JOURNAL OF DIFFERENTIAL EQUATIONS 8, 431–447 (1970)

Singular Perturbations of a Boundary Value Problem for a System of Nonlinear Differential Equations*

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Received June 19, 1969

1. INTRODUCTION

Consider the nonlinear boundary value problem

$$\begin{cases} \dfrac{dx}{dt} = f(x, y, t, \epsilon) \equiv f_1(x, t, \epsilon) + f_2(x, t, \epsilon)\, y \\ \epsilon \dfrac{dy}{dt} = g(x, y, t, \epsilon) \equiv g_1(x, t, \epsilon) + g_2(x, t, \epsilon)\, y \end{cases} \tag{1.1}$$

for $0 \leqslant t \leqslant 1$ where

$$a_1(\epsilon)\, x(0; \epsilon) + a_2(\epsilon)\, y(0; \epsilon) = \alpha(\epsilon) \tag{1.2a}$$

and

$$b_1(\epsilon)\, x(1; \epsilon) + b_2(\epsilon)\, y(1; \epsilon) = \beta(\epsilon). \tag{1.2b}$$

Here, x and y scalars, ϵ is a small positive parameter, the f_i and g_i are infinitely differentiable with respect to x and t, and the f_i, g_i, a_i, b_i, α and β all have asymptotic power series expansions valid as $\epsilon \to 0$.

Our object is to prove the existence of solutions of the boundary value problem (1.1)–(1.2) when the "reduced" system ((1.1) with $\epsilon = 0$) has a solution that satisfies one of the limiting boundary conditions and to determine asymptotic expansions for these solutions which are valid throughout the interval $0 \leqslant t \leqslant 1$ as $\epsilon \to 0$. Under appropriate hypotheses, we will show that the solutions converge away from $t = 0$ to the solution of the "reduced boundary value problem"

$$\frac{dx_0}{dt} = f_1(x_0, t, 0) + f_2(x_0, t, 0)\, y_0 \tag{1.3a}$$

$$0 = g_1(x_0, t, 0) + g_2(x_0, t, 0)\, y_0 \tag{1.3b}$$

* This research was supported by the Air Force Office of Scientific Research under Grant AFOSR-537-67. Reproduction in whole or part is permitted for any purpose of the U.S. Government.

where

$$b_1(0)\, x_0(1) + b_2(0)\, y_0(1) = \beta(0). \tag{1.4}$$

Nonuniform convergence (or "boundary layer" behavior) will then generally occur at $t = 0$.

We shall assume that

$$g_2(x, t, 0) \leqslant -\kappa < 0 \tag{H1}$$

for all x and for $0 \leqslant t \leqslant 1$. Then,

$$y_0(t) = -g_1(x_0, t, 0)/g_2(x_0, t, 0) \tag{1.5}$$

and $x_0(t)$ satisfies the terminal value problem

$$\begin{aligned} \frac{dx_0}{dt} &= f(x_0, y_0, t, 0) \\ b_1(0)\, x_0(1) + b_2(0)\, y_0(1) &= \beta(0). \end{aligned} \tag{1.6}$$

We shall assume that

$$\text{problem (1.5)–(1.6) has a solution} \quad x_0(t) \quad \text{for } 0 \leqslant t \leqslant 1 \tag{H2}$$

such that

$$b_1(0) + b_2(0)\, g_x(x_0(1), y_0(1), 1, 0)/g_2(x_0(1), 1, 0) \neq 0. \tag{H3}$$

Similar two-point boundary value problems were discussed by Harris [2] and Macki [3]. Both authors were forced by their construction procedures to limit the size of the boundary layer jump (i.e., the difference between the limiting value $\alpha(0)$ prescribed by (1.2a) and the limiting value

$$a_1(0)\, x_0(0) + b_1(0)\, y_0(0)$$

attained by the solution of the reduced problem (1.5)–(1.6)). However, an analogous two-point problem for second-order scalar equations was solved in O'Malley [4] without such a limitation using a simplified version of the Visik and Lyusternik approach (see Visik and Lyusternik [6], Vasil'eva [5], and Wasow [7]). Here, we have been able to construct an asymptotic solution for the problem (1.1)–(1.2) and to prove its asymptotic correctness. It should be noted that hypothesis (H1) need only hold for x in a neighborhood of $x_0(t)$, but to keep the presentation relatively simple we shall not alter (H1).

2. Preliminary Transformations

In order to obtain an asymptotic solution $(X(t; \epsilon), Y(t; \epsilon))$ of the system (1.1) which satisfies the boundary condition (1.2b) prescribed at $t = 1$, we formally set

$$X(t\,;\,\epsilon) = \sum_{k=0}^{\infty} x_k(t)\,\epsilon^k$$
$$Y(t\,;\,\epsilon) = \sum_{k=0}^{\infty} y_k(t)\,\epsilon^k. \tag{2.1}$$

