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Legendre polynomials, Legendre–Stirling numbers, and the left-definite spectral analysis of the Legendre differential expression

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Abstract

In this paper, we develop the left-definite spectral theory associated with the self-adjoint operator $A(k)$ in $L^2(-1,1)$, generated from the classical second-order Legendre differential equation

$$\ell_{\mathrm{L},k}[y](t) = -((1-t^2)y')' + ky = \lambda y \quad (t \in (-1,1)),$$

that has the Legendre polynomials $\{P_m(t)\}_{m=0}^{\infty}$ as eigenfunctions; here, k is a fixed, nonnegative constant. More specifically, for $k > 0$, we explicitly determine the unique left-definite Hilbert–Sobolev space $W_n(k)$ and its associated inner product $(\cdot,\cdot)_{n,k}$ for each $n \in \mathbb{N}$. Moreover, for each $n \in \mathbb{N}$, we determine the corresponding unique left-definite self-adjoint operator $A_n(k)$ in $W_n(k)$ and characterize its domain in terms of another left-definite space. The key to determining these spaces and inner products is in finding the explicit Lagrangian symmetric form of the integral composite powers of $\ell_{\mathrm{L},k}[\cdot]$. In turn, the key to determining these powers is a remarkable new identity involving a double sequence of numbers which we call *Legendre–Stirling numbers*.
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1. Introduction

In a recent paper [9], Littlejohn and Wellman developed a general abstract left-definite theory for a self-adjoint, bounded below operator A in a Hilbert space $(H,(\cdot,\cdot))$. More specifically, they construct a continuum of unique Hilbert spaces $\{(W_r,(\cdot,\cdot)_r)\}_{r>0}$ and, for each $r>0$, a unique self-adjoint restriction A_r of A in W_r. The Hilbert space W_r is called the rth *left-definite Hilbert space* associated with the pair (H,A) and the operator A_r is called the rth *left-definite operator* associated with (H,A); further details of these constructions, spaces, and operators is given in Section 2. Left-definite theory (the terminology *left-definite* is due to Schäfke and Schneider (who used the German *Links-definit*) [16] in 1965) has its roots in the classic treatise of Weyl [20] on the theory of formally symmetric second-order differential expressions. We remark, however, that even though our motivation for the general left-definite theory developed in [9] arose through our interest in certain self-adjoint differential operators (having orthogonal polynomial eigenfunctions), the theory developed in [9] can be applied to an arbitrary self-adjoint operator (bounded or unbounded) that is bounded below in a Hilbert space.

In this paper, we apply this left-definite theory to the self-adjoint Legendre differential operator $A(k)$, generated by the classical second-order formally symmetric Legendre differential expression

$$\begin{aligned}\ell_{\mathrm{L},k}[y](t) &:= -((1-t^2)y'(t))' + ky(t)\\ &= -(1-t^2)y'' + 2ty'(t) + ky(t) \quad (t\in(-1,1))\end{aligned} \tag{1.1}$$

having the Legendre polynomials as eigenfunctions. Here, k is a fixed, nonnegative constant. The right-definite setting in this case is the Hilbert space $H=L^2(-1,1)$. Historically, it was Titchmarsh (see [18,19]) who first studied in detail the analytical properties of (1.1); in particular, he showed that the Legendre polynomials $\{P_n\}_{n=0}^{\infty}$ are eigenfunctions of a self-adjoint operator in $L^2(-1,1)$ generated by the singular differential expression $\ell_{\mathrm{L},k}[\,\cdot\,]$.

This paper may be seen as a continuation of the results obtained in [3,6]. In [3], the first three left-definite spaces associated with the Legendre expression are obtained as well as new characterizations of the domains $\mathscr{D}(A(k))$ and $\mathscr{D}(A_1(k))$ and a new proof of the Everitt–Marić result [8]. In [6], the authors obtain further new characterizations of $\mathscr{D}(A(k))$, including the one given in [3], using different techniques than those used in [3] or in this paper; we discuss these characterizations in Section 3.

Even though the theory obtained in [9] guarantees the existence of a continuum of left-definite spaces $\{W_r(k)\}_{r>0}$ and left-definite operators $\{A_r(k)\}_{r>0}$, we can only effectively determine these spaces and operators in this Legendre situation for $r\in\mathbb{N}$; see Remark 2.1 in Section 2. The key to obtaining these explicit characterizations of $\{W_r(k)\}_{r\in\mathbb{N}}$ and $\{A_r(k)\}_{r\in\mathbb{N}}$ is in obtaining the explicit Lagrangian symmetric form for each integral power $\ell^r_{\mathrm{L},k}[\,\cdot\,]$ of the Legendre differential expression $\ell_{\mathrm{L},k}[\,\cdot\,]$, given in (1.1). In turn, the key to obtaining these integral powers is a remarkable, and yet somewhat mysterious, combinatorial identity involving a function that can be viewed as a generating function for these integral powers of $\ell_{\mathrm{L},k}[\,\cdot\,]$. In our discussion of the combinatorics of these integral powers of $\ell_{\mathrm{L},k}[\cdot]$, we introduce a double sequence $\{PS_n^{(j)}\}$ of real numbers that we call the *Legendre–Stirling numbers*; these numbers, as we will see, share similar properties with the classical Stirling numbers of the second kind $\{S_n^{(j)}\}$.

