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On a Riemann–Liouville Generalized Taylor's Formula*

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In this paper, a generalized Taylor's formula of the kind

$$f(x) = \sum_{j=0}^{n} a_j (x-a)^{(j+1)\alpha - 1} + T_n(x),$$

where $a_j \in \mathbb{R}$, $x > a$, $0 \leq \alpha \leq 1$, is established. Such expression is precisely the classical Taylor's formula in the particular case $\alpha = 1$. In addition, detailed expressions for $T_n(x)$ and a_j, involving the Riemann–Liouville fractional derivative, and some applications are also given. © 1999 Academic Press

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1. INTRODUCTION

The ordinary Taylor's formula has been generalized by many authors. Riemann [6], had already written a formal version of the generalized Taylor's series:

$$f(x+h)=\sum_{m=-\infty}^{\infty}\frac{h^{m+r}}{\Gamma(m+r+1)}(D_a^{m+r}f)(x), \tag{1.1}$$

where for $\alpha \le 0$, $D_a^{\alpha}f(x)=I_a^{-\alpha}f(x)$ is the Riemann–Liouville fractional integral of order $-\alpha$. This fractional integral operator is defined for $\beta \in \mathbb{R}^+$, $a \in \mathbb{R}$ and $x > a$ as follows:

$$I_a^{\beta}f(x)=\frac{1}{\Gamma(\beta)}\int_a^x (x-t)^{\beta-1}f(t)dt, \tag{1.2}$$

and for $n \in N$ and $(n-1) < \alpha \le n$ is $D_a^{\alpha}f(x)=D^nI_a^{n-\alpha}f(x)$, where D denotes the classical derivative. Moreover, $I_a^0f(x)=f(x)$.

The proof of the validity of such an expansion for certain classes of functions was undertaken by Hardy [4], both for finite and infinite a.

Afterwards, Watanabe [8] obtained the following relation:

$$f(x)=\sum_{k=-m}^{n-1}\frac{(x-x_0)^{\alpha+k}}{\Gamma(\alpha+k+1)}(D_a^{\alpha+k}f)(x_0)+R_{n,m} \tag{1.3}$$

with $m < \alpha$, $x > x_0 \ge a$, and:

$$R_{n,m}=(I_{x_0}^{\alpha+n}D_a^{\alpha+n}f)(x)+\frac{1}{\Gamma(\alpha-m)}\int_a^{x_0}(x-t)^{-\alpha-m-1}(D_a^{\alpha-m-1}f)(t)dt.$$

On the other hand, a variant of the generalized Taylor's series was given by Dzherbashyan and Nersesyan [3]. For f having all of the required continuous derivatives, they obtained:

$$f(x)=\sum_{k=0}^{m-1}\frac{(\mathbf{D}^{(\alpha_k)}f)(0)}{\Gamma(1+\alpha_k)}x^{\alpha_k}+\frac{1}{\Gamma(1+\alpha_m)}\int_0^x (x-t)^{\alpha_m-1}(\mathbf{D}^{(\alpha_m)}f)(t)dt. \tag{1.4}$$

where $x>0$, $\alpha_0, \alpha_1, \ldots, \alpha_m$ is an increasing sequence of real numbers such that $0<\alpha_k-\alpha_{k-1}\le 1$, $k=1,\ldots,m$ and $\mathbf{D}^{(\alpha_k)}f = I_0^{1-(\alpha_k-\alpha_{k-1})}D_0^{1+\alpha_{k-1}}f$.

In this paper, under certain conditions for f and $\alpha \in [0,1]$, the following generalized Taylor's formula:

$$f(x)=\sum_{j=0}^{n}\frac{c_j}{\Gamma((j+1)\alpha)}(x-a)^{(j+1)\alpha-1}+R_n(x,a) \tag{1.5}$$

is obtained, with

$$R_n(x, a) = \frac{D_a^{(n+1)\alpha} f(\xi)}{\Gamma((n+1)\alpha + 1)} (x - a)^{(n+1)\alpha}, \quad a \le \xi \le x$$

$$c_j = \Gamma(\alpha)[(x - a)^{1-\alpha} D_a^{j\alpha} f(x)](a^+), \quad \forall j = 0, \dots, n$$

and the sequential fractional derivative is denoted by

$$D_a^{n\alpha} = D_a^{\alpha} \cdot \overset{n}{\smile} \cdots D_a^{\alpha}, \tag{1.6}$$

where $n \in \mathbb{N}$, according to the definition introduced by Miller and Ross [5]. Also a generalized mean value theorem and some applications of above generalized Taylor's formula are given.