Then,

$$f(X(t\,;\,\epsilon), Y(t\,;\,\epsilon), t, \epsilon) = \sum_{k=0}^{\infty} f_k(t)\,\epsilon^k$$
$$g(X(t\,;\,\epsilon), Y(t\,;\,\epsilon), t, \epsilon) = \sum_{k=0}^{\infty} g_k(t)\,\epsilon^k \tag{2.2}$$

where

$$f_0(t) = f(x_0(t), y_0(t), t, 0)$$
$$g_0(t) = g(x_0(t), y_0(t), t, 0) \tag{2.3}$$

and for $k \geqslant 1$

$$f_k(t) = f_x(x_0(t), y_0(t), t, 0)\,x_k(t) + f_2(x_0(t), t, 0)\,y_k(t) + p_{k-1}(t)$$
$$g_k(t) = g_x(x_0(t), y_0(t), t, 0)\,x_k(t) + g_2(x_0(t), t, 0)\,y_k(t) + q_{k-1}(t) \tag{2.4}$$

where p_{k-1} and q_{k-1} are determined in the obvious manner in terms of those y_j and x_j with $j < k$. Substituting into (1.1) and formally equating coefficients, we ask that

$$\frac{dx_0}{dt} = f(x_0\,, y_0\,, t, 0)$$
$$0 = g(x_0\,, y_0\,, t, 0) \tag{2.5}$$

and for $k \geqslant 1$

$$\frac{dx_k}{dt} = f_k(t) \tag{2.6a}$$

$$g_k(t) = \frac{dy_{k-1}}{dt}. \tag{2.6b}$$

Likewise, from the boundary condition (1.2b), we ask that

$$b_1(0)\, x_0(1) + b_2(0)\, y_0(1) = \beta(0) \tag{2.7}$$

and for $k \geqslant 1$

$$b_1(0)\, x_k(1) + b_2(0)\, y_k(1) = \tilde{\beta}_{k-1} \tag{2.8}$$

where $\tilde{\beta}_{k-1}$ becomes known successively in terms of the coefficients of $\beta(\epsilon)$, $b_1(\epsilon)$, and $b_2(\epsilon)$ and those $x_j(1)$ and $y_j(1)$ for $j < k$.

Thus, we take $x_0(t)$ to be a solution of the terminal value problem (1.6) and

$$y_0(t) = -g_1(x_0(t), t, 0)/g_2(x_0(t), t, 0).$$

We then proceed to obtain higher order coefficients x_k and y_k as the solutions of linear terminal value problems. First, note that (2.6b) can be rewritten as

$$y_k(t) = A(t)\, x_k(t) + B_{k-1}(t) \tag{2.9}$$

where

$$A(t) = -g_x(x_0(t), y_0(t), t, 0)/g_2(x_0(t), t, 0)$$

and

$$B_{k-1}(t) = \left[\frac{dy_{k-1}}{dt} - q_{k-1}(t)\right] \Big/ g_2(x_0(t), t, 0).$$

Thus, (2.6a) implies

$$\frac{dx_k}{dt} = M(t)\, x_k(t) + N_{k-1}(t) \tag{2.10}$$

where

$$M(t) = f_x(x_0(t), y_0(t), t, 0) + f_2(x_0(t), t, 0)\, A(t)$$

and

$$N_{k-1}(t) = f_2(x_0(t), t, 0)\, B_{k-1}(t) + p_{k-1}(t).$$

Lastly, the boundary condition (2.8) yields the terminal values $x_k(1)$ since

$$b_1(0) - b_2(0)\, A(1) \neq 0$$

by hypothesis (H3). Thus, the functions $x_k(t), y_k(t)$ may be determined successively and we have:

LEMMA. *Under hypotheses* (H1)–(H3), *a solution* $(X(t;\epsilon), Y(t;\epsilon))$ *of the system* (1.1) *exists which satisfies the boundary condition* (1.2b) *and has the asymptotic expansion*

$$X(t\,;\epsilon) \sim \sum_{k\geqslant 0} x_k(t)\,\epsilon^k$$

$$Y(t\,;\epsilon) \sim \sum_{k\geqslant 0} y_k(t)\,\epsilon^k$$

as $\epsilon \to 0$ *throughout the interval* $0 \leqslant t \leqslant 1$.

Proof. This result is obtained by standard arguments. For analogous proofs, see Harris [2], Vasil'eva [5], and Wasow [7].

Now, considering X and Y as known functions,

$$(\xi, \eta) = (x - X, y - Y) \tag{2.11}$$

will solve the boundary value problem

$$\begin{aligned} \frac{d\xi}{dt} &= [f_1(X+\xi, t, \epsilon) - f_1(X, t, \epsilon)] + Y(t) \\ &\quad \times [f_2(X+\xi, t, \epsilon) - f_2(X, t, \epsilon)] + \eta f_2(X+\xi, t, \epsilon) \\ \epsilon\frac{d\eta}{dt} &= (g_1(X+\xi, t, \epsilon) - g_1(X, t, \epsilon)) + Y(t) \\ &\quad \times (g_2(X+\xi, t, \epsilon) - g_2(X, t, \epsilon)) + \eta g_2(X+\xi, t, \epsilon) \end{aligned} \tag{2.12}$$

on $0 \leqslant t \leqslant 1$ with

$$\begin{aligned} &a_1(\epsilon)\,\xi(0\,;\epsilon) + a_2(\epsilon)\,\eta(0\,;\epsilon) = \alpha(\epsilon) - a_1(\epsilon)\,X(0\,;\epsilon) - a_2(\epsilon)\,Y(0\,;\epsilon) \\ &b_1(\epsilon)\,\xi(1\,;\epsilon) + b_2(\epsilon)\,\eta(1\,;\epsilon) = 0. \end{aligned} \tag{2.13}$$

