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Stable dominating circuits in snarks

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Abstract

Snarks are cyclically 4-edge-connected cubic graphs with girth at least 5 and with no 3-edge-coloring. We construct snarks with a (dominating) circuit C so that no other circuit C' satisfies $V(C) \subseteq V(C')$. These graphs are of interest because two known conjectures about graphs can be reduced on them. The first one is Sabidusi's Compatibility Conjecture which suggests that given an eulerian trail T in an eulerian graph G without 2-valent vertices, there exists a decomposition of G into circuits such that consecutive edges in T belong to different circuits. The second conjecture is the Fixed-Circuit Cycle Double-Cover Conjecture suggesting that every bridgeless graph has a cycle double cover which includes a fixed circuit. © 2001 Elsevier Science B.V. All rights reserved.

1. Introduction

By a *circuit* we mean a connected graph where each vertex has valency 2. Let G be a circuit in a graph G. C is called *dominating* if each edge of G is incident with a vertex from C. C is called *hamiltonian* if $V(C) = V(G)$. By a *cycle double cover* (CDC) of a graph G we mean a family $\mathscr{L}$ of circuits in G so that each edge of G is contained in just two circuits from $\mathscr{L}$.

The well-known CDC-*conjecture* states that any bridgeless graph has a CDC. There exist several variants and strengthenings of this conjecture. One of them is the *Fixed Circuit Cycle Double Cover Conjecture* (see [4,8]):

Conjecture 1. Given any circuit C in a bridgeless graph G, there exists a cycle double cover of G which includes C.

This generalizes the following conjecture of Fleischner.

Conjecture 2. Given a dominating circuit C in a cubic graph G, there exists a cycle double cover of G which includes C.

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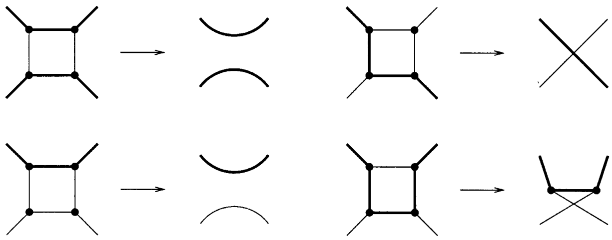


Fig. 1.

Fleischner in fact proved (see [7], and also [3,5,6]) that this conjecture is equivalent to *Sabidusi's Compatibility Conjecture* [22]:

Conjecture 3. Given an eulerian trail T in an eulerian graph G without 2-valent vertices, there exists a decomposition $\mathcal{S}$ of G into circuits so that the consecutive edges in T belong to different circuits in $\mathcal{S}$.

More detailed discussion about Sabidusi's conjecture and its relation to the CDC-conjecture can be found in the survey article of Jaeger [12] and in the book of Fleischner [6].

By a *snark* we mean a cyclically 4-edge-connected cubic graph with girth at least 5 and with no 3-edge-coloring. Note that a graph is called *cyclically k-edge-connected* if deleting fewer than k edges does not disconnect the graph into components so that at least two of them have circuits. The *girth* of a graph is the length of its smallest circuit.

It is well known that the CDC-conjecture remains to verify for snarks (see, e.g., [2,12,13,21,24–26]). We show that also Conjecture 2 remains to verify if G is a snark. We only sketch the proof because we use the well-known arguments. Suppose a cubic graph G with a dominating circuit C presents a counterexample to Conjecture 2. Let G has an edge cut of cardinality $\leqslant 3$ so that after deleting it we get two components H_1 and H_2 having circuits (if there are more than two components, then G must have a bridge, what is not possible, or we can take a smaller edge cut and get only two components). After contracting H_1 and H_2 into one vertex we get from G new graphs G_1 and G_2 and from C new circuits C_1 and C_2, respectively. Then at least one of them is (or is homeomorphic with) a smaller counterexample (if not, then neither G can be). If G has a circuit of length 4, then we get a smaller counterexample after applying the reductions indicated in Fig. 1 (the edges of the circuit C are depicted by bold lines in this figure). If G is 3-edge-colorable, then, by [10, Lemma 1] (see also [15,23]), it cannot be a counterexample to this conjecture. Therefore, if a cubic graph G with a dominating circuit C is a smallest counterexample to Conjecture 2, then G is a snark.

