Exponential decay of the local energy for the solutions of critical wave equations outside a convex obstacle

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Abstract

In this paper, we prove the exponential decay of local energy for the critical wave equation outside a convex obstacle with localized semilinearity. The proof relies on generalized Strichartz estimates, and microlocal defect measures.

Résumé

Dans cet article, on démontre la décroissance exponentielle de l’énergie locale des solutions de l’équation des ondes critiques à l’extérieur d’un obstacle convexe lorsque la semi linéarité est localisée. La preuve est basée sur les inégalités de Strichartz généralisées, et les mesures de défaut microlocales.

Keywords: Critical wave; Microlocal defect measures; Exponential decay

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1. Introduction and position of the problem

The aim of this article is to study the following nonlinear wave equation,

$$
\begin{align*}
\Box u + \chi(x)u^5 &= 0 \quad \text{on } \mathbb{R} \times \Omega, \\
u &= 0 \quad \text{in } \mathbb{R} \times \partial \Omega, \\
u(0, x) &= u^0(x) \in H_D(\Omega) \quad \text{and} \quad \partial_t u(0, x) = u^1(x) \in L^2(\Omega),
\end{align*}
$$

(1.1)

where $\Omega = \mathbb{R}^3 \setminus O$ and $O$ is a strictly convex compact with smooth boundary $\partial \Omega$, $O \subset B_R$ for some $R > 0$. The function $\chi$ is a positive and of class $C^1$, with compact support such that $\text{supp} \chi \subset B_R$. Here the function $\chi$ is allowed to be equal to 1 near $\partial \Omega$. We denote $H = H_D(\Omega) \times L^2(\Omega)$ the completion of $(C^\infty_0(\Omega))^2$ with respect to the norm

$$
\|\varphi_1, \varphi_2\|_H^2 = \int_\Omega (|\nabla \varphi_1|^2 + |\varphi_2|^2) dx.
$$

Global existence and uniqueness of the solutions to the Cauchy problem (1.1) has been studied in [4]. Consequently, for every initial data $(u^0, u^1)$ in the energy space $H$, system (1.1) admits a unique solution $u$ in the “Shatah–Struwe” class, that is

$$
u \in C(\mathbb{R}, H_D(\Omega)) \cap L^5_{\text{loc}}(\mathbb{R}, L^{10}(\Omega)), \quad \partial_t u \in C(\mathbb{R}, L^2(\Omega)).
$$

The global energy of $u$ at time $t$ is defined by

$$
E(u(t)) = \frac{1}{2} \int_\Omega \left( |\partial_t u(t)|^2 + |\nabla_x u(t)|^2 \right) dx + \frac{1}{6} \int_\Omega \chi(x) |u(t)|^6 dx
$$

(1.2)

which is time independent.

We also define the local energy by

$$
E_{\rho}(u(t)) = \frac{1}{2} \int_{\Omega \cap B_\rho} \left( |\partial_t u(t)|^2 + |\nabla_x u(t)|^2 \right) dx + \frac{1}{6} \int_{\Omega \cap B_\rho} \chi(x) |u(t)|^6 dx
$$

(1.3)

where $B_\rho$ is a ball of radius $\rho$ containing the obstacle $O$.

For every $t \in \mathbb{R}$, we define the wave operator $U(t)$ by

$$
U(t) : H \longrightarrow H
$$

$$(\varphi_1, \varphi_2) \mapsto U(t)(\varphi_1, \varphi_2) = (u(t), \partial_t u(t)),$$

where $u$ is the solution of (1.1) in the “Shatah–Struwe” class with initial data $\varphi = (\varphi_1, \varphi_2)$. $(U(t))_{t \in \mathbb{R}}$ forms a one parameter continuous group on $H$, which we will refer to as the nonlinear wave group.

We first recall the following result due to the author [4] who prove that $u$ is equivalent for the energy norm, as $t \to +\infty$, to a solution of the linear equation:

**Theorem 1.** The nonlinear wave group outside a compact convex obstacle is asymptotically complete with respect to the linear wave group in the same domain. More precisely, with the notations defined above we have

(a) If $u$ is the solution in the “Shatah–Struwe” class of (1.1) then there exists a unique finite energy solution $u_+$ of

$$
\begin{align*}
\Box u_+ &= 0 \quad \text{on } \mathbb{R} \times \Omega, \\
u_+ &= 0 \quad \text{in } \mathbb{R} \times \partial \Omega
\end{align*}
$$

(1.4)
such that $E_c((u_+ - u)(t)) \xrightarrow{t \to +\infty} 0$, where

$$E_c((u_+ - u)(t)) = \frac{1}{2} \int_{\Omega} \left( |\partial_t (u_+ - u)(t)|^2 + |\nabla_x (u_+ - u)(t)|^2 \right) dx.$$

(b) The wave operator defined by

$$\Omega_+: H \longrightarrow H$$

$$(u_{t=0}, \partial_t u_{t=0}) \longmapsto (u_{+t=0}, \partial_t u_{+t=0})$$

is a bijection.

The main result of this paper is to prove that the decay of the local energy of the solutions of (1.1) is of exponential type. More precisely we have the following theorem:

**Theorem 2.** Given $R$ and $R_0$ two positive real numbers, there exist $C > 0$ and $\alpha > 0$ such that the inequality

$$E_R(u(t)) \leq Ce^{-\alpha t} E(u(0))$$

(1.4)

holds for every $u$ solution to (1.1) in the “Shatah–Struwe” class with initial data $\varphi = (\varphi_1, \varphi_2)$ supported in $B_R$ and satisfying

$$E(\varphi) = \frac{1}{2} \int_{\Omega} \left( |\varphi_2|^2 + |\nabla_x \varphi_1|^2 \right) dx + \int_{\Omega} \chi(x) \frac{|\varphi_1|^6}{6} dx \leq R_0.$$  (1.5)

For the literature we quote essentially the results of Morawetz [13], Strauss [15] and Jeng-Eng-Lin [11] which obtained various rates of decay (from polynomial to exponential) in free space. Concerning the semilinear waves on unbounded domains (essentially the exterior of bounded obstacles), we mention the work of Daoulatli and the author [5], which establishes an exponential decay of the local energy for the solutions of subcritical wave equation outside convex obstacle. Finally the author obtains in [3] a polynomial decay for the local energy of the solutions of semilinear wave equation with exponent $p > 1 + \sqrt{2}$ but with small data.