Thus, we may restrict further study to the problem

$$\begin{aligned} \frac{d\xi}{dt} &= \xi F_1(\xi, t, \epsilon) + \eta F_2(\xi, t, \epsilon) \\ \epsilon\frac{d\eta}{dt} &= \xi G_1(\xi, t, \epsilon) + \eta G_2(\xi, t, \epsilon) \end{aligned} \tag{2.14}$$

for $0 \leqslant t \leqslant 1$ with

$$a_1(\epsilon)\,\xi(0;\epsilon) + a_2(\epsilon)\,\eta(0;\epsilon) = \gamma(\epsilon) \tag{2.15a}$$

$$b_1(\epsilon)\,\xi(1;\epsilon) + b_2(\epsilon)\,\eta(1;\epsilon) = 0. \tag{2.15b}$$

Here, the F_i and G_i are infinitely differentiable with respect to ξ and t, and the F_i, G_i and γ have asymptotic power series expansions valid as $\epsilon \to 0$. Moreover,

$$G_2(\xi, t, 0) \leqslant -\kappa < 0 \tag{2.16}$$

for all ξ and $0 \leqslant t \leqslant 1$.

Under appropriate additional hypotheses, we might expect the solution to be asymptotically zero as $\epsilon \to 0$ on any interval $[\delta, 1]$, $\delta > 0$, since $(\xi, \eta) = (0, 0)$ satisfies (2.14) and the boundary condition (2.15b) prescribed at $t = 1$. Then, the solution would exhibit boundary layer behavior at $x = 0$. In order to investigate the solution in this boundary layer, we "blow up" the neighborhood of $x = 0$ (for small values of ϵ) by introducing the "stretched variable"

$$\tau = t/\epsilon. \tag{2.17}$$

Note that τ ranges over $[0, \infty)$ as t ranges over $[0, 1]$. In terms of τ, then, the system (2.17) becomes

$$\begin{aligned} \frac{d\xi}{d\tau} &= \epsilon\xi F_1(\xi, \epsilon\tau, \epsilon) + \epsilon\eta F_2(\xi, \epsilon\tau, \epsilon) \\ \frac{d\eta}{d\tau} &= \xi G_1(\xi, \epsilon\tau, \epsilon) + \eta G_2(\xi, \epsilon\tau, \epsilon) \end{aligned} \tag{2.18}$$

and (since $\tau = 0$ when $t = 0$), (2.15a) becomes

$$a_1(\epsilon)\, \xi(0; \epsilon) + a_2(\epsilon)\, \eta(0; \epsilon) = \gamma(\epsilon). \tag{2.19}$$

We seek solutions (ξ, η) of (2.18)–(2.19) which are asymptotically zero as $\tau \to \infty$. The asymptotic solutions, however, are quite different in the two cases (a). $a_2(0) \neq 0$ and (b) $a_2(0) = 0$. Thus, they will be treated separately.

3. Case (a): $a_2(0) \neq 0$

Motivated by the solution of the linear problem, let us seek solutions of (2.18)–(2.19) of the form

$$\begin{aligned} \xi(\tau\,;\epsilon) &= \sum_{k=1}^{\infty} \xi_k(\tau)\, \epsilon^k \\ \eta(\tau\,;\epsilon) &= \sum_{k=0}^{\infty} \eta_k(\tau)\, \epsilon^k. \end{aligned} \tag{3.1}$$

Substituting into (2.1) and equating lowest order coefficients, we ask that

$$\begin{aligned}\frac{d\xi_1}{d\tau} &= \eta_0 F_2(0,0,0)\\ \frac{d\eta_0}{d\tau} &= \eta_0 G_2(0,0,0).\end{aligned} \tag{3.2}$$

Setting $G_2(0, 0, 0) = -c < 0$, then,

$$\eta_0(\tau) = \eta_0(0)\, e^{-c\tau} \qquad \text{and} \qquad \xi_1(\tau) = \frac{-\eta_0(0)}{c} F_2(0,0,0)\, e^{-c\tau} \tag{3.3}$$

so both ξ_1 and η_0 decay exponentially to zero as $\tau \to \infty$. Moreover, equating lowest order coefficients in the boundary condition, we find that

$$\eta_0(0) = \gamma(0)/a_2(0).$$

From the next highest coefficients, we have

$$\begin{aligned}\frac{d\xi_2}{d\tau} = \eta_1 F_2(0,0,0) &+ \xi_1 F_1(0,0,0)\\ &+ \eta_0[\xi_1 F_{2\xi}(0,0,0) + \tau F_{2t}(0,0,0) + F_{2\epsilon}(0,0,0)]\end{aligned}$$

and

$$\begin{aligned}\frac{d\eta_1}{d\tau} = -c\eta_1 &+ \xi_1 G_1(0,0,0)\\ &+ \eta_0[\xi_1 G_{2\xi}(0,0,0) + \tau G_{2t}(0,0,0) + G_{2\epsilon}(0,0,0)]\end{aligned}$$

and

$$\eta_1(0) = (\gamma'(0) - a_1(0)\,\xi_1(0) - a_2'(0)\,\eta_0(0))/a_2(0). \tag{3.4}$$

