psi = 0. \tag{53}$$

The perturbation is resolved into independent normal modes by writing

$$\psi = \delta A(t) e^{i\tilde{k}x} + \text{cc} \tag{54}$$

with $\tilde{k}$ real and positive. Here, and subsequently, cc is used to mean "the complex conjugate of all the preceding terms." Substitution of (54) into (53) gives

$$L\delta A = -\tilde{k}^2 \delta A, \tag{55}$$

where

$$L \equiv \frac{d^2}{dt^2} + V''(A_0).$$

Since $V''(A_0)$ is periodic with period τ, Floquet theory predicts (55) has a solution of the form

$$\delta A = p(t) e^{-i\tilde{\omega}t},$$

where p is periodic with period τ, $\tilde{\omega}$ is a root of

$$2\cos(\tilde{\omega}\tau) = v_1(\tau, \tilde{k}) + v_{2t}(\tau, \tilde{k}) \equiv F(\tilde{k}, B), \qquad -\pi < \mathrm{Re}(\tilde{\omega}\tau) \leq \pi, \tag{56}$$

and $v_t, v_2(t, \tilde{k})$ are two independent solutions of (55) determined by the initial values $v_1 = 1$, $v_{1t} = 0$, $v_2 = 0$, $v_{2t} = 1$. The relation (56) is an implicit quasi-dispersion relation for the small perturbations and may be solved numerically to obtain $\tilde{\omega} = \tilde{\omega}(\tilde{k}, B)$. A useful pictorial representation of the stability properties of the carrier is given by the transition curves ($|F| = 2$) that divide $\tilde{k} - B$ space into stable ($|F| < 2$, $\tilde{\omega}$ real) and unstable ($|F| > 2$, $\tilde{\omega}$ complex) regimes.

Analytical results may be obtained for the stability of the carrier to long wavelength (small $\tilde{k}$) perturbations by expanding F as a Maclaurin series in $\tilde{k}^2$. With $\tilde{k} = 0$, (55) has solutions A_{0B} and A_{0t}. Hence, with $A_0 = A_+$ at $t = 0$, we have $v_1(t, 0) = A_{0B}(t)/A_{0B}(0)$ and $v_2(t, 0) = A_{0t}(t)/A_{0tt}(0)$, so that $v_1(\tau, 0) = 1$, $v_2(\tau, 0) = 0$, $v_{1t}(\tau, 0) = \tau' A_{0tt}^2(0)$, and $v_{2t}(\tau, 0) = 1$. With these results it is found that

$$F(\tilde{k}, B) = 2 + R\tilde{k}^2 + O(\tilde{k}^4), \tag{57}$$

where

$$R = v_{1t}(\tau, 0) \int_0^\tau v_2^2(t, 0)\, dt = \tau' \tau \langle A_{0t}^2 \rangle.$$

For $\tilde{k} \neq 0$, (56) yields two distinct values of $\tilde{\omega}$. If $F > 2$ these will be imaginary and imply linear instability. From (57) it is seen that, for small $\tilde{k}$, the condition for this is $\tau' > 0$ or equivalently $\omega' < 0$. As $k = 0$ corresponds to $\tilde{\omega} = 0$, and in view of (55) and (56), we expand $\tilde{\omega}$ in odd powers of $\tilde{k}$:

$$\tilde{\omega} = \omega_1 \tilde{k} + \omega_3 \tilde{k}^3 + \cdots . \tag{58}$$

On substituting (58) into (56), with F given by (57), and equating coefficients of $\tilde{k}^2$ we obtain the long wavelength limit of (56), namely

$$\omega_1^2 = -\frac{\tau'}{\tau} \langle A_{0t}^2 \rangle. \tag{59}$$

It has been shown by Parkes (1991) that the long wavelength limit of (58) may be obtained by the two standard techniques for nonlinear stability analysis in this regime due to Infeld and Rowlands (1979) and Whitham's averaged Lagrangian technique (Whitham, 1974).

For the particular example of (52)

$$c_1 \frac{\partial^2 A}{\partial T_1^2} - c_2 \frac{\partial^2 A}{\partial Z_1^2} - GA - \bar{N} A^3 = 0, \tag{60}$$

which corresponds to (50) for $c_1 = 1$, $c_2 = 0$, and to (51) for $c_1 = 0$, $c_2 = -1$. Rescaling (60) as

$$Z = |G|^{1/2} Z_1, \qquad T = |G|^{1/2} T_1,$$
$$A = \left(|\bar{N}|/|G|\right)^{1/2} \phi, \qquad g = G/|G|,$$
$$n = \bar{N}/|\bar{N}|,$$

reduces (60) to

$$\phi_{TT} - \phi_{ZZ} - g\phi - n\phi^3 = 0. \tag{61}$$

The corresponding pseudo-potential is (Weissman, 1979)

$$V(\phi) = -\frac{g}{2}\phi^2 - \frac{n}{4}\phi^4. \tag{62}$$

There are four cases to consider, namely (a) $g = -1$, $n = -1$; (b) $g = +1$, $n = -1$; (c) $g = -1$, $n = +1$; (d) $g = +1$, $n = +1$. Sketches of $V(\phi_0)$ for these four cases are given in Weissman (1979) and Murakami (1986). The former also gives phase plane trajectories for the four cases. Note that the phase plane trajectories associated with cases (a) and (c) are those of the undamped, unforced Duffing's equation. There are periodic solutions (the carrier waves) for cases (a), (b), and (c) only. The carriers that satisfy $\phi_0 = \phi_+$ at $t = 0$ with $\phi_+ > 0$, together with some associated quantities, are given below; cn, dn, and sn are elliptic functions and $E(m)$ and $K(m)$ are complete elliptic integrals (Abramowitz and Stegun, 1964)