**Remark 1.1.** The results of this article remain true if $O = \phi$, that is the free space.

We discuss now the methods used to establish the main result of this paper. We remind that in [5] the authors proved that the nonlinear Lax–Phillips semigroup $Z(t)$ is compact for some $T > 0$. Their proof was based on the properties of the microlocal defect measures of Gérard [8] and used in crucial way, the subcritical nature of the equation. Obviously, this is not possible in the present work; we will overcome this difficulty with the help of the “energy balance Theorem ” proved by Dehman and Gérard [6] which is adapted in our case. We will actually prove that for some sequences of initial data $Z(T)$ is compact “at infinity”.

**2. Nonlinear Lax–Phillips theory**

Let us consider the wave equation in exterior domain

$$(E_L) \begin{cases} \square u = 0 & \text{on } \mathbb{R} \times \Omega \\ u = 0 & \text{in } \mathbb{R} \times \partial \Omega \\ u(0, x) = u_0(x) \in H_D(\Omega) & \text{and} \quad \partial_t u(0, x) = u_1(x) \in L^2(\Omega). \end{cases}$$

(2.1)

We denote $U_L(t)$ the linear wave group.
In order to study the influence of the obstacle, Lax and Phillips introduced the spaces of outgoing and incoming data associated to solutions of problem \((E_L)\):

\[
D^R_+ = \{ \varphi = (\varphi_1, \varphi_2) \in H; U_L(t)\varphi = 0 \text{ on } |x| \leq t + R, \ t \geq 0 \}
\]  
(2.2)

\[
D^R_- = \{ \varphi = (\varphi_1, \varphi_2) \in H; U_L(t)\varphi = 0 \text{ on } |x| \leq -t + R, \ t \leq 0 \}
\]  
(2.3)

These spaces satisfy the following properties

(a) \(D^R_+\) and \(D^R_-\) are closed in \(H\).

(b) \(D^R_+\) and \(D^R_-\) are orthogonal and

\[
D^R_+ \oplus D^R_- \oplus \left( (D^R_+)^\perp \cap (D^R_-)^\perp \right) = H.
\]  
(2.4)

**Remarks 2.1.** (1) The solutions of \((E_L)\) and (1.1) verify the finite speed propagation property.

(2) The nonlinearity being localized in a ball \(B_R\), it is easy to see that \(U(t) = \tilde{U}_L(t)\) on \(D^R_+\) and \(U(-t) = U_L(-t)\) on \(D^R_-\) for every \(t \geq 0\). In particular this yields

\[
U(t) \text{ operates on } D^R_+ \text{ and } U(-t) \text{ operates on } D^R_- \text{ for every } t \geq 0.
\]  
(2.5)

(3) We remind that \(P^+[P^-]\) is the orthogonal projection of \(H\) onto the orthogonal complement of \(D^R_+ [D^R_-]\) and thanks to (2.4), it is clear that

\[
P^+\varphi \in \left( (D^R_+)^\perp \cap (D^R_-)^\perp \right) \text{ if } \varphi \in (D^R_-)^\perp.
\]  
(2.6)

(4) \(U(t)\) operates on \(D^R_+\) for \(t \geq 0\), so \(\text{supp}(U(t)\varphi) \cap \text{supp}(\chi) = \emptyset\) for every \(t \geq 0\) and \(\varphi \in D^R_+\).

Using then the uniqueness for the Cauchy problem in the “Shatah–Struwe” class, we obtain: for every \(\varphi\) in \(H\) and for every \(t \in \mathbb{R}_+\),

\[
U(t)\varphi = U(t)P^+\varphi + U(t)(I-P^+)\varphi
\]  
(2.7)

\[
= U(t)P^+\varphi + U_0(t)(I-P^+)\varphi,
\]  
(2.8)

where \((I-P^+)\) denotes the orthogonal projection on \(D^R_+\) and \(U_0(t)\) is the free wave group.

(5) \(U(t)\) operates on \((D^R_-)^\perp \left[ (D^R_+)^\perp \right]\) for every \(t \geq 0 [t \leq 0]\).

By analogy with the linear case, we define the nonlinear Lax–Phillips semigroup by

\[
Z(t) = P^+U(t)P^- \quad \text{for } t \geq 0.
\]  
(2.9)

Then the following proposition holds (see [5] for a proof).

**Proposition 2.1.** (1) \(Z(t)D^R_+ = Z(t)D^R_- = \{0\}\), for every \(t \geq 0\).

(2) \(Z(t)\) operates on \(K = (D^R_+)^\perp \cap (D^R_-)^\perp\).

(3) \((Z(t))_{t \geq 0}\) is a continuous semigroup on \(K\), satisfying \(E(Z(t)\varphi) \leq E(\varphi)\) for every \(t \geq 0\), and \(\varphi \in K\).

**Proof.** (1) It is clear that \(Z(t)\varphi = 0\) if \(\varphi \in D^R_-\). Furthermore, let \(\varphi \in D^R_+\), i.e. \(\varphi \in (D^R_-)^\perp\) since \(D^R_+\) and \(D^R_-\) are orthogonal. This implies that \(Z(t)\varphi = P^+U(t)\varphi = 0\), for \(t \geq 0\), thanks to (2.5).
Proposition 2.2. By virtue of (2.6) and in order to prove that $Z(t)$ operates on $D^R_+$; for every $t \geq 0$.