But what can be said about C? We claim that there does not exist another circuit C' satisfying $V(C) \subseteq V(C')$. In this case we say that C is *stable*. Really, if C is not stable and we have a circuit C' so that $V(C) \subseteq V(C')$, then also C' is dominating, but neither C nor C' are hamiltonian (cubic hamiltonian graphs are 3-edge-colorable, thus, by [10, Lemma 1], Conjecture 2 is satisfied for them). Take G' so that $V(G') = V(G)$ and $E(G') = E(G) - [E(C) - E(C')]$. All vertices of G' have valency 2 or 3 (because $E(C) - E(C')$ is a matching) and C' is dominating in G'. G' is not a circuit, because C' is not hamiltonian. Furthermore, G' and C' present a counterexample to Conjecture 1. If not, then G' has a CDC $\mathscr{L}'$ so that $C' \in \mathscr{L}'$, and taking $\mathscr{L} = (\mathscr{L}' - C') \cup \{C, C_1, \ldots, C_k\}$, where $C_1, \ldots, C_k$ are the circuits induced by the set of edges $[E(C) \cup E(C')] - [E(C) \cap E(C')]$, we get a CDC of G which includes C — a contradiction. Thus, the cubic graph G'' homeomorphic with G' contains a dominating circuit C'' (arising from C' after suppressing the vertices of valency two) so that G'' and C'' present a smaller counterexample to Conjecture 2 than are G and C.

Thus, by induction, Conjecture 2 remains to verify for the cases when G is a snark and C is a stable dominating circuit in G.

Clearly, any bridgeless graph can be obtained from a bridgeless cubic graph after contracting some edges. Using this fact and similar arguments as are presented above, we can show that Conjecture 1 remains to verify for the cases when G is a snark and C is a stable circuit in G. Now it is natural to ask the following questions.

Problem 1. Do there exist snarks with stable circuits?

Problem 2. Do there exist snarks with stable dominating circuits?

If the answer to Problem 1 (resp. 2) is negative, then Conjecture 1 (resp. 2) is valid. In this paper we construct an infinite family of snarks with stable dominating circuits, thereby obtaining a positive answer to both problems. By the way, our methods can be used also for constructions of snarks with a stable circuit that is not dominating.

Problem 1 was given by Huck [9], who wanted to use the above-mentioned arguments for proving Conjecture 1. But we have a suspicion that also other authors have been aware of the significance of this problem with respect to Conjecture 1. For instance, in 1990 Seymour asked the following question (see [7]): *Does there exist a cubic* 3*-connected graph with a stable circuit?* This problem was solved by Fleischner [7] who constructed graphs with stable dominating circuits. Unfortunately, these graphs are not cyclically 4-edge-connected, and, therefore, they are not snarks. In this paper we present another construction and obtain snarks with stable (dominating) circuits.

2. Construction

Following the notation from [18], by an *abstract network*, simply a *network*, we mean a couple $\mathscr{N} = (G, U)$ where G is a graph and $U \subseteq V(G)$. The vertices from U and $V(G) - U$ are called *outer* and *inner* vertices of $\mathscr{N}$, respectively.

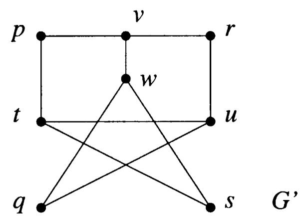


Fig. 2. $\mathcal{N}' = (G', \{p,q\}, \{r,s\})$.

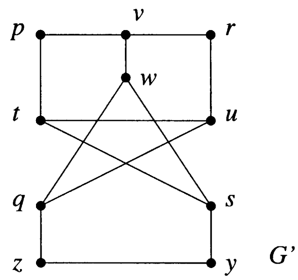


Fig. 3. $\mathcal{N}'' = (G'', \{p,z\}, \{r,y\})$.