The contents of this paper are as follows. In Section 2, we state some of the main left-definite results developed in [9]. In Section 3, we review some of the properties of the Legendre differential equation, the Legendre polynomials, and the right-definite self-adjoint operator $A(k)$, generated by the second-order Legendre expression (1.1), having the Legendre polynomials as eigenfunctions, including some new properties obtained in [3,6]. In Section 3, we determine the Lagrangian symmetric form of each integral composite power of the second-order Legendre expression (see Theorem 4.2) using some new combinatorial identities (see Theorem 4.1). Lastly, in Section 4, we establish the left-definite theory associated with the pair $(L^2(-1,1), A(k))$. Specifically, we determine explicitly

(a) the sequence $\{W_n(k)\}_{n=1}^{\infty}$ of left-definite spaces associated with the pair $(L^2(-1,1), A(k))$,
(b) the sequence of left-definite self-adjoint operators $\{A_n(k)\}_{n=1}^{\infty}$ and the domains $\{\mathscr{D}(A_n(k))\}_{n=1}^{\infty}$ of these operators, associated with $(L^2(-1,1), A(k))$, and
(c) the domains $\mathscr{D}((A(k))^n)$ of each integral composite power $(A(k))^n$ of $A(k)$.

These results culminate in Theorem 5.4.

Throughout this paper, $\mathbb{N}$ will denote the set of positive integers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{R}$ and $\mathbb{C}$ will denote, respectively, the real and complex number fields. The term AC will denote absolute continuity; the notation $\mathrm{AC}_{\mathrm{loc}}(I)$ will denote those functions $f : I \to \mathbb{C}$ that are absolutely continuous on all compact subintervals of an interval $I \subset \mathbb{R}$. The space of all polynomials $p : \mathbb{R} \to C$ will be denoted by $\mathscr{P}$. Further notations are introduced as needed throughout the paper.

2. Left-definite Hilbert spaces and left-definite operators

Let V denote a vector space (over the complex field $\mathbb{C}$) and suppose that $(\cdot,\cdot)$ is an inner product with norm $\|\cdot\|$ generated from $(\cdot,\cdot)$ such that $H = (V, (\cdot,\cdot))$ is a Hilbert space. Suppose V_r (the subscripts will be made clear shortly) is a linear manifold (subspace) of the vector space V and let $(\cdot,\cdot)_r$ and $\|\cdot\|_r$ denote an inner product and associated norm, respectively, over V_r (quite possibly different from $(\cdot,\cdot)$ and $\|\cdot\|$). We denote the resulting inner product space by $W_r = (V_r, (\cdot,\cdot)_r)$.

Throughout this section, we assume that $A : \mathscr{D}(A) \subset H \to H$ is a self-adjoint operator that is bounded below by kI, for some $k > 0$; that is,

$$(Ax, x) \geqslant k(x, x) \quad (x \in \mathscr{D}(A)).$$

It follows that A^r, for each $r > 0$, is a self-adjoint operator that is bounded below in H by $k^r I$.

We now define an rth left-definite space associated with (H, A).

Definition 2.1. Let $r > 0$ and suppose V_r is a linear manifold of the Hilbert space $H = (H, (\cdot,\cdot))$ and $(\cdot,\cdot)_r$ is an inner product on V_r. Let $W_r = (V_r, (\cdot,\cdot)_r)$. We say that W_r is an rth *left-definite space* associated with the pair (H, A) if each of the following conditions hold:

(1) W_r is a Hilbert space,
(2) $\mathscr{D}(A^r)$ is a linear manifold of V_r,
(3) $\mathscr{D}(A^r)$ is dense in W_r,

(4) $(x,x)_r \geqslant k^r(x,x) \quad (x \in V_r)$, and
(5) $(x,y)_r = (A^r x, y) \quad (x \in \mathscr{D}(A^r),\ y \in V_r)$.

It is not clear, from the definition, if such a self-adjoint operator A generates a left-definite space for a given $r > 0$. However, in [9], the authors prove the following theorem; the Hilbert space spectral theorem plays a prominent role in establishing this result.