Let $\varphi \in (D^R_+)^\perp$ and $\psi \in D^R_-$, we have

$$\langle U(t)\varphi, U(t)\psi \rangle_H - \langle \varphi, \psi \rangle_H = -\int_0^t \left\langle \chi(x) u^5(s), \partial_t v(s) \right\rangle_{L^2} ds.$$

Consequently, we obtain

$$\langle U(t)\varphi, \psi \rangle_H - \langle \varphi, U(-t)\psi \rangle_H = -\int_0^t \left\langle \chi(x) u^5(s), \partial_t v(s-t) \right\rangle_{L^2} ds.$$

Thanks to (2.5), $U(s-t)\psi \in D_R^-$ for every $s \leq t$ so that supp $(\chi) \cap$ supp $(U(s-t)\psi) = \emptyset$, and

$$\langle U(t)\varphi, \psi \rangle_H = \langle \varphi, U(-t)\psi \rangle_H = 0.$$

(3) $Z(t)$ is obviously continuous, we only have to prove that

$$Z(t_1 + t_2) = Z(t_1)Z(t_2) \text{ for } t_1, t_2 \geq 0.$$

Let $\varphi \in K$; by (2.7), we find

$$Z(t_1 + t_2)\varphi = P^+ U(t_1 + t_2)\varphi = P^+ U(t_1)P^+ U(t_2)\varphi + P^+ U_0(t_1) (I - P^+) U(t_2)\varphi.$$

Since

$$P^+ U_0(t_1) (I - P^+) U(t_2)\varphi \in D^R_+,$$

we obtain

$$Z(t_1 + t_2)\varphi = P^+ U(t_1)Z(t_2)\varphi = Z(t_1)Z(t_2).$$

Moreover, we easily deduce that

$$E(Z(t)\varphi) = \frac{1}{2} \| P^+ U(t)\varphi \|_H^2 + \frac{1}{6} \int_\Omega \chi(x) \left| (U(t)\varphi)_1 \right|^6 dx \leq E(U(t)\varphi) = E(\varphi). \quad (2.10)$$

The proposition below shows that $Z(t)\varphi$ goes to 0 as $t \to +\infty$ for all $\varphi \in K$. This result is useful to deduce the exponential decay for the local energy of the solutions to (1.1).

**Proposition 2.2.** (1) For all $\rho \geq R$ and $\varphi \in H$

$$\lim_{t \to +\infty} \| U(t)\varphi \|_{H(B_\rho \cap \Omega)} = 0.$$

(2) For all $\varphi \in K$, $\lim_{t \to +\infty} \| Z(t)\varphi \|_H = 0$.

**Proof.** (1) Taking $\varphi$ in $H$, and applying Theorem 1, we can find $\psi$ in $H$ such that,

$$\| U(t)\varphi - U_L(t)\psi \|_H \to 0$$
then
\[\|U(t)\varphi\|_{H(B_R \cap \Omega)} \leq \|U(t)\varphi - U_L(t)\psi\|_{H(B_R \cap \Omega)} + \|U_L(t)\psi\|_{H(B_R \cap \Omega)} \xrightarrow{t \to +\infty} 0, \quad (2.11)\]
since the last term of the right-hand side of (2.11) converges to 0, by the classical Lax–Phillips theory [9].

(2) For all \( \varphi \in K = (D^\perp)^{-1} \cap (DR)^{-1} \) and \( t \geq 2R \) we have
\[Z(t)\varphi = P^+ MU(t - 2R)\varphi + P^+ U_0(2R)U(t - 2R)\varphi\]
where \( M = U(2R) - U_0(2R) \).

By Remarks 2.1, \( U(t - 2R)\varphi \in (DR)^{-1} \). Moreover
\[U_0(2R)(DR)^{-1} \subset D^+_R \quad \text{(see [1, Lemma 4.2])}\]
hence
\[Z(t)\varphi = P^+ MU(t - 2R)\varphi.\]

Using the finite speed propagation property and the fact that the nonlinearity is supported in \( B_R \), we get
\[
\|Z(t)\varphi\|_H = \|Z(t)\varphi\|_{H_0} \\
= \|P^+ MU(t - 2R)\varphi\|_{H_0} \\
= \|MU(t - 2R)\varphi\|_{H_0(B_3)} \\
= \|U(t)\varphi - U_0(2R)U(t - 2R)\varphi\|_{H_0(B_3)} \\
\leq \|U(t)\varphi\|_{H(B_3)} + \|U(t - 2R)\varphi\|_{H(B_3)} \xrightarrow{t \to +\infty} 0. \quad (2.12)
\]

Here \( H_0 \) denotes the completion of \( \left(C^\infty_0(\mathbb{R}^3)\right)^2 \) with respect to the norm
\[\|\varphi\|^2 = \|(\varphi_2, \varphi_2)\|^2 = \int_{\mathbb{R}^3} (|\nabla \varphi_1(x)|^2 + |\varphi_2|^2)dx. \square\]

3. Exponential decay of the local energy

**Definition 3.1.** We denote by \( T_R \) the minimal time needed by all the “generalized” geodesics starting from \( B_R \) at \( t = 0 \) to leave the ball \( B_R \): \( T_R \) is called the escape time.