For technical reasons, we shall deal with networks where the set U is partitioned into nonempty sets $U_1, \ldots, U_n$. In this case we write $\mathcal{N} = (G, U_1, \ldots, U_n)$ and call $\mathcal{N}$ the *partitioned network*. Sets $U_1, \ldots, U_n$ are called *connectors* of $\mathcal{N}$. For example in Figs. 2 and 3 partitioned networks $\mathcal{N}' = (G', \{p,q\}, \{r,s\})$ and $\mathcal{N}'' = (G'', \{p,z\}, \{r,y\})$ are indicated, respectively.

Remark 1. Adding to G'' edges (p,z) and (r,y) we get a Petersen graph P. Furthermore, deleting from P the vertices z and y we get the graph G'.

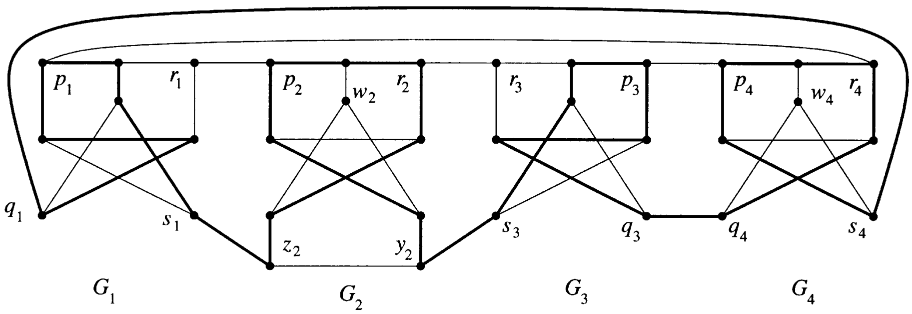
Remark 2. $\varphi = (pq)(tw)(uv)$ and $\psi = (pq)(rs)(tu)(vw)$ are automorphisms of G'.

Let $\mathcal{N} = (G, U)$ be a network. Any path $v_1 \ldots v_n$ in G we shall call *v_1–v_n-path*. Furthermore, if $v_1, v_n \in U$, then it is called *open* in $\mathcal{N}$. By a *k-polygon* in $\mathcal{N}$ (briefly a *polygon*) we mean k vertex disjoint open paths in $\mathcal{N}$.

Furthermore, if $\mathcal{N}$ is partitioned, then an open *v_1–v_n-path* in $\mathcal{N}$ is called *crossing* if the vertices v_1 and v_n do not belong to the same connector. A polygon in $\mathcal{N}$ is called *crossing* if it is composed from crossing paths. For instance, paths *pvr* and *qws* form a crossing polygon in $\mathcal{N}'$.

Lemma 1. *Let $\mathcal{N}'$ be the partitioned network from Fig.* 2 *and $X = V(G') - \{r\}$. Then qutpvws is the only q–s-path containing all vertices from X and pvwquts is the only p–s-path containing X. Furthermore*, *these two* 1*-polygons are the only crossing polygons in $\mathcal{N}'$ containing all vertices from X.*

Proof. Let A be a q–s-path containing X. If it contains r, then A covers all vertices from G' and, using Remark 1, this path can be extended into a hamiltonian circuit in P, which is a contradiction. Thus, A cannot contain r and neither the edges incident with it, and, therefore, (w,v), (v,p), (q,u), $(u,t) \in A$. Also $(p,t) \in A$ because p has valency 2. Since $(q,u) \in A$, then $(q,w) \notin A$, and, therefore $(w,s) \in A$. Thus $A = qutpvws$. Furthermore, applying the automorphism φ we get that *pvwquts* is the only p–s-path containing X.



G

Fig. 4.