Theorem 2.1 (Littlejohn and Wellman [9, Theorem 3.1]). *Suppose $A : \mathscr{D}(A) \subset H \to H$ is a self-adjoint operator that is bounded below by kI, for some $k > 0$. Let $r > 0$. Define $W_r = (V_r, (\cdot,\cdot)_r)$ by*

$$V_r = \mathscr{D}(A^{r/2}) \tag{2.1}$$

and

$$(x,y)_r = (A^{r/2}x, A^{r/2}y) \quad (x,y \in V_r).$$

Then W_r is a left-definite space associated with the pair (H,A). Moreover, suppose $W_r' := (V_r', (\cdot,\cdot)_r')$ is another rth left-definite space associated with the pair (H,A). Then $V_r = V_r'$ and $(x,y)_r = (x,y)_r'$ for all $x,y \in V_r = V_r'$; i.e., $W_r = W_r'$. That is to say, $W_r = (V_r,(\cdot,\cdot)_r)$ is the unique *left-definite space associated with (H,A).*

Remark 2.1. Although all five conditions in Definition 2.1 are necessary in the proof of Theorem 2.1, the most important property, in a sense, is the one given in (5). Indeed, this property asserts that the rth left-definite inner product is generated from the rth power of A. If A is generated from a Lagrangian symmetric differential expression $\ell[\cdot]$, we see that the rth powers of A are then determined by the rth powers of $\ell[\cdot]$. Consequently, in this case, it is possible to obtain these powers only when r is a positive integer. We refer the reader to [9] where an example is discussed in which the entire continuum of left-definite spaces is explicitly obtained.

Definition 2.2. For $r > 0$, let $W_r = (V_r, (\cdot,\cdot)_r)$ denote the rth left-definite space associated with (H,A). If there exists a self-adjoint operator $A_r : \mathscr{D}(A_r) \subset W_r \to W_r$ that is a restriction of A,

$$A_r f = Af \quad (f \in \mathscr{D}(A_r) \subset \mathscr{D}(A)),$$

we call such an operator an rth *left-definite operator associated with* (H,A).

Again, it is not immediately clear that such an A_r exists for a given $r > 0$; in fact, however, as the next theorem shows, A_r exists and is unique.

Theorem 2.2 (Littlejohn and Wellman [9, Theorem 3.2]). *Suppose A is a self-adjoint operator in a Hilbert space H that is bounded below by kI, for some $k > 0$. For any $r > 0$, let $W_r = (V_r, (\cdot,\cdot)_r)$ be the rth left-definite space associated with (H,A). Then there exists a unique left-definite operator A_r in W_r associated with (H,A). Moreover,*

$$\mathscr{D}(A_r) = V_{r+2} \subset \mathscr{D}(A).$$

The last theorem that we state in this section shows that the point spectrum, continuous spectrum, and resolvent set of a self-adjoint operator A and each of its associated left-definite operators A_r $(r>0)$ are identical.

Theorem 2.3 (Littlejohn and Wellman [9, Theorem 3.6]). *For each $r>0$, let A_r denote the rth left-definite operator associated with the self-adjoint operator A that is bounded below by kI, where $k>0$. Then*

(a) *the point spectra of A and A_r coincide*; *i.e.*, $\sigma_{\mathrm{p}}(A_r)=\sigma_{\mathrm{p}}(A)$,
(b) *the continuous spectra of A and A_r coincide*; *i.e.*, $\sigma_{\mathrm{c}}(A_r)=\sigma_{\mathrm{c}}(A)$,
(c) *the resolvent sets of A and A_r are equal*; *i.e.*, $\rho(A_r)=\rho(A)$.

We refer the reader to [9] for other theorems, and examples, associated with the general left-definite theory of self-adjoint operators A that are bounded below.

3. Preliminary information on the Legendre polynomials and the Legendre differential equation

When

$$\lambda=r(r+1)+k \quad (r\in\mathbb{N}_0), \tag{3.1}$$

the Legendre equation $\ell_{\mathrm{L},k}[y](t)=(\lambda+k)y(t)$, where $\ell_{\mathrm{L},k}[\cdot]$ is defined in (1.1), has a polynomial solution $P_r(t)$ of degree r, called the rth Legendre polynomial. For later purposes, we call the term $r(r+1)$ in (3.1) the *principal part* of the eigenvalue $\lambda_r=r(r+1)+k$.