In general, for $k \geqslant 0$, we have

$$\begin{aligned}\frac{d\xi_{k+1}}{d\tau} &= \eta_k F_2(0,0,0) + P_{k-1}(\tau)\\ \frac{d\eta_k}{d\tau} &= -c\eta_k + Q_{k-1}(\tau)\end{aligned} \tag{3.5}$$

where P_{k-1} and Q_{k-1} are functions which are known successively and decay exponentially to zero as $\tau \to \infty$. Moreover,

$$\eta_k(0) = \tilde{\gamma}_k/a_2(0) \tag{3.6}$$

where $\tilde{\gamma}_k$ is known successively. Thus,

$$\eta_k(\tau) = \frac{\tilde{\gamma}_k}{a_2(0)} e^{-c\tau} + \int_0^\tau Q_{k-1}(s)\, e^{-c(\tau-s)}\, ds$$

and

$$\xi_{k+1}(\tau) = -\frac{\tilde{\gamma}_k F_2(0, 0, 0)}{ca_2(0)} e^{-c\tau} - F_2(0, 0, 0) \int_\tau^\infty \times \left[\int_0^r Q_{k-1}(s)\, e^{-c(r-s)}\, ds\right] P_{k-1}(r)\, dr.$$

Thus, η_k and ξ_{k+1} both decay exponentially to zero as $\tau \to \infty$. Summarizing our results, we have:

THEOREM 1. *Consider the boundary value problem* (1.1)–(1.2) *under the hypotheses* (H1)–(H3). *Assume further that* $a_2(0) \neq 0$. *Then, for* ϵ *sufficiently small, the boundary-value problem has a solution* $[x(t, \epsilon), y(t, \epsilon)]$ *such that for each integer* $N \geqslant 1$

$$x(t, \epsilon) = x_0(t) + \sum_{k=1}^{N} (x_k(t) + \xi_k(t/\epsilon))\, \epsilon^k + \epsilon^{N+1} r_N(t, \epsilon)$$

and

$$y(t, \epsilon) = \sum_{k=0}^{N} \left[y_k(t) + \eta_k\left(\frac{t}{\epsilon}\right)\right] \epsilon^k + \epsilon^{N+1} s_N(t, \epsilon).$$

where $r_N(t, \epsilon)$ *and* $s_N(t, \epsilon)$ *are both bounded for all* t *with* $0 \leqslant t \leqslant 1$.

Proof. The proof follows more simply, but in the same manner, as the proof of Theorem 2 given below, and shall be omitted.

The reader should note that the solutions have the asymptotic series expansions

$$x(t, \epsilon) \sim x_0(t) + \sum_{k=1}^{\infty} (x_k(t) + \xi_k(t/\epsilon))\, \epsilon^k$$

and

$$y(t, \epsilon) \sim \sum_{k=0}^{\infty} (y_k(t) + \eta_k(t/\epsilon))\, \epsilon^k.$$

Since the coefficients depend on ϵ, these expansions are asymptotic in the general sense of van der Corput (cf., O'Malley [4]).

4. Case (b): $a_2(0) = 0$

Let us suppose that

$$a_2(\epsilon) = \epsilon \tilde{a}_2(\epsilon) \tag{4.1}$$

where $\tilde{a}_2(\epsilon)$ has an asymptotic power series expansion as $\epsilon \to 0$. For $\gamma(0) \neq 0$, then, we cannot satisfy the condition (2.19) by assuming a solution (ξ, η) of the form (3.1). Instead, we introduce

$$\xi(\tau\,;\epsilon) = \sum_{k=0}^{\infty} \xi_k(\tau)\,\epsilon^k \qquad \text{and} \qquad \eta(\tau\,;\epsilon) = \sum_{k=-1}^{\infty} \eta_k(\tau)\,\epsilon^k. \tag{4.2}$$

(This ansatz, too, can be motivated by considering the linear problem.) Substituting into the differential equation and formally equating coefficients of powers of ϵ, we ask that

$$\frac{d\xi_0}{d\tau} = \eta_{-1} F_2(\xi_0\,, 0, 0) \tag{4.3a}$$

$$\frac{d\eta_{-1}}{d\tau} = \eta_{-1} G_2(\xi_0\,, 0, 0), \tag{4.3b}$$

$$\begin{aligned} \frac{d\xi_1}{d\tau} &= \xi_1 F_{2\xi}(\xi_0\,, 0, 0)\,\eta_{-1} + \eta_0 F_2(\xi_0\,, 0, 0) \\ &\quad + [\tau F_{2t}(\xi_0\,, 0, 0) + F_{2\epsilon}(\xi_0\,, 0, 0)]\,\eta_{-1} + F_1(\xi_0\,, 0, 0)\,\xi_0 \\ \frac{d\eta_0}{d\tau} &= \xi_1 G_{2\xi}(\xi_0\,, 0, 0)\,\eta_{-1} + \eta_0 G_2(\xi_0\,, 0, 0) \\ &\quad + [\tau G_{2t}(\xi_0\,, 0, 0) + G_{2\epsilon}(\xi_0\,, 0, 0)]\,\eta_{-1} + G_1(\xi_0\,, 0, 0)\,\xi_0 \end{aligned}$$

and, generally, for $k \geqslant 1$,

$$\frac{d\xi_k}{d\tau} = \xi_k F_{2\xi}(\xi_0\,, 0, 0)\,\eta_{-1} + \eta_{k-1} F_2(\xi_0\,, 0, 0) + P_{k-1}(\tau) \tag{4.4a}$$

and

$$\frac{d\eta_{k-1}}{d\tau} = \xi_k G_{2\xi}(\xi_0\,, 0, 0)\,\eta_{-1} + \eta_{k-1} G_2(\xi_0\,, 0, 0) + Q_{k-1}(\tau) \tag{4.4b}$$

where P_{k-1} and Q_{k-1} are known successively.