In the following section we identify \( U(t)\varphi \) and \( Z(t)\varphi \) with their first components. Let \( \varphi \in H \) with support in \( B_R \); clearly \( \varphi \in K \). Moreover for all \( h \in H \), we have \( P^+ h = h \) on \( B_R \). Consequently \( U(t)\varphi = Z(t)\varphi \) on \( B_R \), so
\[E_R(U(t)\varphi) = E_R(Z(t)\varphi) \leq E(Z(t)\varphi).\]

Thus it is enough to prove the exponential decay of \( E(Z(t)\varphi) \). Furthermore, by the semigroup property it suffices to prove: for every \( E_0 > 0 \) there exist \( T > 0 \) and \( 0 < C < 1 \) such that,
\[E(Z(T)\varphi) \leq CE(\varphi) \quad \text{for every} \ \varphi \in K \text{ satisfying} \ E(\varphi) \leq E_0.\]
For that we argue by contradiction: We fix $E_0 > 0$ and we suppose that for every $T$ and for every $0 < C < 1$, there exists $\phi$ such that,

$$E(Z(T)\phi) \geq CE(\phi) \quad \text{and} \quad E(\phi) \leq E_0.$$  \hfill (3.1)

Then we obtain two sequences $C_n \xrightarrow{n \to +\infty} 1$, and $(\phi_n)_n$ with

$$E (Z (n) \phi_n) \geq C_n E (\phi_n).$$

Therefore for every $t \leq n$

$$E(Z(t)\phi_n) \geq C_n E (\phi_n) = C_n E(U(t)\phi_n) = C_n \left( \frac{1}{2} \|U(t)\phi_n\|_H^2 + \frac{1}{6} \int_{\Omega} \chi(x) \left| \left| \left| U(t) \phi_n \right| \right| \right|^6 \, dx \right)$$

$$= C_n \left( E(Z(t)\phi_n) + \frac{1}{2} \| (P^- I)U(t)\phi_n \|_H^2 \right),$$

then

$$\frac{1}{2} C_n \left\| (P^- I)U(t)\phi_n \right\|_H^2 \leq (1 - C_n) E(Z(t)\phi_n) \xrightarrow{n \to +\infty} 0.$$  \hfill (3.2)

$(\phi_n)$ is a bounded sequence in $H$, so there exists a subsequence, still denoted $(\phi_n)$ and $\phi \in K$ such that $\phi_n \xrightarrow{n \to +\infty} \phi$ in $K$. And thanks to Corollary A.1

$$(P^- I)U(t)\phi_n \xrightarrow{n \to +\infty} (P^- I)U(t)\phi, \quad \text{for every} \ t \geq 0.$$ Combining with (3.2), we obtain $E(Z(t)\phi) = E(U(t)\phi) = E(\phi)$, for every $t \geq 0$. Using then Proposition 2.2, we easily obtain that the weak limit $\phi$ of the sequence $\phi_n$ is 0.

To finish the proof of Theorem 2 we need the following proposition.

**Proposition 3.1.** Let $(\phi_n)_n$ a bounded sequence in $K$ such that $\phi_n \rightharpoonup 0$ then there exists a positive and nondecreasing sequence $(\alpha_j)$ satisfying

$$\lim_{j \to +\infty} \lim_{n \to +\infty} \| U(\alpha_j)\phi_n \|_{H(B_{3R})} = 0.$$  \hfill (3.3)

We postpone the proof of this proposition.

**End of Proof of Theorem 2.** We write as in (2.12)

$$\| Z(\alpha_j + 2R)\phi_n \|_H = \| Z(\alpha_j + 2R)\phi_n \|_{H_0}$$

$$= \| P^+ MU(\alpha_j)\phi_n \|_{H_0}$$

$$\leq \| MU(\alpha_j)\phi_n \|_{H_0}$$

$$= \| MU(\alpha_j)\phi_n \|_{H_{0(B_{3R})}}$$

$$\leq \| U(\alpha_j + 2R)\phi_n \|_{H(B_{3R})} + \| U(\alpha_j)\phi_n \|_{H(B_{3R})},$$
where \( M = U(2R) - U_0(2R) \). As the first term of the last inequality is controlled by the second (the finite speed propagation property) we deduce that

\[
\lim_{n \to +\infty} E(Z(\alpha_j + 2R)\varphi_n)
= \lim_{n \to +\infty} \left( \frac{1}{2} \| Z(\alpha_j + 2R)\varphi_n \|_H^2 + \frac{1}{6} \int_\Omega \chi(x) |(U(\alpha_j + 2R)\varphi_n)_1|^6 \, dx \right)
\to 0.
\]

Now we rewrite the right hand term of (3.4) as

\[
E(Z(\alpha_j + 2R)\varphi_n) = E(\varphi_n) - \frac{1}{2} \| (P^+ - I)U(\alpha_j + 2R)\varphi_n \|_H^2.
\]

Passing then to the limit, first as \( n \to +\infty \), then as \( j \to +\infty \) we obtain \( \lim_{n \to +\infty} E(\varphi_n) = 0 \).

Let \( \beta_n^2 = E(\varphi_n) \cdot \nu_n = \frac{\nu_n}{\beta_n} \) satisfies

\[
\begin{aligned}
\Box v_n + \beta_n^4 \chi(x)v_n^5 &= 0 & \text{on } \mathbb{R} \times \Omega \\
v_n &= 0 & \text{in } \mathbb{R} \times \partial \Omega \\
(v_n(0), \partial_t v_n(0)) &= \frac{\varphi_n}{\beta_n} = \psi_n \in K
\end{aligned}
\]

\[ (S) \]

\[
\tilde{E}_n(v_n) = \frac{1}{2} \int_{\Omega} (|\partial_t v_n|^2 + |\nabla x v_n|^2) \, dx + \frac{1}{6} \int_{\Omega} \beta_n^4 \chi(x)v_n^6 \, dx = 1.
\]

Denote \( V_n(t) = (v_n(t), \partial_t v_n(t)) = V^0_n(t) + W_n(t) \), where \( V^0_n(t) = (v^0_n(t), \partial_t v^0_n(t)) \), and \( W_n(t) = (w_n(t), \partial_t w_n(t)) \) with

\[
\begin{aligned}
\Box v^0_n &= 0 & \text{on } \mathbb{R} \times \Omega \\
v^0_n &= 0 & \text{in } \mathbb{R} \times \partial \Omega \\
(v^0_n(0), \partial_t v^0_n(0)) &= \frac{\varphi_n}{\beta_n} \\
w_n &= 0 & \text{in } \mathbb{R} \times \partial \Omega \\
(w_n(0), \partial_t w_n(0)) &= 0.
\end{aligned}
\]