The Legendre polynomials $\{P_m\}_{m=0}^{\infty}$ form a complete orthogonal set in the Hilbert space $L^2(-1,1)$ of Lebesgue measurable functions $f:(-1,1)\to\mathbb{C}$ satisfying $\|f\|<\infty$, where $\|\cdot\|$ is the norm generated from the inner product $(\cdot,\cdot)$, defined by

$$(f,g):=\int_{-1}^{1} f(t)\bar{g}(t)\,\mathrm{d}t \quad (f,g\in L^2(-1,1)). \tag{3.2}$$

In fact, with the mth Legendre polynomial defined by

$$P_m(t)=\sqrt{\frac{2m+1}{2}}\sum_{j=0}^{[m/2]}\frac{(-1)^j(2m-2j)!}{2^m j!(m-j)!(m-2j)!}t^{m-2j} \quad (m\in\mathbb{N}_0), \tag{3.3}$$

it is the case that the sequence $\{P_m\}_{m=0}^{\infty}$ is orthonormal in $L^2(-1,1)$; that is,

$$(P_m,P_n)=\delta_{m,n} \quad (m,n\in\mathbb{N}_0), \tag{3.4}$$

where $\delta_{m,n}$ is the Kronecker delta function. The derivatives of the Legendre polynomials satisfy the identity

$$\frac{\mathrm{d}^j(P_m(t))}{\mathrm{d}t^j}=c(m,j)P_{m-j}^{(j,j)}(t) \quad (m,j\in\mathbb{N}_0), \tag{3.5}$$

where

$$c(m,j) := \sqrt{\frac{(m+j)!}{(m-j)!}} \quad (j = 0, 1, \ldots, m) \tag{3.6}$$

and where $P_m^{(j,j)}(t)$ is the Gegenbauer polynomial of degree m, defined by

$$P_m^{(j,j)}(t) = \frac{k_m(j)}{2^m} \sum_{r=0}^{m} \binom{m+j}{m-r} \binom{m+j}{r} (t-1)^r (t+1)^{m-r} \quad (m \in \mathbb{N}_0); \tag{3.7}$$

here

$$k_m(j) = \frac{(2m+2j+1)^{1/2}((m+2j)!)^{1/2}(m!)^{1/2}}{2^{(2j+1)/2}(m+j)!} \quad (m, j \in \mathbb{N}_0).$$

The Gegenbauer polynomials $\{P_m^{(j,j)}\}_{m=0}^{\infty}$ constitute a complete orthonormal set in the Hilbert space

$$L_j^2(-1,1) := \{f : (-1,1) \to \mathbb{C} \,|\, f \text{ is Lebesgue measurable and } \int_{-1}^{1} |f(t)|^2 (1-t^2)^j \, \mathrm{d}t < \infty\} \tag{3.8}$$

with inner product and norm defined, respectively, by

$$(f,g)_j := \int_{-1}^{1} f(t)\bar{g}(t)(1-t^2)^j \, \mathrm{d}t, \quad \|f\|_j := \left(\int_{-1}^{1} |f(t)|^2 (1-t^2)^j \, \mathrm{d}t \right)^{1/2} \tag{3.9}$$

more specifically,

$$(P_m^{(j,j)}, P_n^{(j,j)})_j = \delta_{m,n} \quad (m, n \in \mathbb{N}_0). \tag{3.10}$$

When $j=0$, these polynomials are the Legendre polynomials and we write $P_m^{(0,0)}(t) = P_m(t)$. Moreover, in this case, we write $L^2(-1,1)$ instead of $L_0^2(-1,1)$ and we will use $(\cdot,\cdot)$ and $\|\cdot\|$ instead of, respectively, $(\cdot,\cdot)_0$ and $\|\cdot\|_0$.

From (3.4) and (3.5), we see that

$$\int_{-1}^{1} \frac{\mathrm{d}^j(P_m(t))}{\mathrm{d}t^j} \frac{\mathrm{d}^j(P_n(t))}{\mathrm{d}t^j} (1-t^2)^j \, \mathrm{d}t = \frac{(m+j)!}{(m-j)!} \delta_{m,n} \quad (m, n, j \in \mathbb{N}_0). \tag{3.11}$$

We refer the reader to [14] or [17] for various properties of the Legendre and Gegenbauer polynomials.

We now turn our attention to discuss some operator-theoretic properties of the Legendre expression $\ell_{\mathrm{L},k}[\cdot]$; for further information, we refer the reader to paper [5] of Everitt, where an in-depth discussion of the Legendre expression is made. The thesis [10] of Loveland contains a summarized, yet detailed, account of the functional-analytic properties of $\ell_{\mathrm{L},k}[\cdot]$. Likewise, the thesis [13] of Onyango–Otieno is an impressive record of the Legendre expression as well as the other classical second-order differential equations of Jacobi, Laguerre, and Hermite. For a general discussion of self-adjoint extensions of formally Lagrangian symmetric differential expressions, the texts of Akhiezer and Glazman [2] and Naimark [11] are recommended.