Now let us assume that for some $\kappa > 0$,

$$G_2(\xi, 0, 0) \leqslant -\kappa \tag{4.5}$$

and

$$|F_2(\xi, 0, 0)| \geqslant \kappa \tag{4.6}$$

for all values ξ. Note that (H1) implies (4.5). In addition, (4.6) requires that

$$|f_2(x, 0, 0)| \geqslant \kappa$$

for all values of x. Then, (4.3) implies that

$$\frac{d\eta_{-1}}{d\tau} = \frac{G_2(\xi_0, 0, 0)}{F_2(\xi_0, 0, 0)} \frac{d\xi_0}{d\tau}.$$

Integrating, then, from τ to ∞ and asking that both ξ_0 and η_{-1} tend to zero as $\tau \to \infty$, we have

$$\eta_{-1}(\tau) = \int_0^{\xi_0(\tau)} \frac{G_2(r, 0, 0)}{F_2(r, 0, 0)} dr. \tag{4.7}$$

Thus, ξ_0 must satisfy the nonlinear equation

$$\frac{d\xi_0}{d\tau} = F_2(\xi_0, 0, 0) \int_0^{\xi_0} \frac{G_2(r, 0, 0)}{F_2(r, 0, 0)} dr \tag{4.8}$$

for $0 \leqslant \tau < \infty$.

Rearranging (4.4), then, for each $k \geqslant 1$, we have

$$\frac{d\eta_{k-1}}{d\tau} = \frac{d}{d\tau}\left(\frac{\xi_k G_2(\xi_0, 0, 0)}{F_2(\xi_0, 0, 0)}\right) + \tilde{Q}_{k-1}(\tau)$$

where

$$\tilde{Q}_{k-1} = Q_{k-1} - \frac{G_2(\xi_0, 0, 0)}{F_2(\xi_0, 0, 0)} P_{k-1}.$$

Asking that both ξ_k and η_{k-1} tend to zero as $\tau \to \infty$, we have

$$\eta_{k-1}(\tau) = \frac{G_2(\xi_0, 0, 0)}{F_2(\xi_0, 0, 0)} \xi_k - \int_\tau^\infty \tilde{Q}_{k-1}(s)\, ds. \tag{4.9}$$

Using (4.3), (4.4a), and (4.9), then,

$$\frac{d}{d\tau}\left(\frac{\xi_k}{F_2(\xi_0, 0, 0)}\right) = \frac{\xi_k G_2(\xi_0, 0, 0)}{F_2(\xi_0, 0, 0)} + \tilde{P}_{k-1}(\tau)$$

where

$$\tilde{P}_{k-1} = \frac{P_{k-1}}{F_2(\xi_0, 0, 0)} - \int_\tau^\infty \tilde{Q}_{k-1}(s)\, ds,$$

so integration implies

$$\xi_k(\tau) = F_2(\xi_0(\tau), 0, 0) \left\{ \frac{\xi_k(0) \exp(\int_0^\tau G_2(\xi_0, 0, 0)\, ds)}{F_2(\xi_0(0), 0, 0)} + \int_0^\tau \exp\left(\int_r^\tau G_2(\xi_0, 0, 0)\, ds\right) \tilde{P}_{k-1}(r)\, dr \right\}. \tag{4.10}$$

Thus, all coefficients in the expansion (4.2) may be calculated directly in terms of $\xi_0(\tau)$ and the initial conditions $\xi_k(0)$.

Using the boundary condition (2.19) and (4.1), we have

$$a_1(0)\, \xi_0(0) + \tilde{a}_2(0)\, \eta_{-1}(0) = \gamma(0)$$

and for each integer $k \geqslant 1$

$$a_1(0)\, \xi_k(0) + \tilde{a}_2(0)\, \eta_{k-1}(0) = \tilde{\gamma}_{k-1}$$

where the $\tilde{\gamma}_{k-1}$ are known successively. Equation (4.7), then, implies that $\xi_0(0)$ must satisfy the nonlinear equation

$$a_1(0)\, \xi_0(0) + \tilde{a}_2(0) \int_0^{\xi_0(0)} \frac{G_2(r, 0, 0)}{F_2(r, 0, 0)}\, dr = \gamma(0) \tag{4.11}$$

while (4.9) implies that successive $\xi_k(0)$ must satisfy

$$\left[a_1(0) + \tilde{a}_2(0) \frac{G_2(\xi_0(0), 0, 0)}{F_2(\xi_0(0), 0, 0)}\right] \xi_k(0) = \tilde{\gamma}_k + \tilde{a}_2(0) \int_0^\infty \tilde{Q}_{k-1}(s)\, ds. \tag{4.12}$$

Now suppose that (4.11) has a solution $\xi_0(0)$ such that

$$a_1(0) + \tilde{a}_2(0) \frac{G_2(\xi_0(0), 0, 0)}{F_2(\xi_0(0), 0, 0)} \neq 0. \tag{4.13}$$

Then, we can determine the $\xi_k(0)$'s successively.