Suppose B is a crossing polygon in $\mathcal{N}'$ covering X and not equal to a q–s- or a p–s-path. Then it must contain also vertex r and, therefore, all vertices form G'. Using Remark 1 we can extend B into a hamiltonian circuit in P — a contradiction. □

Lemma 2. *Let $\mathcal{N}'$ be the partitioned network from Fig.* 2 *and $Y = V(G') - \{w\}$. Then qurvpts is the only crossing* 1*-polygon in $\mathcal{N}'$ containing all vertices from Y.*

Proof. Let A be a q–s-path containing all vertices from Y. If it contains also vertex w, then, by Remark 1, it would imply hamiltonicity of P. Then A cannot contain the edges incident with w and, thus, $(s,t),(q,u),(p,v),(v,r) \in A$. Furthermore $(t,u) \notin A$, otherwise A is not a path, and thus $(p,t),(u,r) \in A$, what implies that $A = qurvpts$.

If a crossing path B in $\mathcal{N}'$ is no q–s-path and covers X, then it contains the edges incident either with q or with s, and, thus, also vertex w. Therefore, B contains all vertices from G', what, by Remark 1, implies hamiltonicity of P — a contradiction. □

Lemma 3. *Let $\mathcal{N}''$ be the partitioned network from Fig.* 3 *and $Z = V(G'') - \{w\}$. Then zqurvptsy is the only z–y-path in $\mathcal{N}''$ containing all vertices from Z.*

Proof. Let A be an z–y-path containing Z. It cannot contain (z,y) (otherwise, $A = zy$) and, therefore, it contains the edges (z,q) and (s,y). Then, the statement follows from Lemma 2. □

Take the graph G depicted in Fig. 4. It arises from three copies of G' (G_1, G_3, G_4) and one copy of G'' (G_2) after joining the vertices of valency 2 as indicated in the figure. Let $T = X_1 \cup Z_2 \cup X_3 \cup Y_4$, where X_1, X_3 (Y_4, Z_2) are the sets arising from X (Y,Z) after adding appropriate indices. More formally, $T = V(G) - \{r_1, w_2, r_3, w_4\}$. The edges depicted in Fig. 4 by bold lines induce a dominating circuit C satisfying $V(C) = T$. We show that C is stable.

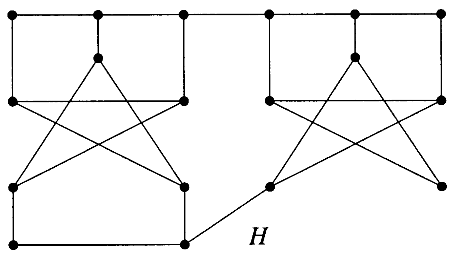


Fig. 5.

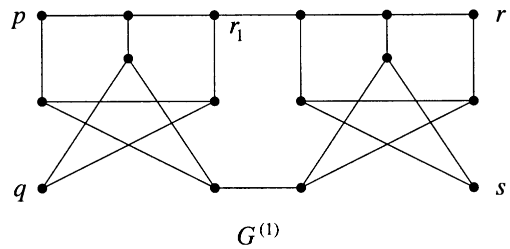


Fig. 6. $\mathcal{N}^{(1)} = (G^{(1)}, \{p,q\}, \{r,s\})$.

Theorem 1. *Graph G is a snark with a stable dominating circuit C so that $V(C)=T$.*

Proof. Graph G contains the graph H indicated in Fig. 5 as an induced subgraph, thus, by [20, Lemma 2], G is a snark.

Let C be a circuit in G so that $T \subseteq V(C)$. If $C \cap G_1$ is a 2-polygon, then, by Lemma 1, it must be composed from a p_1–q_1- and an r_1–s_1-paths, and, therefore, $C \cap G_i$ must be a crossing 2-polygon for any $i = 2,3,4$, which contradicts Lemma 1 in the case $i = 3$.