The maximal domain $\Delta(k)$ of $\ell_{L,k}[\cdot]$ in $L^2(-1,1)$ is defined to be

$$\Delta(k) = \{f \in L^2(-1,1) \mid f, f' \in \mathrm{AC}_{\mathrm{loc}}(-1,1);\ \ell_{L,k}[f] \in L^2(-1,1)\}. \tag{3.12}$$

The *maximal operator* $T_{\max}(k)$, generated by $\ell_{L,k}[\cdot]$ in $L^2(-1,1)$ is defined by

$$\mathscr{D}(T_{\max}(k)) = \Delta(k),$$

$$T_{\max}(k)(f) = \ell_{L,k}[f].$$

The *minimal operator* $T_{\min}(k)$ is then defined as $T_{\min}(k) = T^*_{\max}(k)$, the Hilbert space adjoint of $T_{\max}(k)$. This operator $T_{\min}(k)$ is closed, symmetric, and satisfies $T^*_{\min}(k) = T_{\max}(k)$. The deficiency index of $T_{\min}(k)$ is (2,2); consequently, from the Glazman–Krein–Naimark theory of self-adjoint extensions of symmetric operators, $T_{\min}(k)$ has (uncountably many) self-adjoint extensions in $L^2(-1,1)$.

In particular, the operator $A(k) : \mathscr{D}(A(k)) \subset L^2(-1,1) \to L^2(-1,1)$ defined by

$$\mathscr{D}(A(k)) = \{f \in \Delta(k) \mid \lim_{t\to\pm 1^\mp} (1-t^2) f'(t) = 0\},$$

$$A(k)f = \ell_{L,k}[f] \tag{3.13}$$

is self-adjoint in $L^2(-1,1)$. Furthermore, the Legendre polynomials $\{P_r\}_{r=0}^{\infty}$ are a (complete) set of eigenfunctions of $A(k)$ and the spectrum of $A(k)$ is given by

$$\sigma(A(k)) = \{r(r+1) + k \mid r \in \mathbb{N}_0\}.$$

We note that $A(k)$ is the so-called Friedrich's extension; see [12] for further details.

We note that, in [3,6], the authors obtain new characterizations of $\mathscr{D}(A(k))$. The following results are proved in [6].

Theorem 3.1. *Let the domain* $\mathscr{D}(A(k))$ *of the Legendre self-adjoint operator be as given in* (3.13). *Then*

(i) $f \in \mathscr{D}(A(k))$ *if and only if* $f \in \Delta(k)$ *and* $f' \in L^1(-1,1)$,
(ii) $f \in \mathscr{D}(A(k))$ *if and only if* $f \in \Delta(k)$ *and* $f' \in L^2(-1,1)$,
(iii) $f \in \mathscr{D}(A(k))$ *if and only if* $f \in \Delta(k)$ *and* f *is bounded on* $(-1,1)$,
(iv) $f \in \mathscr{D}(A(k))$ *if and only if* $f \in \Delta(k)$ *and* $(1-t^2)^{1/2} f' \in L^2(-1,1)$,
(v) $f \in \mathscr{D}(A(k))$ *if and only if* $f, f' \in \mathrm{AC}_{\mathrm{loc}}(-1,1)$ *and* $(1-t^2) f'' \in L^2(-1,1)$.

In (i) above, $L^1(-1,1)$ is the well-known Banach space of all Lebesgue measurable functions $f : (-1,1) \to \mathbb{C}$ satisfying $\int_{(-1,1)} |f| < \infty$. Property (ii) was first obtained by Everitt and Marić in their unpublished notes [8]; details of this proof can be found in [6] and a partial proof can be found in [3]. Property (v) of this theorem was first proved in [3] and, by different means, in [6]. We note that other characterizations of $\mathscr{D}(A(k))$ can be found in [2, Appendix 1].

For $f,g\in\mathscr{D}(A(k))$, it is known that

$$(A(k)f,g)=\int_{-1}^{1}\ell_{\mathrm{L},k}[f](t)\bar{g}(t)\,\mathrm{d}t=\int_{-1}^{1}\{(1-t^2)f'(t)\bar{g}'(t)+kf(t)\bar{g}(t)\}\,\mathrm{d}t; \tag{3.14}$$

this is known as *Dirichlet's identity*. Observe from (3.14) that

$$(A(k)f,f)\geqslant k(f,f)\quad (f\in\mathscr{D}(A(k))); \tag{3.15}$$

that is to say, $A(k)$ is bounded below in $L^2(-1,1)$ by kI so the left-definite theory discussed in Section 2 can be applied to this self-adjoint operator. Furthermore, notice that the right-hand side of (3.14) satisfies the conditions of an inner product. Consequently, we define the inner product $(\cdot,\cdot)_1$ on $\mathscr{D}(A(k))\times\mathscr{D}(A(k))$ by