2. DEFINITIONS AND PROPERTIES

Let Ω be an real interval and $\alpha \in [0, 1]$.

DEFINITION 2.1. Let f a Lebesgue measurable function in Ω, $\alpha \in [0, 1)$ and $x_0 \in \Omega$. f is called α-continuous in x_0 if there exists $\lambda \in [0, 1 - \alpha)$ for which $g(x) = |x - x_0|^\lambda f(x)$ is a continuous function in x_0. Moreover, f is called 1-continuous in x_0 if it is continuous in x_0.

As usualy it is said that "f is a α-continuous function on Ω if f is α-continuous for every x in Ω," and it is denoted:

$$\mathbf{C}_\alpha(\Omega) = \{f \in F(\Omega) : f \text{ is } \alpha\text{-continuous in } \Omega\},$$

and so $\mathbf{C}_1(\Omega) = \mathbf{C}(\Omega)$.

DEFINITION 2.2. Let $a \in \Omega$. The function f is called a-singular of order α if

$$\lim_{x \to a} \frac{f(x)}{|x - a|^{\alpha - 1}} = k < \infty \textit{ and } k \ne 0.$$

Let $\alpha \in \mathbb{R}^+$, $a \in \Omega$, E an interval, $E \subset \Omega$, such that $a \le x$, $\forall x \in E$. Then we write

$${}_a\mathbf{I}_\alpha(E) = \{f \in \mathbf{F}(\Omega) : I_a^\alpha f(x) \text{ exists and it is finite } \forall x \in E\}$$

$${}_a\mathbf{D}_\alpha(E) = \{f \in \mathbf{F}(\Omega) : D_a^\alpha f(x) \text{ exists and it is finite } \forall x \in E\},$$

where $\mathbf{F}(\Omega)$ stands for the set of real functions of a single real variable with domain in Ω.

Some properties of the Riemann–Liouville derivative and integral operators will be extensively used. They are collected in the next two propositions.

PROPOSITION 2.1. *Let* $\alpha \in [0, 1]$, $[a, b] \subset \mathbb{R}$. *Then*

(i) *If* $f \in \mathbf{C}((a, b])$ *and* $f \in {}_a\mathbf{I}_\alpha((a, b])$, *then*

$$D_a^\alpha I_a^\alpha f(x) = f(x), \quad \forall x \in (a, b]. \tag{2.1}$$

(ii) *If* $f, D_a^\alpha f \in \mathbf{C}((a, b])$ *and* $D_a^\alpha f \in {}_a\mathbf{I}_\alpha([a, b])$, *then*

$$I_a^\alpha D_a^\alpha f(x) = f(x) + k\frac{(x-a)^{\alpha-1}}{\Gamma(\alpha)}, \quad \forall x \in (a, b], \tag{2.2}$$

where

$$k = -\Gamma(\alpha)[(x-a)^{1-\alpha} f(x)](a^+) = -[I_a^{1-\alpha} f(x)](a^+). \tag{2.3}$$

Proof. (i) It follows from its corresponding ordinary case ($\alpha = 1$).

(ii) First, it is assumed that $I_a^\alpha D_a^\alpha f(x) = f(x) + \Psi(x)$, where $\Psi(x)$ is a suitable function. From this $D_a^\alpha \Psi(x) = 0$ and then $\Psi(x) = k[(x - a)^{\alpha-1}/\Gamma(\alpha)]$. Therefore

$$\begin{aligned} k &= -\Gamma(\alpha)\lim_{x\longrightarrow a^+}[(x-a)^{1-\alpha}\{f(x) - I_a^\alpha D_a^\alpha f(x)\}] \\ &= -\Gamma(\alpha)[(x-a)^{1-\alpha} f(x)](a^+). \end{aligned}$$

PROPOSITION 2.2. *Let* $\alpha \in [0, 1]$, $m \in \mathbb{N}$ *and* f *a function. If one of the following conditions is satisfied,*

(i) $f \in L(a, b)$ *and* $(m+1)\alpha \geq 1$.