Now observe that all functions $\xi_j(\tau)$, $\eta_{j-1}(\tau)$ satisfying (4.7)–(4.10) decay exponentially (in absolute value) on the interval $0 \leqslant \tau < \infty$. For every finite value $\xi_0(0)$, $|G_2(r, 0, 0)|$ will be bounded for $|r| < |\xi_0(0)|$. Thus, (4.6)–(4.7) imply that $|\eta_{-1}(0)|$ is bounded. Moreover, since

$$\eta_{-1}(\tau) = \eta_{-1}(0) \exp\left[\int_0^\tau G_2(\xi_0(s), 0, 0)\, ds\right],$$

(4.5) implies that $\eta_{-1}(\tau)$ decays exponentially to zero. Likewise, by (4.8) and sign conditions (4.5)–(4.6), we see that $|\xi_0(\tau)|$ decreases monotonically to zero. Therefore, $F_2(\xi_0, 0, 0)$ remains bounded and (4.3a) implies that $d\xi_0/d\tau$

decays exponentially to zero as $\tau \to \infty$. Thus, so does $\xi_0(\tau) = \int_\tau^\infty (d\xi_0/d\tau)\, d\tau$. The exponential decay of ξ_0 and η_{-1} implies that of P_0, Q_0, $\tilde{P}_0$, and $\tilde{Q}_0$. By (4.9)–(4.10), then, ξ_1 and η_0 also decay exponentially as $\tau \to \infty$. Proceeding by induction, we show that each ξ_j and η_{j-1} for $j \geq 0$ decays exponentially to zero.

Finally, we observe that for any value $\xi_0'(0)$, the solution of the nonlinear equation (4.8) can be obtained on the semi-infinite τ interval by successive approximations (see Erdélyi [1] regarding successive approximations and O'Malley [4] for an analogous problem). Knowing $\xi_0(\tau)$ enables us to calculate all other η_{j-1} and ξ_j successively.

Summarizing, we have:

THEOREM 2. *Consider the boundary value problem* (1.1)–(1.2) *under the hypotheses* (H1)–(H3) *where* $a_2(\epsilon) = \epsilon \tilde{a}_2(\epsilon)$. *In addition, suppose* (H4) *there exists a constant* $\kappa > 0$ *such that*

$$|f_2(x, t, 0)| \geq \kappa$$

for all values of x, *and* (H5) *there exists a finite number* λ *such that*

$$\int_0^\lambda \left[a_1(0) + \tilde{a}_2(0) \frac{g_2(x_0(0) + r, 0, 0)}{f_2(x_0(0) + r, 0, 0)}\right] dr = \alpha(0) - a_1(0)\, x_0(0)$$

and

$$a_1(0) + \tilde{a}_2(0) \frac{g_2(x_0(0) + \lambda, 0, 0)}{f_2(x_0(0) + \lambda, 0, 0)} \neq 0.$$

Then, for ϵ *sufficiently small, the boundary value problem has a solution* $[x(t, \epsilon), y(t, \epsilon)]$ *such that for each integer* $n \geq 0$

$$x(t, \epsilon) = \sum_{k=0}^{n} \left[x_k(t) + \xi_k\left(\frac{t}{\epsilon}\right)\right] \epsilon^k + \epsilon^{n+1} r_n(t, \epsilon)$$

and

$$y(t, \epsilon) = \frac{\eta_{-1}(t/\epsilon)}{\epsilon} + \sum_{k=0}^{n} \left[y_k(t) + \eta_k\left(\frac{t}{\epsilon}\right)\right] \epsilon^k + \epsilon^{n+1} s_n(t, \epsilon)$$

where $r_n(t, \epsilon)$ *and* $s_n(t, \epsilon)$ *are both bounded for all* t *within* $[0, 1]$.

Note:

1. Hypothesis (H4) is satisfied when the scalar equation considered in O'Malley [4] (and elsewhere) is put in system form.

2. Hypothesis (H5) is automatically satisfied for the problems considered

in Harris [2] and Macki [3] since, then, (in case (b)) $a_2(\epsilon) \equiv a_2(0) = 0$ and $a_1(0) \neq 0$. Note that (H5) implies (4.11) and (4.13).

3. Examples can be easily found for which several values λ exist satisfying (H5). Each such determination λ will lead to a different asymptotic solution of the problem (1.1)–(1.2) with $\xi_0(0) = \lambda$. An example is furnished by the problem

$$\frac{dx}{dt} = y$$

$$\epsilon \frac{dy}{dt} = -\frac{1}{2}(1 + 3x^2)y$$

$$x(0, \epsilon) + \epsilon y(0, \epsilon) = 0$$

$$x(1, \epsilon) = 0$$

which satisfies the hypotheses of Theorem 2 for the three values $\lambda = 0$, $\lambda = 1$, and $\lambda = -1$. These values of λ correspond to the three solutions

$$x(t, \epsilon) = \frac{\lambda e^{-t/2\epsilon}}{\sqrt{(1 + \lambda^2) - \lambda^2 e^{-t/\epsilon}}}$$

$$y(t, \epsilon) = \frac{-\lambda(1 + \lambda^2)\, e^{-t/2\epsilon}}{2\epsilon[\sqrt{(1 + \lambda^2) - \lambda^2 e^{-t/\epsilon}}]^3},$$

all of which converge to the trivial solution of the reduced problem for $t > 0$.