$$(f,g)_1:=\int_{-1}^{1}\{(1-t^2)f'(t)\bar{g}'(t)+kf(t)\bar{g}(t)\}\,\mathrm{d}t\quad (f,g\in\mathscr{D}(A(k))), \tag{3.16}$$

later in this paper, we extend this inner product to the set $V_1\times V_1$, where V_1 is a vector space of functions (specifically, the first left-definite space) properly containing $\mathscr{D}(A(k))$. This inner product $(\cdot,\cdot)_1$ is called the *first left-definite inner product* associated with $(H,A(k))$. Notice that the weights in this inner product are precisely the terms in the Lagrangian symmetric differential expression $\ell_{\mathrm{L},k}[\cdot]$; see (1.1) and Remark 2.1.

4. The combinatorics of powers of the Legendre differential expression

We now turn our attention to the explicit construction of the sequence of left-definite inner products $(\cdot,\cdot)_{n,k}$ $(n\in\mathbb{N})$ associated with the pair $(L^2(-1,1),A(k))$, where $A(k)$ is the self-adjoint Legendre differential operator defined in (3.13). As we will see, these inner products are generated from the integral powers $\ell^n_{\mathrm{L},k}[\cdot]$ $(n\in\mathbb{N})$ of the Legendre expression $\ell_{\mathrm{L}}[\cdot]$, inductively given by

$$\ell^1_{\mathrm{L},k}[y]=\ell_{\mathrm{L},k}[y],\ \ell^2_{\mathrm{L},k}[y]=\ell_{\mathrm{L},k}(\ell_{\mathrm{L},k}[y]),\ldots,\ell^n_{\mathrm{L},k}[y]=\ell_{\mathrm{L},k}(\ell^{n-1}_{\mathrm{L},k}[y])\quad (n\in\mathbb{N}).$$

A key to the explicit determination of these powers of $\ell_{\mathrm{L},k}[\cdot]$ are certain positive numbers $\{c_j(n,k\}\}_{j=0}^{n}$, whose properties are discussed in the following theorem.

Theorem 4.1. *Suppose $k\geqslant 0$ and $n\in\mathbb{N}$. For each $m\in\mathbb{N}_0$, the recurrence relations*

$$(m(m+1)+k)^n=\sum_{j=0}^{n}c_j(n,k)\frac{(m+j)!}{(m-j)!} \tag{4.1}$$

have unique, nonnegative solutions $c_j(n,k)$ $(j=0,1,\ldots,n)$, independent of m, given explicitly by

$$c_0(n,k)=\begin{cases}0 & \text{if } k=0,\\ k^n & \text{if } k>0\end{cases} \tag{4.2}$$

and

$$c_j(n,k) := \begin{cases} PS_n^{(j)} & \text{if } k = 0 \\ \displaystyle\sum_{s=0}^{n-j} \binom{n}{s} PS_{n-s}^{(j)} k^s & \text{if } k > 0 \end{cases} \qquad (j \in \{1,\dots,n\}), \tag{4.3}$$

where each $PS_n^{(j)}$ *is positive and given by*

$$PS_n^{(j)} = \sum_{r=1}^{j} (-1)^{r+j} \frac{(2r+1)(r^2+r)^n}{(r+j+1)!(j-r)!}. \tag{4.4}$$

Moreover, $PS_n^{(j)}$ *is the coefficient of* t^{n-j} *in the Taylor series expansion of*

$$f_j(t) = \prod_{r=1}^{j} \frac{1}{1-r(r+1)t} \qquad \left(|t| < \frac{1}{j(j+1)}\right). \tag{4.5}$$

For an explanation of the notation $\{PS_n^{(j)}\}$, see Remark 4.1.

Before proceeding to a proof of Theorem 4.1, we prove the following lemma.

Lemma 4.1. *Let* $n \in \mathbb{N}$, $j \in \{1,\dots,n\}$, *and suppose* $r = n-j+i$ *for some* $i \in \{1,\dots,j\}$; *i.e.*, $j > n-r$. *Then*

$$\sum_{s=0}^{j} (-1)^s \left(\binom{2j}{j-s} - \binom{2j}{j-s-1} \right) (s^2+s)^{n-r} = 0. \tag{4.6}$$

Proof. If $i = j$, i.e., $r = n$, the sum in (4.6) can be written as

$$\frac{1}{(2j)!} \left(\binom{2j}{j} + 2\sum_{s=1}^{j} (-1)^s \binom{2j}{j+s} \right). \tag{4.7}$$