(ii) $f \in \mathbf{C}_\gamma([a, b])$ *with* $0 \leq 1 - (m+1)\alpha \leq \gamma \leq 1$.

(iii) $f \in \mathbf{C}_\gamma((a, b])$ *with* $0 \leq 1 - (m+1)\alpha \leq \gamma \leq 1$ *and it is a-singular of order* α,

then the relation:

$$I_a^{(m+1)\alpha} f(x) = I_a^\alpha I_a^{m\alpha} f(x) = I_a^{m\alpha} I_a^\alpha f(x), \ \forall x \in [a, b] \tag{2.4}$$

holds true.

Proof. See Bonilla–Trujillo–Rivero [2] and Samko–Kilbas–Marichev [7].

3. A GENERALIZED MEAN VALUE THEOREM

THEOREM 3.1. *Let* $\alpha \in [0, 1]$ *and* $f \in \mathbf{C}((a, b])$ *such that* $D_a^\alpha f \in \mathbf{C}([a, b])$. *Then*

$$f(x) = [(x-a)^{1-\alpha} f(x)](a^+)(x-a)^{\alpha-1} + D_a^\alpha f(\xi)\frac{(x-a)^\alpha}{\Gamma(\alpha+1)}, \quad \forall x \in (a, b] \tag{3.1}$$

with $a \leq \xi \leq x$.

Proof. Since

$$I_a^\alpha D_a^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} D_a^\alpha f(t)dt,$$

by using the integral mean value theorem, we have

$$I_a^\alpha D_a^\alpha f(x) = D_a^\alpha f(\xi) \frac{(x-a)^\alpha}{\Gamma(\alpha+1)},$$

where $a \le \xi \le x$. Now (3.1) is obtained from (2.2).

Remark 3.1.

(i) If $b = a + h$ and $f \in \mathbf{C}((a, a+h])$ such that $D_a^\alpha f \in \mathbf{C}([a, a+h])$, then

$$f(a+h) = [h^{1-\alpha} f(x)](a^+) h^{\alpha-1} + D_a^\alpha f(\xi) \frac{h^\alpha}{\Gamma(\alpha+1)},$$

with $a \le \xi \le a + h$.

(ii) If $f \in L(a, b)$ and $D_a^\alpha f \in \mathbf{C}([a, b])$ then (3.1) holds almost everywhere in $(a, b]$.

(iii) Another variant of the above theorem can be given as follows:

$$f(x) = [I_a^{1-\alpha} f(x)](a^+)(x-a)^{\alpha-1} + D_a^\alpha f(\xi) \frac{(x-a)^\alpha}{\Gamma(\alpha+1)}, \quad \forall x \in (a, b],$$

with $a \le \xi \le x$, under the same conditions for f as those in Theorem 3.1.

COROLLARY 3.1. *Let $\alpha \in [0, 1]$ and $g \in \mathbf{C}((a, b])$ such that*

$$D_a^\alpha[(x-a)^{\alpha-1} g(x)] \in \mathbf{C}([a, b]).$$

Then

$$g(x) = g(a^+) + \frac{[D_a^\alpha((x-a)^{\alpha-1} g(x))](\xi)}{\Gamma(\alpha+1)} (x-a), \quad \forall x \in (a, b] \tag{3.2}$$

for some ξ, $a \le \xi \le x$.

Proof. The function $f(x) = (x-a)^{\alpha-1} g(x)$ satisfies the conditions of the above theorem. So

$$f(x) = g(a^+)(x-a)^{\alpha-1} + \frac{[D_a^\alpha((x-a)^{\alpha-1} g(x))](\xi)}{\Gamma(\alpha+1)} (x-a)^\alpha$$

and (3.2) is obtained.