Strichartz inequality (see Corollary 2.2 in [14] or Proposition 2.1 in [4]) applied to system (S) gives

\[
\| v_n \|_{L^5([0,T],L^{10}(\Omega))} \leq C \left( E\left( \frac{\varphi_n}{\beta_n} \right) + \beta_n^4 \| \chi(x)v_n^5 \|_{L^1([0,T],L^2(\Omega))} \right)
\]

\[
\leq C \left( 1 + \beta_n^4 \| v_n \|^5_{L^5([0,T],L^{10}(\Omega))} \right).
\]

Since \( \beta_n \to 0 \) as \( n \to +\infty \), a classical bootstrap argument shows that \( \chi(x)v_n^5 \) is bounded in \( L^1([0,T],L^2(\Omega)) \) for every \( T \geq 0 \), which yields due to the hyperbolic inequality

\[
\sup_{0 \leq t \leq T} \left| E \left( v_n(t) - v^0_n(t) \right) \right| \to 0.
\]

Now, for \( t \geq 0 \), \( (V_n(t)) \) is bounded in \( H \), and admits then a subsequence weakly converging to \( V(t) \). Moreover (3.5) gives

\[
V_n^0(t) \to V^0(t) = V(t) \quad \text{for every } t \geq 0,
\]
and by the compactness of $Z_L(t)$ [12], we have
\[ P^+V_n^0(t) \xrightarrow{n \to +\infty} P^+V^0(t), \quad \text{for every } t \geq T_R + 9R. \]

Then, according to (3.5)
\[ P^+V_n(t) \xrightarrow{n \to +\infty} P^+V^0(t), \quad \forall t \geq T_R + 9R. \]

Coming back to the contradiction argument developed above, we have
\[ C_n \leq \tilde{E}_n(P^+V_n(t)) \leq 1, \]
and passing to the limit we get
\[ \frac{1}{2} \left\| P^+V^0(t) \right\|^2_H = 1. \] (3.6)

Using again the fact $\tilde{E}_n(v_n) = 1$, we obtain
\[ \| V_n(t) \|_H \leq \sqrt{2} \]
then
\[ \left\langle V_n(t), V^0(t) \right\rangle \leq \| V_n(t) \|_H \left\| V^0(t) \right\|^2_H \leq \sqrt{2} \left\| V^0(t) \right\|^2_H, \]
and using $V_n(t) \xrightarrow{n \to +\infty} V^0(t)$ which implies in particular
\[ \left\langle V_n(t), V^0(t) \right\rangle \xrightarrow{n \to +\infty} \| V^0(t) \|^2_H, \]
then we obtain $\| V^0(t) \|_H \leq \sqrt{2}$. Combining this with (3.6) we deduce that we can find $\psi = V(0) \in K$ such that
\[ \| Z_L(t)\psi \|_H = \| \psi \|_H = \sqrt{2} \quad \text{for every } t \geq 0, \]
which contradicts the result of Melrose (see [12]). □

In order to prove Proposition 3.1 we will need the following Proposition due to Dehman and Gérard. They proved this result for $\Omega = \mathbb{R}^3$, but one can see that, with slight modifications, the proof remains valid when $\Omega$ is the exterior of convex obstacle.

**Proposition 3.2** (Adapted from [6]). Let $(r_n)$ be a sequence of solutions of
\[ \Box r_n + \chi(x) r_n^5 = f_n, \]
in the “Shatah–Struwe” class and we assume that $(r_n(0), \partial_t r_n(0)) \to (r_0, r_1)$ in $H_D(\Omega) \times L^2(\Omega)$ and $f_n \to 0$ strongly in $L^1_{\text{loc}}(\mathbb{R}^+ \times L^2(\Omega))$.

Let $r$ be the “Shatah–Struwe” solution of
\[ \Box r + \chi(x) r^5 = 0, \quad r(0) = r_0, \quad \partial_t r(0) = r_1 \]
and $\tilde{r}_n$ the “Shatah–Struwe” solution of
\[ \Box \tilde{r}_n + \chi(x) \tilde{r}_n^5 = 0, \quad \tilde{r}_n(0) = r_n(0) - r_0, \quad \partial_t \tilde{r}_n(0) = \partial_t r_n(0) - r_1. \]

Then for every $T > 0$,
\[ \sup_{0 \leq t \leq T} \| \nabla_x t r_n - \nabla_x t r - \nabla_x t \tilde{r}_n \|_{L^2(\Omega)} + \| r_n - r - \tilde{r}_n \|_{L^5(\mathbb{R}, L^{10}(\Omega))} \xrightarrow{n \to +\infty} 0. \]
We come back now to the proof of Proposition 3.1.

**Proof of Proposition 3.1.** Let \((u_n)_{n \in \mathbb{N}}\) (resp.\((v_n)_{n \in \mathbb{N}}\)) the sequence of solutions to (1.1) (resp.\((E_L)\)) associated to the sequence of initial data \((\varphi_n)_{n \in \mathbb{N}}\), in the sense that

\[
\begin{align*}
\square u_n + \chi(x)u_n^5 &= 0 \quad \text{on } \mathbb{R} \times \Omega, \\
\varphi_n &= 0 \quad \text{in } \mathbb{R} \times \partial \Omega, \\
(u_n(0), \partial_t u_n(0)) &= \varphi_n,
\end{align*}
\]

where

\[
\begin{align*}
\square v_n &= 0 \quad \text{on } \mathbb{R} \times \Omega, \\
v_n &= 0 \quad \text{in } \mathbb{R} \times \partial \Omega, \\
(v_n(0), \partial_t v_n(0)) &= \varphi_n.
\end{align*}
\]

Due to Proposition 5.1 in [5] (which is easily adapted in our context) \(v_n \longrightarrow 0\) on \(H^1_{loc}(\tilde{K}(T))\) for \(T \geq T_0 = T_R + 3R\), where \(\tilde{K}(T) = \{(t, x) \in \mathbb{R} \times \Omega / |x| \leq t - T + R, \ t \geq T\}\), then

\[
f_n = \chi(x)\sum_{p=0}^{4} C_p^5 u_n^p v_n^{5-p} \longrightarrow 0 \quad \text{in } L^1_{loc}([T_0, +\infty[ , L^2(\Omega)).
\]

Indeed by Hölder’s inequality, Strichartz estimates and Corollary A.2 one can see that the sequence of solutions to (3.7) converges to 0.