Thus $C \cap G_1$ is a 1-polygon and analogously can be shown that $C \cap G_3$ is a 1-polygon as well. Then we can check that $C \cap G_i$ is a crossing 1-polygon for any $i = 1,\dots,4$. Therefore, by Lemma 2, $C \cap G_4$ is a q_4–s_4-path, and, by Lemma 1, $C \cap G_1$ and $C \cap G_3$ are q_1–s_1- and q_3–s_3-paths, respectively, what together with Lemma 3 gives that $C \cap G_2$ is a z_2–y_2-path. From Lemmas 1–3 it follows that these paths are unique and that C is the circuit depicted in Fig. 4, concluding the proof. $\square$

Fig. 6 depicts a partitioned network $\mathcal{N}^{(1)}$. If $X^{(1)} = V(G^{(1)}) - \{r, r_1\}$, then using Lemma 1 we can check that there exist a q–s- and a p–s-paths each containing $X^{(1)}$ and that they are the only crossing polygons in $\mathcal{N}^{(1)}$ containing $X^{(1)}$, which is similar to Lemma 1.

Fig. 7 depicts a partitioned network $\mathcal{N}^{(2)}$. If $Z^{(2)} = V(G^{(2)}) - \{w_1, w_2, w_3\}$, then using Lemmas 2 and 3 we can check that there exists just one z–y-path containing all vertices from $Z^{(2)}$. This is similar to Lemma 3.

Thus we can replace G_1 by a copy of $G^{(1)}$ and G_2 by a copy of $G^{(2)}$ in the graph G. Furthermore, we can recursively repeat this operations. All graphs obtained

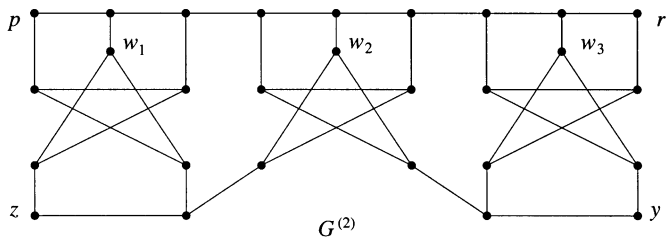


Fig. 7. $\mathcal{N}^{(2)} = (G^{(2)}, \{p,z\}, \{r,y\})$.

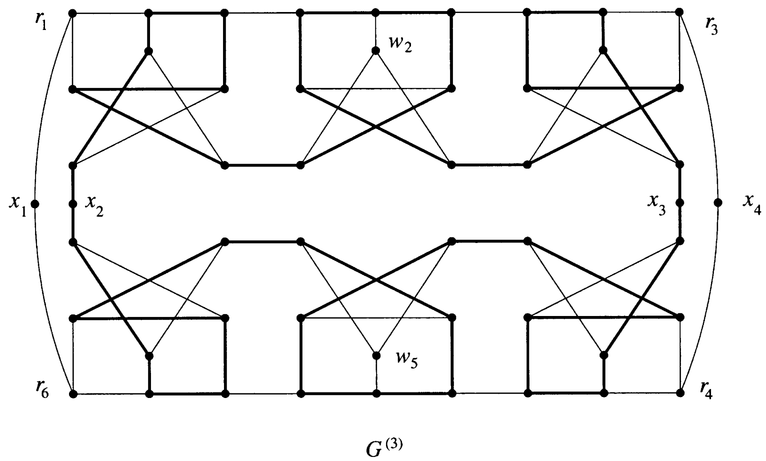


Fig. 8. $\mathcal{N}^{(3)} = (G^{(3)}, \{x_1,x_2,x_3,x_4\})$.

in this process are snarks and have a stable dominating circuit. This can be checked analogously as in Theorem 1 using the above-mentioned properties of $\mathcal{N}^{(1)}$ and $\mathcal{N}^{(2)}$. Therefore, we can conclude.

Theorem 2. *For every nonnegative integers k, m there exists a snark of order* $34 + 8k + 18m$ *having a stable dominating circuit of length* $30 + 7k + 16m$.

It is only an easy exercise to prove that Theorem 2 implies the following.