COROLLARY 3.2. *Let* $\alpha \in [0,1]$. *Let* g *be a continuous function on* $(a,b]$, *differentiable in the ordinary sense when* $x > a$ *and such that* $D_a^\alpha[(x-a)^{\alpha-1}g(x)] \in \mathbf{C}([a,b])$. *Then*

$$D_a^\alpha((x-a)^{\alpha-1}g(x))(a^+) = \Gamma(\alpha+1)Dg(a^+). \tag{3.3}$$

Proof. Applying the above theorem to the function $f(x) = (x-a)^{\alpha-1}g(x)$, we obtain

$$\begin{aligned} D_a^\alpha f(\xi) &= \frac{\Gamma(\alpha+1)}{(x-a)^\alpha}[f(x) - [(x-a)^{1-\alpha}f(x)](a^+)(x-a)^{\alpha-1}] \\ &= \frac{\Gamma(\alpha+1)}{(x-a)}[g(x) - g(a^+)]. \end{aligned}$$

Now, (3.3) is obtained by taking limits when $x \to a^+$.

COROLLARY 3.3. *Let* $\alpha \in [0,1]$. *Let* g *be a continuous function on* $(a,b]$ *such that* $g(a^+) = g(b)$ *and*

$$D_a^\alpha[(x-a)^{\alpha-1}g(x)] \in \mathbf{C}([a,b]).$$

Then there exists ξ, $a \le \xi \le b$, *such that* $[D_a^\alpha(x-a)^{\alpha-1}g(x)](\xi) = 0$.

Proof. It follows from Corollary 3.1.

4. A GENERALIZATION OF TAYLOR'S FORMULA

PROPOSITION 4.1. *Set* $\alpha \in [0,1]$ *and* $m \in \mathbb{N} - \{0\}$. *Let* f *be a function such that*

(i) $D_a^{m\alpha}f$ *and* $D_a^{(m+1)\alpha}f$ *are continuous in* $(a,b]$,

(ii) $D_a^{(m+1)\alpha}f \in {}_a\mathbf{I}_\alpha([a,b])$,

(iii) *If* $\alpha < 1/2$ *and* $(m+1)\alpha < 1$, *then* $D_a^{(m+1)\alpha}f$ *is* γ*-continuous in* a, *with* $1-(m+1)\alpha \le \gamma \le 1$, *or* $D_a^{(m+1)\alpha}f$ *is a-singular of order* α.

Then

$$I_a^{m\alpha}D_a^{m\alpha}f(x) - I_a^{(m+1)\alpha}D_a^{(m+1)\alpha}f(x) = c_m\frac{(x-a)^{(m+1)\alpha-1}}{\Gamma((m+1)\alpha)}, \quad \forall x \in (a,b]. \tag{4.1}$$

where $c_m = \Gamma(\alpha)[(x-a)^{1-\alpha}D_a^{m\alpha}f(x)](a^+) = I_a^{1-\alpha}D_a^{m\alpha}f(a^+)$.

If $m = 0$, *and* f *is a continuous function such that* $D_a^\alpha f \in \mathbf{C}((a,b])$ *and* $D_a^\alpha f \in {}_a\mathbf{I}_\alpha([a,b])$ *then*, (4.1) *also holds*.

Proof. For $m = 0$ see Samko–Kilbas–Marichev [7]. For $m > 0$, we find from (2.4) that

$$I_a^{m\alpha} D_a^{m\alpha} f - I_a^{(m+1)\alpha} D_a^{(m+1)\alpha} f = I_a^{m\alpha} D_a^{m\alpha} f - I_a^{m\alpha} [I_a^{\alpha} D_a^{\alpha}] D_a^{m\alpha} f.$$

Now (4.1) follows from (1.6), (2.1) and (2.2).

THEOREM 4.1. *Set $\alpha \in [0, 1]$, $n \in \mathbb{N}$. Let f be a continuous function in $(a, b]$ satisfying the following conditions*:

(i) $\forall j = 1, \ldots, n$, $D_a^{j\alpha} f \in \mathbf{C}((a, b])$ *and* $D_a^{j\alpha} f \in {}_a\mathbf{I}_\alpha([a, b])$.

(ii) $D_a^{(n+1)\alpha} f$ *is continuous on* $[a, b]$.