Applying then Proposition 3.2, we obtain

\[
\|r_n - \tilde{r}_n\|_{H^1_{loc}([T_0, T] \times \Omega)} \longrightarrow 0 \quad \text{for every } T \geq T_0
\]

where \(\tilde{r}_n\) satisfies

\[
\begin{align*}
\square \tilde{r}_n + \chi(x)\tilde{r}_n^5 &= 0 \quad \text{on } \mathbb{R} \times \Omega, \\
\tilde{r}_n &= 0 \quad \text{in } \mathbb{R} \times \partial \Omega, \\
(\tilde{r}_n, \partial_t \tilde{r}_n)_{t=T_0} &= (r_n, \partial_t r_n)_{t=T_0}.
\end{align*}
\]

Combining then (3.7) with the fact that \(\text{supp}(r_n(t)) \subset B_{R+t}\) for every \(t \geq 0\), we see that

\[
\tilde{r}_n \longrightarrow 0 \quad \text{in } H^1_{loc}(|x| > R + t, \ t \geq T_0).
\]

Moreover, we recall that the energy density of \((\tilde{r}_n)\) is given by

\[
e_n(t, x) = \frac{1}{2} \left| \partial_t \tilde{r}_n(t, x) \right|^2 + \frac{1}{6} \chi(x) |\tilde{r}_n(t, x)|^6,
\]

and \(e(t, x)\) the weak limit of \(e_n(t, x)\).

We note that the conclusion of Theorem 7 in [6] remains valid in our situation, that is

\[
e(t, x) = \sum_{j=1}^{+\infty} e^{(j)}(t, x) + e_f(t, x),
\]

where \(e^{(j)}\) is the limit energy density of the nonlinear concentrating wave \(q_n^{(j)}\) solution to

\[
\begin{align*}
\square q_n^{(j)} + \chi(x)\left(q_n^{(j)}\right)^5 &= 0 \quad \text{on } \mathbb{R} \times \Omega, \\
q_n^{(j)}(0, \partial_t q_n^{(j)}(0)) &= (p_n^{(j)}(0), \partial_t p_n^{(j)}(0)),
\end{align*}
\]
Applying then the linear result of Lebeau (see [10]) for propagation of the support of a sequence of solutions of the linear wave equation $\tilde{w}_n$, namely

$$e_f(t, x) = \int_{\xi \in S^2} \mu(t, x, d\xi)$$

with $\mu(t, x, d\xi) = \mu_+(t, x, d\xi) + \mu_-(t, x, d\xi)$ and $\mu_\pm$ are positive measures on $\Omega \times S^2$.

Consequently, using (3.9) we obtain

$$\lim_{n \to +\infty} \|q_n^{(j)}\|_{H^1_{loc}(|x| > R + t, t \geq T_0)} = 0$$

(3.10)

On the other hand, taking $\chi = \chi(x^{(j)})$, where $x^{(j)} = \lim_{n \to +\infty} x_n^{(j)}$ and using Theorem 1 in [4] (or also Theorem 2 in [7] with slight modifications), we obtain

$$\int_{\Omega} \left( |\partial_t (q_n^{(j)} - v_n^{(j)}) (t, x)|^2 + |\nabla_x (q_n^{(j)} - v_n^{(j)}) (t, x)|^2 \right) dx \longrightarrow 0,$$

$$t \in \left]\bar{t}^{(j)}_\infty, T \right[, (3.11)$$

for every $T > \bar{t}^{(j)}_\infty$, where $v_n^{(j)}$ is a sequence of finite energy solutions of the linear wave equation and $\bar{t}^{(j)}_\infty = \lim_{n \to +\infty} \bar{t}^{(j)}_n$ which verifies $\bar{t}^{(j)}_\infty \geq T_0$, in fact the decomposition of the energy density is only made in the region $t \geq T_0$.

Denote $\mu_j$, $j \geq 1$ (resp.$\mu$) the microlocal defect measures associated to $\left(q_n^{(j)}\right)_n$ (resp.$\tilde{w}_n$).

The result (3.11) implies that $\mu_j$ is also attached to the sequence $v_n^{(j)}$ on the time interval $\left]\bar{t}^{(j)}_\infty, T \right[$. Let $q \in T^*(\tilde{K} \left(\bar{t}^{(j)}_\infty + R, t^{(j)}_\infty\right))$ (recall that $T^{(j)}_\infty + R$ is given by Definition 3.1) and $\lambda$ a generalized bicharacteristic starting at $q$. The obstacle is strictly convex and then nontrapping; so if $\lambda$ is traced backwards in time, it does not meet $\partial \Omega$ or meets $\partial \Omega$.