Theorem 3. *For every even integer* $n \geqslant 82$, *there exists a snark of order n having a stable dominating circuit.*

Remark 3. We can also construct snarks with a stable circuit that is not dominating. For instance, let $\mathcal{N}^{(3)}$ be the network from Fig. 8 and $C^{(3)}$ be the circuit in it indicated by bold lines. Denote $T^{(3)} = V(C^{(3)})$. Then using the ideas from Theorem 1 we can

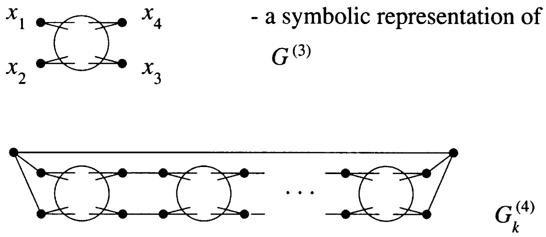


Fig. 9.

show the following:

- there does not exist a polygon in $\mathcal{N}^{(3)}$ containing all vertices from $T^{(3)}$;
- $C^{(3)}$ is the only circuit in $G^{(3)}$ containing all vertices from $T^{(3)}$.

Now add new vertices o_1 and o_2 to $G^{(3)}$ together with edges (o_1, o_2), (x_1, o_1), (x_2, o_1), (x_3, o_2), (x_4, o_2), getting a new graph $G^{(4)}$. $G^{(4)}$ is a snark. This follows either from the methods presented in [14,19], or from the fact that $G^{(4)}$ arises as dot product (see [1,2,11,24,26]) of two copies of Petersen graph and a cubic graph with a 1-edge-cut (that is not 3-edge-colorable — see, e.g., [1,2,24,26]). Thus $G^{(4)}$ is a snark containing a circuit $C^{(3)}$, which is stable but not dominating. Using dot products of $G^{(4)}$ and other snarks we get an infinite class of snarks with this property.

Remark 4. Suppose a family $\mathscr{C} = \{C_1, \dots, C_k\}$ of circuits in a graph satisfies the following condition: if $\mathscr{C}' \subseteq \mathscr{C}$ and there exists a family of circuits $\mathscr{C}''$ covering all vertices from $\bigcup_{C \in \mathscr{C}'} V(C)$ so that $|\mathscr{C}''| \leqslant |\mathscr{C}'|$, then $\mathscr{C}'' = \mathscr{C}'$. In this case we say that $\mathscr{C}$ is *stable*. Take a graph $G_k^{(4)}$ arising from $k \geqslant 1$ copies of $G^{(3)}$ as indicated in Fig. 9 ($G_1^{(4)}$ is identical with $G^{(4)}$). Similar to $G^{(4)}$, $G_k^{(4)}$ is also a snark. Take a family $\mathscr{C}_k^{(4)}$ of k circuits in $G_k^{(4)}$ arising as copies of $C^{(3)}$. From Remark 3 it follows that $\mathscr{C}_k^{(4)}$ is stable. Therefore, we can conclude: for every positive integer k, there exists a snark of order $52k + 2$ with a stable family of k circuits. This, in certain sense, generalizes Theorem 1.

Graph G from Fig. 4 has in fact two stable dominating circuits. The second one can be obtained from C after applying the permutations φ, ψ and φ to G_1, G_4 and G_3, respectively. Similarly, the graphs from Theorem 2 have at least two (some of them more) stable dominating circuits. Analogously the network from Fig. 8 has at least four stable circuits whose vertices cannot be covered by a polygon.

Let us note that using more general results from [14,19] we can check that the snarks from Theorem 2 can have arbitrary large oddness (see [14,19] for more details and definitions). This fact is also of some interest, because snarks with oddness 2 have a CDC (see [10,15]).

All snarks presented here are cyclically 4-edge-connected. But there are known constructions of cyclically 5- and 6-edge-connected snarks (see, e. g., [11,14,16,17,19,26]).

Thus it is natural to set the following problem: Construct cyclically 5- or 6-edge connected snarks with a stable (dominating) circuit.

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