(iii) *If $\alpha < 1/2$ then, for each $j \in N$, $1 \le j \le n$, such that $(j+1)\alpha < 1$, $D_a^{(j+1)\alpha} f(x)$ is γ-continuous in $x = a$ for some γ, $1 - (j+1)\alpha \le \gamma \le 1$, or a-singular of order α.*

Then, $\forall x \in (a, b]$,

$$f(x) = \sum_{j=0}^{n} \frac{c_j (x-a)^{(j+1)\alpha - 1}}{\Gamma((j+1)\alpha)} + R_n(x, a), \tag{4.2}$$

with

$$R_n(x, a) = \frac{D_a^{(n+1)\alpha} f(\xi)}{\Gamma((n+1)\alpha + 1)} (x-a)^{(n+1)\alpha}, \quad a \le \xi \le x \tag{4.3}$$

and for each $j \in N$, $0 \le j \le n$,

$$c_j = \Gamma(\alpha)[(x-a)^{1-\alpha} D_a^{j\alpha} f(x)](a^+) = I_a^{1-\alpha} D_a^{j\alpha} f(a^+). \tag{4.4}$$

Proof. By using (4.1), for $j = 0, \ldots, n$, it follows that

$$f(x) = \sum_{j=0}^{n} \frac{c_j (x-a)^{(j+1)\alpha - 1}}{\Gamma((j+1)\alpha)} + I_a^{(n+1)\alpha} D_a^{(n+1)\alpha} f(x).$$

Applying the integral mean value theorem, we have

$$\begin{aligned} I_a^{(n+1)\alpha} D_a^{(n+1)\alpha} f(x) &= \frac{1}{\Gamma(\alpha(n+1))} \int_a^x (x-t)^{(n+1)\alpha - 1} D_a^{(n+1)\alpha} f(t) dt \\ &= D_a^{(n+1)\alpha} f(\xi) \int_a^x (x-t)^{(n+1)\alpha - 1} dt \\ &= D_a^{(n+1)\alpha} f(\xi) \frac{(x-a)^{(n+1)\alpha}}{\Gamma((n+1)\alpha + 1)} \end{aligned}$$

with $a \le \xi \le x$, and so (4.2) is obtained.

COROLLARY 4.1. *Set $\alpha \in [0,1]$ and $n \in \mathbb{N}$. Let g be a continuous function on $(a, b]$ such that the function*

$$f(x) = (x-a)^{\alpha-1} g(x)$$

satisfies the conditions of the above theorem. Then, $\forall x \in (a, b]$,

$$g(x) = \sum_{j=0}^{n} \frac{c_j (x-a)^{j\alpha}}{\Gamma((j+1)\alpha)} + R'_n(x, a) \tag{4.5}$$

with

$$R'_n(x,a) = \frac{[D_a^{(n+1)\alpha}(x-a)^{\alpha-1}g(x)](\xi)}{\Gamma((n+1)\alpha+1)}(x-a)^{n\alpha+1}, \quad a \le \xi \le x$$

and where

$$\begin{aligned} c_j &= \Gamma(\alpha)[(x-a)^{1-\alpha} D_a^{j\alpha}(x-a)^{\alpha-1} g(x)](a^+) \\ &= [I_a^{1-\alpha} D_a^{j\alpha}(x-a)^{\alpha-1} g(x)](a^+) \end{aligned}$$

for each $j \in N$, $1 \le j \le n$.

Proof. It follows from the above theorem.

Remark 4.1. In a natural way, all functions $f(x)$ satisfying the conditions of Theorem 4.1 could be expanded in a generalized Taylor's series as follows:

$$f(x) = (x-a)^{\alpha-1} \sum_{j=0}^{\infty} \frac{c_j (x-a)^{j\alpha}}{\Gamma((j+1)\alpha)}, \tag{4.6}$$

with c_j given above, holds for all $x \in (a, b]$, where the series converges and

$$\lim_{n\to\infty} R_n(x, a) = 0.$$

The functions which can be expanded as in (4.6) will be called *α-analytic* in $x = a$.

5. APPLICATIONS

1. Let us consider the fractional differential equation:

$$D_0^{\alpha} y(x) = \lambda y(x), \tag{5.1}$$

where $0 < \alpha \le 1$, λ a real number and $x > 0$.

It is assumed that y(x) is 0-singular of order α, and continuous $\forall x > 0$.

Since $D_0^{j\alpha} y(x) = \lambda^j y(x)$, $\forall j \in \mathbb{N}$, and $\forall x > 0$,

$$\lim_{j\to\infty} \frac{x^{(j+1)\alpha}}{\Gamma((j+1)\alpha+1)} = 0.$$

We obtain

$$y(x) = \sum_{j=0}^{\infty} c_j \frac{x^{(j+1)\alpha-1}}{\Gamma((j+1)\alpha)},$$

where $c_0 = \Gamma(\alpha)[x^{1-\alpha} y(x)](0^+)$, $c_j = \lambda^j c_0$.