(a) $\phi_0 = \phi_+ \operatorname{cn}(\alpha t|m), \qquad B > 0$

$$\phi_+^2 = -1 + (1 + 4B)^{1/2}, \qquad \phi_- = -\phi_+$$

$$\tau = 4K/\alpha, \qquad \alpha^2 = (1 + 4B)^{1/2},$$

$$m = \left[-1 + (1 + 4B)^{1/2}\right] / \left[2(1 + 4B)^{1/2}\right]$$

$$\langle \phi_0^2 \rangle = \frac{\phi_+^2}{mK}\left[E - (1 - m)K\right],$$

$$\langle \phi_{0t}^2 \rangle = \frac{\alpha^2 \phi_+^2}{3mK}\left[(1 - m)K - (1 - 2m)E\right].$$

(b) $\phi_0 = \phi_+ \operatorname{cn}(\alpha t|m)$, $\quad B > 0$ (trajectories outside the separatrix)

$\phi_+^2 = 1 + (1 + 4B)^{1/2}, \qquad \phi_- = -\phi_+$

$\tau = 4K/\alpha, \qquad \alpha^2 = (1 + 4B)^{1/2},$

$m = \left[1 + (1 + 4B)^{1/2}\right] / \left[2(1 + 4B)^{1/2}\right]$

$\langle \phi_0^2 \rangle$ and $\langle \phi_{0t}^2 \rangle$ as for case (a).

(c) $\phi_0 = \phi_+ \operatorname{dn}(\alpha t|m)$,

$-1/4 < B < 0$ (trajectories inside the separatrix)

$\phi_+^2 = 1 + (1 + 4B)^{1/2}, \qquad \phi_-^2 = 1 - (1 + 4B)^{1/2}\ (\phi_- > 0)$

$\tau = 2K/\alpha, \qquad \alpha^2 = \left[1 + (1 + 4B)^{1/2}\right]/2,$

$m = \left[2(1 + 4B)^{1/2}\right] / \left[1 + (1 + 4B)^{1/2}\right]$

$$\langle \phi_0^2 \rangle = \phi_+^2 \frac{E}{K}, \qquad \langle \phi_{0t}^2 \rangle = \frac{\alpha^2 \phi_+^2}{3K}\left[(2 - m)E - 2(1 - m)K\right].$$

(d) $\phi_0 = \phi_+ \operatorname{sn}(\alpha t + K|m)$, $\quad 0 < B < 1/4$

$\tau = 4K/\alpha, \qquad \alpha^2 = \left[1 + (1 - 4B)^{1/2}\right]/2,$

$m = \left[1 - (1 - 4B)^{1/2}\right] / \left[1 + (1 - 4B)^{1/2}\right]$

$$\langle \phi_0^2 \rangle = \frac{\phi_+^2}{mK}\left[K - E\right], \quad \langle \phi_{0t}^2 \rangle = \frac{\alpha^2 \phi_+^2}{3mK}\left[(1 + m)E - (1 - m)K\right].$$

Here, B is the constant of integration in the equation

$$\frac{c_1}{2} A_{0T_1}^2 + V(A_0) = B \tag{63}$$

obtained using uniform solution $A = A_0(T_1)$ in (52). Murakami (1986) presents transition curves in $\tilde{k} - B$ space for cases (a), (b), and (c). He concluded that cases (a) and (b) are always stable while case (c) is unstable. Parkes (1991) discovered some errors in his paper. In particular, case (a) may sometimes be unstable, while case (c) of the "nonlinear" or "subcritical" instability has a second unstable regime in addition to that found by Murakami. Case (d) possesses no bounded solutions since both the linear and nonlinear terms g and n are destabilizing.

Nonlinear envelope solutions (of permanent form) of (60) are also considered by Weissman (Section 5) for the case (b) corresponding to the most physically interesting case, where the linear term g is destabilizing while the nonlinear n term is stabilizing, so that the linear instability is equilibrated by nonlinear effects. In this case, the amplitude A of the linear fields of (20)–(23) could evolve to permanent waveforms with the forms on the *sn* and *dn* elliptic functions given there under the slow nonlinear evolution on the T_1 and Z_1 scales. These solutions are analogous to final permanent wave solutions considered by Newell (1974) for the Ginzburg–Landau (Newell–Whitehead) canonical nonlinear evolution equation in the alternate setting of dissipative instabilities. Such permanent waveforms may correspond to trains of vortices for the present supersonic instability. The different evolutions for $x > 0$, $x < 0$ seen in (50) and (51) would imply two families of such vortices on the two sides of the original discontinuity at $x = 0$. Similar behavior would exist for most regimes of case (a) where both G and N (the linear and nonlinear terms) are stabilizing, save for the narrow unstable wedge in the $(\tilde{k}, B)$ plane found by Parkes. For subsonic two-dimensional shear layers, there is detailed experimental (Brown and Roshko, 1974; Winant and Browand, 1974), and simulation (Aref and Siggia, 1981) information on the nonlinear evolution giving rise to an ensemble of plus and minus vortices (somewhat like the von Karman vortex street). There are less conclusive indications of similar ensembles of vortices in simulations for the supersonic case (Norman *et al.*, 1982). Subsequent investigations will seek to correlate nonlinear analyses of the supersonic instability for more realistic spatially nonuniform equilibria relevant to space and astrophysical configurations to the results of detailed simulations.

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