5. Proof of Theorem 2

By Lemma 1, it suffices to consider the problem

$$\frac{d\xi}{dt} = F(\xi, \eta, t, \epsilon) \equiv \xi F_1(\xi, t, \epsilon) + \eta F_2(\xi, t, \epsilon)$$
$$\epsilon \frac{d\eta}{dt} = G(\xi, \eta, t, \epsilon) \equiv \xi G_1(\xi, t, \epsilon) + \eta G_2(\xi, t, \epsilon) \tag{2.17}$$

for $0 \leqslant t \leqslant 1$ with

$$a_1(\epsilon)\, \xi(0; \epsilon) + \epsilon \tilde{a}_2(\epsilon)\, \eta(0; \epsilon) = \gamma(\epsilon) \tag{2.15a}$$

and

$$b_1(\epsilon)\, \xi(1; \epsilon) + b_2(\epsilon)\, \eta(1; \epsilon) = 0. \tag{2.15b}$$

Instead, we shall consider an initial value problem for (2.14) where we obtain the initial conditions from the formal expansions (4.2) constructed above. Thus, we define $A(\epsilon)$ and $B(\epsilon)$ to be functions having the asymptotic power series expansions

$$A(\epsilon) \sim \sum_{k=0}^{\infty} \xi_k(0)\, \epsilon^k \tag{5.1}$$

and

$$B(\epsilon) \sim \sum_{k=-1}^{\infty} \eta_k(0)\, \epsilon^{k+1}. \tag{5.2}$$

Such functions $A(\epsilon)$ and $B(\epsilon)$ exist by the Borel–Ritt Theorem (cf., Wasow [7], p. 43). Thus, we shall consider the system (2.14) subject to the initial conditions

$$\begin{aligned} \xi(0; \epsilon) &= A(\epsilon) \\ \eta(0; \epsilon) &= B(\epsilon)/\epsilon. \end{aligned} \tag{5.3}$$

Further, for every integer $n \geqslant 1$, let us define

$$\xi^n(t; \epsilon) = \sum_{k=0}^{n} \xi_k(\tau)\, \epsilon^k \tag{5.4a}$$

and

$$\frac{\eta^n(t; \epsilon)}{\epsilon} = \sum_{k=-1}^{n-1} \eta_k(\tau)\, \epsilon^k \tag{5.4b}$$

where $\tau = t/\epsilon$. We shall show that a solution $(\xi(t; \epsilon), \eta(t; \epsilon))$ of the initial value problem (2.14)–(5.3) exists for ϵ sufficiently small so that

$$\begin{aligned} \xi(t; \epsilon) &= \xi^n(t; \epsilon) + \epsilon^{n+1} R_n(t; \epsilon) \\ \eta(t; \epsilon) &= \frac{\eta^n(t; \epsilon)}{\epsilon} + \epsilon^n S_n(t; \epsilon) \end{aligned} \tag{5.5}$$

where $R_n(t; \epsilon)$ and $S_n(t; \epsilon)$ are bounded throughout $0 \leqslant t \leqslant 1$.

Note that, by construction,

$$a_1(\epsilon)\, \xi^n(0; \epsilon) + \epsilon \tilde{a}_2(\epsilon)\, \frac{\eta^n(0; \epsilon)}{\epsilon} = \gamma(\epsilon) + 0(\epsilon^{n+1}).$$

Moreover, both $\xi^n(t;\epsilon)$ and $\eta^n(t;\epsilon)$ decay exponentially to zero away from $t = 0$, so

$$b_1(\epsilon)\,\xi^n(1;\epsilon) + b_2(\epsilon)\,\frac{\eta^n(1;\epsilon)}{\epsilon} = 0(\epsilon^l)$$

for l arbitrarily large. Thus, it follows that the solution (5.5) of the initial value problem will also asymptotically satisfy the original boundary value problem (1.1)–(1.2).

Recall that the coefficients $\xi_j(\tau)$ and $\eta_j(\tau)$ were determined so that

$$\frac{d\xi^n}{d\tau} = \epsilon F(\xi^n, \eta^n/\epsilon, \epsilon\tau, \epsilon) + \epsilon^{n+1}\tilde{\tilde{P}}_n(\tau, \epsilon)$$

$$\frac{d}{d\tau}(\eta^n/\epsilon) = G(\xi^n, \eta^n/\epsilon, \epsilon\tau, \epsilon) + \epsilon^n\tilde{\tilde{Q}}_n(\tau, \epsilon)$$

where

$$\xi^n(0) = A(\epsilon) - \epsilon^{n+1}\alpha(\epsilon)$$

and

$$\frac{\eta^n(0)}{\epsilon} = \frac{B(\epsilon)}{\epsilon} - \epsilon^n\beta(\epsilon).$$

Here, $\alpha(\epsilon)$ and $\beta(\epsilon)$ are bounded and $\tilde{\tilde{P}}_n$ and $\tilde{\tilde{Q}}_n$ are bounded on every finite τ interval and they decay exponentially to zero as $\tau \to \infty$ (since each ξ_j and η_j does so). In terms of $t = \epsilon\tau$, then,

$$\frac{d\xi^n}{dt} = F(\xi^n, \eta^n/\epsilon, t, \epsilon) + \epsilon^n\tilde{\tilde{P}}_n(t/\epsilon, \epsilon)$$
$$\epsilon\,\frac{d}{dt}(\eta^n/\epsilon) = G(\xi^n, \eta^n/\epsilon, t, \epsilon) + \epsilon^n\tilde{\tilde{Q}}_n(t/\epsilon, \epsilon). \tag{5.6}$$