Then

$$y(x) = c_0 x^{\alpha-1} \sum_{j=0}^{\infty} \frac{\lambda^j x^{j\alpha}}{\Gamma((j+1)\alpha)}$$

which converges $\forall x > 0$.

If it is assumed that

$$\lim_{x\to 0^+} (x^{1-\alpha} y(x)) = \frac{1}{\Gamma(\alpha)},$$

it may define the α-exponential function:

$$e_\alpha^{\lambda x} = x^{\alpha-1} \sum_{j=0}^{\infty} \lambda^j \frac{x^{j\alpha}}{\Gamma((j+1)\alpha)} = x^{\alpha-1} E_{\alpha,\alpha}(\lambda x^\alpha) \tag{5.2}$$

where $E_{\alpha,\beta}(z)$ is the Mittag–Leffler function.

The general solution of (5.1) is then, $y(x) = k e_\alpha^{\lambda x}$.

2. Let us consider now the fractional differential equation

$$D_0^{2\alpha} y(x) = -y(x) \tag{5.3}$$

with $0 < \alpha \leq 1$, $x > 0$ and $D_0^2 \alpha = D_0^\alpha D_0^\alpha$ [see (1.6)].

Assuming that $y(x)$ is 0-singular of order α and continuous $\forall x > 0$, then $\forall n \in \mathbb{N}$,

$$D_0^{4n\alpha} y(x) = y(x),\ D_0^{(4n+1)\alpha} y(x) = D_0^\alpha y(x),\ D_0^{(4n+2)\alpha} y(x) = -y(x),$$

$$D_0^{(4n+3)\alpha} y(x) = -D_0^\alpha y(x)$$

and using the generalized Taylor's formula, we obtain

$$y(x) = c_0 x^{\alpha-1} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n\alpha}}{\Gamma((2n+1)\alpha)} + c_1 x^{\alpha-1} \sum_{n=0}^{\infty} (-1)^n \frac{x^{(2n+1)\alpha}}{\Gamma((2n+2)\alpha)}. \tag{5.4}$$

(A) If it is assumed that

$$[x^{1-\alpha} y(x)](0^+) = 0 \quad \text{and} \quad [x^{1-\alpha} D_0^\alpha y(x)](0^+) = \frac{1}{\Gamma(\alpha)},$$

it may define,

$$\sin_\alpha(x) = x^{\alpha-1}\sum_{n=0}^{\infty}(-1)^n \frac{x^{(2n+1)\alpha}}{\Gamma((2n+2)\alpha)} = x^{2\alpha-1}E_{2\alpha,2\alpha}(-x^{2\alpha}) \tag{5.5}$$

which converges $\forall x > 0$.

(B) If it is assumed that

$$[x^{1-\alpha}y(x)](0^+) = \frac{1}{\Gamma(\alpha)} \quad \text{and} \quad [x^{1-\alpha}D_0^\alpha y(x)](0^+) = 0$$

it may similarly define:

$$\cos_\alpha(x) = x^{\alpha-1}\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n\alpha}}{\Gamma((2n+1)\alpha)} = x^{\alpha-1}E_{2\alpha,\alpha}(-x^{2\alpha}) \tag{5.6}$$

which converges $\forall x > 0$.

The general solution of (5.2) is $y(x) = c_0\sin_\alpha(x) + c_1\cos_\alpha(x)$.

The above functions satisfy the following relations:

$$\sin_\alpha(x) = \frac{e_\alpha^{ix} - e_\alpha^{-ix}}{2i}, \quad \cos_\alpha(x) = \frac{e_\alpha^{ix} + e_\alpha^{-ix}}{2}$$

and

$$e_\alpha^{ix} = \cos_\alpha(x) + i\sin_\alpha(x), \quad \sin_\alpha^2(x) + \cos_\alpha^2(x) = e_\alpha^{ix}e_\alpha^{-ix}$$

just as in the ordinary case, where i is the complex imaginary unit ($i^2 = -1$), using the notation of Euler, and $e_\alpha^{\lambda x}$ is the natural extension of (5.2) to complex values of λ.

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