Since $\binom{2j}{j-s} = \binom{2j}{j+s}$, we see that

$$0 = (1-1)^{2j} = \sum_{s=0}^{2j} (-1)^s \binom{2j}{s} = (-1)^j \sum_{s=-j}^{j} (-1)^s \binom{2j}{j-s}$$

$$= (-1)^j \left(\sum_{s=-j}^{-1} (-1)^s \binom{2j}{j-s} + \binom{2j}{j} + \sum_{s=1}^{j} (-1)^s \binom{2j}{j-s} \right)$$

$$=(-1)^j\left(2\sum_{s=1}^{j}(-1)^s\binom{2j}{j-s}+\binom{2j}{j}\right)$$

and this proves (4.7).

Suppose $i\in\{1,\dots,j-1\}$.

$$\sum_{s=0}^{j}(-1)^s\left(\binom{2j}{j-s}-\binom{2j}{j-s-1}\right)(s^2+s)^{n-r}$$
$$=\sum_{s=1}^{j}(-1)^s((s^2-s)^{n-r}+(s^2+s)^{n-r})\binom{2j}{j+s}. \tag{4.8}$$

On the other hand,

$$(-1)^j\sum_{s=0}^{2j}(-1)^s(((s-j)^2-(s-j))^{n-r}+((s-j)^2+(s-j))^{n-r})\binom{2j}{s}$$
$$=\sum_{s=-j}^{j}(-1)^s((s^2-s)^{n-r}+(s^2+s)^{n-r})\binom{2j}{j+s}$$
$$=2\sum_{s=1}^{j}(-1)^s((s^2-s)^{n-r}+(s^2+s)^{n-r})\binom{2j}{j+s}, \tag{4.9}$$

consequently, by comparing (4.8) and (4.9), it suffices to prove

$$0=\sum_{s=0}^{2j}(-1)^s(((s-j)^2-(s-j))^{n-r}+((s-j)^2+(s-j))^{n-r})\binom{2j}{s}$$

or, equivalently,

$$0=\sum_{s=0}^{2j}(-1)^s(s-j)^{n-r}((s-j-1)^{n-r}+(s-j+1)^{n-r})\binom{2j}{s}. \tag{4.10}$$

Now $t^{-j}(1-t)^{2j}=\sum_{s=0}^{2j}(-1)^s\binom{2j}{s}t^{s-j}$ so that

$$t(t^{-j}(1-t)^{2j})'=\sum_{s=0}^{2j}(-1)^s\binom{2j}{s}(s-j)t^{s-j}.$$

In fact, if we introduce the notation $t_i = t$ $(i = 1, 2, \ldots)$, we see that

$$v_{n-r-1}(t) := (t_{n-r-1}(t_{n-r-2}(t_{n-r-3} \ldots (t_2(t_1(t^{-j}(1-t)^{2j})')')' \ldots)')')'$$
$$= \sum_{s=0}^{2j} (-1)^s \binom{2j}{s} (s-j)^{n-r} t^{s-j-1}, \tag{4.11}$$

we note that there are $(n-r)$ derivatives taken in (4.11) to define $v_{n-r-1}(t)$.

Similarly, it is straightforward to show that

$$t^2(t_{n-r-1}(t_{n-r-2}(t_{n-r-3} \ldots (t_3(t_2(t_1 v'_{n-r-1}(t))')')' \ldots)')')'$$
$$= \sum_{s=0}^{2j} (-1)^s \binom{2j}{s} (s-j)^{n-r}(s-j-1)^{n-r} t^{s-j} \tag{4.12}$$

and

$$(t_{n-r-1}(t_{n-r-2}(t_{n-r-3} \ldots (t_2(t_1(t^2 v_{n-r-1}(t))')')' \ldots)')')'$$
$$= \sum_{s=0}^{2j} (-1)^s \binom{2j}{s} (s-j)^{n-r}(s-j+1)^{n-r} t^{s-j}, \tag{4.13}$$

where the left-hand sides in both (4.12) and (4.13) each involve $(n-r)$ derivatives. Since $v_{n-r-1}(t)$ also involves $(n-r)$ derivatives, we see that

$$\sum_{s=0}^{2j} (-1)^s (s-j)^{n-r}((s-j-1)^{n-r} + (s-j+1)^{n-r}) \binom{2j}{s} t^{s-j} \tag{4.14}$$

involves a total of $(2n-2r)$ derivatives. Since this sum (4.14) is generated by $(1-t)^{2j}$ and its derivatives, we see that the evaluation of (4.14) at $t = 1$ is 0 since $2j > 2n - 2r$; i.e., $j > n - r$. This completes the proof of the lemma. □

We are now in a position to prove Theorem 4.1.