But in the two cases $\lambda_0 = \lambda_{t = \bar{t}^{(j)}_\infty} \in \left\{ |x| > R + t, t \geq \bar{t}^{(j)}_\infty \right\}$ and in view of (3.10), we get

$$\mu_j = \mu = 0 \quad \text{on} \quad \left\{ |x| > R + t, t \geq \bar{t}^{(j)}_\infty \right\}.$$

Applying then the linear result of Lebeau (see [10]) for propagation of the support of $\mu_j$ (resp.$\mu$) we deduce that $\mu_j = \mu = 0$ on $\tilde{K} \left(\bar{t}^{(j)}_\infty + R, t^{(j)}_\infty\right)$. Hence

$$e^{(j)} = e_f = 0 \quad \text{on} \quad \tilde{K} \left(\bar{t}^{(j)}_\infty + R, t^{(j)}_\infty\right),$$

consequently

$$\sum_{p=1}^j e^{(p)} = 0 \quad \text{on} \quad \tilde{K} \left(\max_{1 \leq p \leq j} \left(\bar{t}^{(p)}_\infty + R, t^{(p)}_\infty\right)\right).$$

(3.12)

On the other hand, by (3.9) we have

$$\forall \varepsilon > 0, \exists j_0 \in \mathbb{N} \text{ such that for every } j \geq j_0 \sum_{p \geq j+1} e^{(p)} \leq \frac{\varepsilon}{\text{vol}(B_{5R})},$$

(3.13)
then (3.12) and (3.13) gives
\[
\lim_{j \to +\infty} \lim_{n \to +\infty} \| \tilde{r}_n(\alpha_j) \|_{H(B_{SR})} = 0,
\]
where \( \alpha_j = \max_{1 \leq p \leq j} T_{t_{(p)}^{(p)} + t_{(p)}^{(p)}}. \)

Consequently, we get by (3.7) the same limit for \( r_n \). Finally, recalling that \( u_n = r_n + v_n \) and \( v_n \to 0 \) on \( H^1_{loc}(\tilde{K}(T)) \) for \( T \geq T_0 \) we obtain the desired result.  

\[\square\]

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Appendix

In this appendix, we give some important theorems of the literature that we have used in several proofs in this article. We give them in our context i.e. for the solutions of the critical wave equation with localized semilinearity near a convex obstacle. All these results are borrowed from [2,6,7], we note that (with slight modifications of their proofs) these results remain valid in the context of our problem.

Let us first introduce some vocabulary.

A scale \( h \) is a sequence \( (h_n)_n \) of positive numbers going to 0 if \( n \) goes to infinity; a core is a convergent sequence \( z = (t_n, x_n) \) of \( \mathbb{R}_t \times \mathbb{R}_x^3 \).

\( (h, z) \) and \( (h', z') \) are orthogonal if
\[
\frac{|h_n - h'_n| + |x_n - x'_n|}{h_n} \to +\infty. \tag{A.1}
\]

The linear concentrating wave associated with \( (\varphi, \psi, h_n, x_n, t_n) \) is the solution of the following linear wave equation
\[
\begin{cases}
\Box p_n = 0 & \text{on } \mathbb{R} \times \Omega, \\
 p_n = 0 & \text{in } \mathbb{R} \times \partial \Omega \\
 (p_n(t_n), \partial_t p_n(t_n)) = \left( \frac{1}{h_n^{1/2}} P_{\Omega} \left( \varphi \left( \frac{\cdot - x_n}{h_n} \right) \right) ; \frac{1}{h_n^{3/2}} 1_{\Omega}(\cdot) \psi \left( \frac{\cdot - x_n}{h_n} \right) \right),
\end{cases} \tag{A.2}
\]

where \( \Omega \) is the exterior of a compact, strictly convex, smooth domain of \( \mathbb{R}^3 \) and \( P_{\Omega} \) is the orthogonal projection from \( H^1(\mathbb{R}^3) \) to \( H^1(\Omega) \).

The nonlinear concentrating wave associated with \( p_n \) is the solution of the following equation
\[
\begin{cases}
\Box q_n + \chi(\cdot) q_n^5 = 0 & \text{on } \mathbb{R} \times \Omega, \\
 q_n = 0 & \text{in } \mathbb{R} \times \partial \Omega \\
 (q_n(0), \partial_t q_n(0)) = (p_n(0), \partial_t p_n(0)).
\end{cases} \tag{A.3}
\]

We recall that the energy of any function \( u \) solution to (A.2) or (A.3) is defined by:
\[
E_0(u)(t) = \frac{1}{2} \int_{\Omega} \left( |\partial_t u(t, x)|^2 + |\nabla_x u(t, x)|^2 \right) dx.
\]
Finally, we assume that the initial data \((\varphi_n, \psi_n)\) is compact at infinity, in the sense that
\[
\lim_{n \to +\infty} \int_{|x| \geq R} \left( |\nabla \varphi_n(x)|^2 + |\psi_n(x)|^2 \right) dx \to 0.
\]  
We recall the following theorem which is adapted from Theorems 1 and 3 in [7].