Thus, (5.5) and (2.14) imply that

$$\epsilon^{n+1}\frac{dR_n}{dt} = F\left(\xi^n + \epsilon^{n+1}R_n, \frac{\eta^n}{\epsilon} + \epsilon^n S_n, t, \epsilon\right) - F(\xi^n, \eta^n/\epsilon, t, \epsilon) + \epsilon^n\tilde{\tilde{P}}_n(t/\epsilon, \epsilon)$$

and

$$\epsilon^{n+1}\frac{dS_n}{dt} = G\left(\xi^n + \epsilon^{n+1}R_n, \frac{\eta^n}{\epsilon} + \epsilon^n S_n, t, \epsilon\right) - G(\xi^n, \eta^n/\epsilon, t, \epsilon) + \epsilon^n\tilde{\tilde{Q}}_n(t/\epsilon, \epsilon) \tag{5.7}$$

where

$$R_n(0;\epsilon) = \alpha(\epsilon) \qquad \text{and} \qquad S_n(0;\epsilon) = \beta(\epsilon). \tag{5.8}$$

Rewriting (5.7), we have

$$\frac{dR_n}{dt} = R_n F_\xi(\xi^n, \eta_n/\epsilon, t, \epsilon) + \frac{S_n}{\epsilon} F_2(\xi^n, t, \epsilon) + \frac{\tilde{\tilde{P}}_n(t/\epsilon, \epsilon)}{\epsilon} + \epsilon^n \tilde{F}_2(R_n, S_n, t, \epsilon)$$

$$\frac{dS_n}{dt} = R_n G_\xi(\xi^n, \eta^n/\epsilon, t, \epsilon) + \frac{S_n}{\epsilon} G_2(\xi^n, t, \epsilon) + \frac{\tilde{\tilde{Q}}_n(t/\epsilon, \epsilon)}{\epsilon} + \epsilon^n \tilde{G}_2(R_n, S_n, t, \epsilon)$$

where the "remainders" $\tilde{F}_2$ and $\tilde{G}_2$ are defined in the obvious manner.

Since $F_2 \neq 0$, using the first equation to obtain an expression for S_n, substituting into the second equation, and integrating, we find that S_n satisfies an equation of the form

$$S_n(t) = \int_{\alpha(\epsilon)}^{R_n(t)} \frac{G_2(\xi^n, u, \epsilon)}{F_2(\xi^n, u, \epsilon)} du + \int_0^t \Big[R_n(u) H(\xi^n, \eta^n/\epsilon, u, \epsilon) + \epsilon^n \tilde{H}(R_n, S_n, u, \xi^n, \epsilon) + \frac{1}{\epsilon} \tilde{\tilde{H}}\left(\frac{u}{\epsilon}, u, \epsilon\right)\Big] du + \beta(\epsilon). \tag{5.9}$$

Thus, the first equation can be rewritten as

$$\frac{dR_n}{dt} + \frac{1}{\epsilon} p(t, R_n, \epsilon) R_n = \left\{R_n F_\xi + \epsilon^n \tilde{F}_2 + \frac{\tilde{\tilde{P}}_n}{\epsilon} + \tilde{\beta}(\epsilon)\right\} + \frac{F_2}{\epsilon} \int_0^t \left\{R_n H + \epsilon^n \tilde{H} + \frac{\tilde{\tilde{H}}}{\epsilon}\right\} du,$$

where

$$p(t, R_n, \epsilon) = -\frac{F_2(\xi^n, t, \epsilon)}{R_n} \int_0^{R_n} \frac{G_2(\xi^n, u, \epsilon)}{F_2(\xi^n, u, \epsilon)} du \tag{5.10}$$

is strictly positive since $G_2 < 0$ and $F_2 \neq 0$. Integrating then, R_n satisfies

$$\begin{aligned} R_n(t) = {} & \alpha(\epsilon) \exp\left(-\frac{1}{\epsilon}\int_0^t p(s, R_n, \epsilon)\, ds\right) \\ & + \int_0^t \exp\left(-\frac{1}{\epsilon}\int_v^t p(s, R_n, \epsilon)\, ds\right)\Big[\Big\{R_n(v) F_\xi\left(\xi^n, \frac{\eta^n}{\epsilon}, v, \epsilon\right) \\ & + \epsilon^n \tilde{F}_2(R_n, S_n, v, \epsilon) + \frac{1}{\epsilon}\tilde{\tilde{P}}_n\left(\frac{v}{\epsilon}, \epsilon\right) + \tilde{\beta}(\epsilon)\Big\} \\ & + \frac{1}{\epsilon} F_2\left(\xi^n, v, \epsilon\right) \int_0^{R_n(v)} \Big\{R_n(u) H\left(\xi^n, \frac{\eta^n}{\epsilon}, u, \epsilon\right) \\ & + \epsilon^n \tilde{H}(R_n, S_n, u, \xi^n, \epsilon) + \frac{1}{\epsilon}\tilde{\tilde{H}}\left(\frac{u}{\epsilon}, u, \epsilon\right)\Big\}\, du\Big]\, dv. \end{aligned} \tag{5.11}$$

The vector Volterra integral equation consisting of (5.9) and (5.11) can be solved by a successive approximations scheme in a straightforward manner. We note that it is critical that p be strictly positive and that $\tilde{\tilde{H}}(u/\epsilon, u, \epsilon) = 0(e^{-\kappa(1-\delta)u/\epsilon})$. Thus, we are able to show the existence of a solution (ξ, η) of the boundary value problem (2.14)–(2.15) with remainders R_n and S_n (defined by (5.5)) which are bounded throughout $0 \leqslant t \leqslant 1$ provided ϵ is sufficient small.

Acknowledgment

The author wishes to acknowledge the encouragement of Professor W. A. Harris, Jr. in undertaking this investigation and the assistance of Mr. William Steinmetz who pointed out an error in an earlier manuscript.

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