**Theorem 3.** Let \(v_n\) be the solution of
\[
\begin{cases}
\square v_n = 0 & \text{on } \mathbb{R} \times \Omega, \\
v_n = 0 & \text{in } \mathbb{R} \times \partial \Omega \\
(v_n(0), \partial_t v_n(0)) = (\varphi_n, \psi_n)
\end{cases}
\]
satisfying \(\sup_n E_0(v_n) < +\infty\) and (A.4). Then there exist a finite energy solution to the linear wave equation \(v\), orthogonal concentrating data \((\varphi^{(j)}, \psi^{(j)}, h_n^{(j)}, x_n^{(j)}, t_n^{(j)})\), for \(j \in \mathbb{N}^*\), such that \(v_n\) can be decomposed as follows, up to the extraction of a subsequence: for any \(l \in \mathbb{N}^*\),
\[
v_n = v + \sum_{j=1}^{l} p_n^{(j)} + w_n^{(l)}
\]
where \(p_n^{(j)}\) is the linear concentrating wave associated with \((\varphi^{(j)}, \psi^{(j)}, h_n^{(j)}, x_n^{(j)}, t_n^{(j)})\) and the remainder \(w_n^{(l)}\) satisfies, for every \(T > 0\),
\[
\lim_{n \to +\infty} \left\| w_n^{(l)} \right\|_{L^\infty([-T, T], L^6(\Omega))} \to 0.
\]
Moreover, denote \(u_n\) a solution in the “Shatah–Struwe” class of
\[
\begin{cases}
\square u_n + \chi(x) u_n^5 = 0 & \text{on } \mathbb{R} \times \Omega, \\
u_n = 0 & \text{in } \mathbb{R} \times \partial \Omega \\
(u_n(0), \partial_t u_n(0)) = (\varphi_n, \psi_n)
\end{cases}
\]
satisfying \(\sup_n E_0(u_n) < +\infty\) and (A.4). Then up to the extraction of a subsequence, we can write, for any \(l \in \mathbb{N}^*\),
\[
u_n = u + \sum_{j=1}^{l} q_n^{(j)} + w_n^{(l)} + r_n^{(l)},
\]
where \(u\) is a solution of a nonlinear wave equation, \(q_n^{(j)}\) is the nonlinear concentrating wave equation associated with \(p_n^{(j)}\) and for every \(T > 0\),
\[
\lim_{n \to +\infty} \left( \sup_{-T \leq t \leq T} E_0(r_n^{(l)}, t)^{1/2} + \left\| r_n^{(l)} \right\|_{L^5([-T, T], L^{10}(\Omega))} \right) \to 0.
\]
Let us note that this result, which describes the high frequency approximation of the solutions of the critical wave equation, is easily applicable in our context i.e. in the presence of the function \(\chi\). Indeed, looking carefully to the proof of Theorem 3, one observes that the behavior of a profile concentrating at \(x_n^{(j)} \to x^{(j)}\) depends locally only on \(\chi(x^{(j)})\) while the behavior is nonlinear and does not have any effect while the profile is close to linear.
Now, notice the following corollaries.

**Corollary A.1** (Adapted from Corollary 1 in [2]). Let \((u_n)\) be a sequence of solution in the “Shatah–Struwe” class to (1.1). We assume that \((\varphi_n, \psi_n) \rightharpoonup (\varphi, \psi)\) in \(H_D(\Omega) \times L^2(\Omega)\). Then \(u_n \rightarrow u\), where \(u\) is the solution in the “Shatah–Struwe” class of

\[
\begin{cases}
\Box u + \chi(x)u^5 = 0 & \text{on } \mathbb{R} \times \Omega \\
(u(0), \partial_t u(0)) = (\varphi, \psi). 
\end{cases}
\]  

(A.9)

**Corollary A.2** (Adapted from Corollary 2 in [2]). There exists a nondecreasing function \(A : [0, +\infty[ \rightarrow [0, +\infty[\) such that, for every Shatah–Struwe solution \(u\) to (1.1),

\[
\|u\|_{L^\infty(\mathbb{R}, L^{10}(\Omega))} \leq A (E(u)).
\]

Let now \((u_n)\) be a sequence of solutions to (A.9). We recall that the energy density of \(u_n\) is given by

\[
e_n(t, x) = \frac{1}{2} \left[ |\partial_t u_n(t, x)|^2 + |\nabla_x u_n(t, x)|^2 \right] + \frac{1}{6} \chi(x) |u_n(t, x)|^6,
\]

and we say that \(e(t, x)\) is the limit energy density of the sequence \((u_n)\) if \(e_n(t, x)\) converges weakly to \(e(t, x)\).

We finally come to the “energy balance theorem” which is adapted from Theorem 7 in [6].

**Theorem 4.** Let \((u_n)\) be a bounded sequence in the “Shatah–Struwe” class, solution of (A.9) and satisfying \(\sup_n E_0(u_n) < +\infty\), \(u_n(0), \partial_t u_n(0)\) are supported in a fixed compact of \(\Omega\) and \(u_n \rightarrow 0\). Then we can write the limit energy density of \((u_n)\) as

\[
e(t, x) = \sum_{j=1}^{+\infty} e^{(j)}(t, x) + e_f(t, x)
\]

(A.10)

where \(e^{(j)}\) is the limit energy density of the nonlinear concentrating wave \(q_n^{(j)}\) and \(e_f\) is the limit energy density of a sequence of solutions of linear wave equation \(\tilde{w}_n\), namely

\[
e_f(t, x) = \int_{\xi \in S^2} \mu(t, x, d\xi)
\]

with \(\mu(t, x, d\xi) = \mu_+ (t, x, d\xi) + \mu_- (t, x, d\xi)\) and \(\mu_+, \mu_-\) are positive measures on \(\Omega \times S^2\).

This theorem remains valid in our case. Indeed its proof is based on Lemma A.3 of [6] which we easily adapt to our work by extending the solutions by 0 outside \(\Omega\). More precisely, using the notations of [6], let \(\varphi(t, x) \in C_0^\infty (\mathbb{R} \times \Omega)\), \(\psi (t, x) \in C_0^\infty (\mathbb{R} \times \Omega)\) such that \(\text{supp} (\varphi) \subset \{(t, x) \mid \psi \equiv 1\}\) and \(\tilde{v}_n^{(j)}\) (resp. \(\tilde{w}_n^{(l)}\)) the extensions by 0 of \(v_n^{(j)}\) (resp. \(w_n^{(l)}\)), outside \(\Omega\). We have

\[
\Box \left( \psi \tilde{w}_n^{(l)} \right) = [\Box, \psi] \, w_n^{(l)} \quad \rightharpoonup_{n \rightarrow +\infty} \quad 0 \quad \text{in } L^2 (\mathbb{R} \times \Omega),
\]

which yields (as in [6]) the desired result i.e.

\[
\varphi b \left( v_{\pm,n}^{(j)}, w_n^{(l)} \right) = \varphi b \left( \psi \tilde{v}_{\pm,n}^{(j)}, \psi \tilde{w}_n^{(l)} \right) \quad \rightharpoonup_{n \rightarrow +\infty} \quad 0 \quad \text{in } L^1 (\mathbb{R} \times \Omega).
\]

Finally, let us indicate that the analogue of the Lemma A2 in [6] is in [7, Lemma 3.7, p. 35].
References