Proof of Theorem 4.1. From the definition of $c_j(n,k)$ in (4.1), we see that

$$c_0(n,k) = k^n,$$

$$c_1(n,k) = \frac{(k+2)^n - k^n}{2!},$$

$$c_2(n,k) = \frac{(k+6)^n - 3(k+2)^n + 2k^n}{4!},$$

etc. in general, it is not difficult to see that $c_j(n,k)$ is unique and given by

$$c_j(n,k) = \frac{1}{(2j)!}\sum_{r=0}^{j}(-1)^r\left(\binom{2j}{r}-\binom{2j}{r-1}\right)(k+(j-r)(j-r+1))^n$$

$$= \frac{1}{(2j)!}\sum_{r=0}^{j}(-1)^{r+j}\left(\binom{2j}{j-r}-\binom{2j}{j-r-1}\right)(k+r(r+1))^n$$

$$= \frac{1}{(2j)!}\sum_{r=0}^{j}\sum_{s=0}^{n}(-1)^{r+j}\left(\binom{2j}{j-r}-\binom{2j}{j-r-1}\right)(r^2+r)^{n-s}\binom{n}{s}k^s$$

$$= \frac{1}{(2j)!}\sum_{s=0}^{n}\left(\sum_{r=0}^{j}(-1)^{r+j}\left(\binom{2j}{j-r}-\binom{2j}{j-r-1}\right)(r^2+r)^{n-s}\right)\binom{n}{s}k^s$$

$$= \frac{1}{(2j)!}\sum_{s=0}^{n-j}\left(\sum_{r=0}^{j}(-1)^{r+j}\left(\binom{2j}{j-r}-\binom{2j}{j-r-1}\right)(r^2+r)^{n-s}\right)\binom{n}{s}k^s$$

(by Lemma 4.1)

$$= \sum_{s=0}^{n-j}\left(\sum_{r=1}^{j}\frac{(-1)^{r+j}(2r+1)(r^2+r)^{n-s}}{(j-r)!(j+r+1)!}\right)\binom{n}{s}k^s.$$

This proves (4.2)–(4.4). We now prove (4.5). Indeed, we decompose

$$\prod_{r=1}^{j}\frac{t}{1-r(r+1)t}$$

into partial fractions as

$$\prod_{r=1}^{j}\frac{t}{1-r(r+1)t} = \sum_{r=1}^{j}\frac{A_r}{1-r(r+1)t}. \tag{4.15}$$

By letting $t = 1/r(r+1)$, we find that

$$A_r = \frac{(-1)^{r+j}(2r+1)}{(r+j+1)!(j-r)!}.$$

Substituting the geometric series for each term

$$\frac{1}{1-r(r+1)t} \quad \left(|t| < \frac{1}{r(r+1)}\right)$$

on the right-hand side of (4.15), we now see that

$$\prod_{r=1}^{j} \frac{t}{1-r(r+1)t} = \sum_{r=1}^{j}\sum_{n=0}^{\infty} \frac{(-1)^{r+j}(2r+1)(r^2+r)^n}{(r+j+1)!(j-r)!}\, t^n$$

$$= \sum_{n=0}^{\infty}\left(\sum_{r=1}^{j} \frac{(-1)^{r+j}(2r+1)(r^2+r)^n}{(r+j+1)!(j-r)!}\right) t^n,$$

yielding

$$\prod_{r=1}^{j} \frac{1}{1-r(r+1)t} = \sum_{n=0}^{\infty}\left(\sum_{r=1}^{j} \frac{(-1)^{r+j}(2r+1)(r^2+r)^n}{(r+j+1)!(j-r)!}\right) t^{n-j}$$

$$= \sum_{n=0}^{\infty} PS_n^{(j)} t^{n-j} \quad \text{as required.}$$

Lastly, since the coefficient of each term of the Taylor (geometric) series of $1/(1-r(r+1)t)$ $(r = 1,\dots,j)$ is positive and each $PS_n^{(j)}$ is obtained from the Cauchy product of these positive coefficients, it is clear that each $PS_n^{(j)}$ is also positive. In turn, $c_j(n,k) > 0$ for $j \in \{1,\dots,n\}$ and $c_0(n,k) = k^n \geqslant 0$. This completes the proof of this theorem. □

We are now in a position to prove the following theorem; recall the definition of $\mathscr{P}$, see the end of Section 1, as the space of all polynomials $p : \mathbb{R} \to \mathbb{C}$.

Theorem 4.2. *Let $k \geqslant 0$. For each $n \in \mathbb{N}$, the nth composite power of the classical Legendre differential expression $\ell_{\mathrm{L},k}[\,\cdot\,]$, defined in* (1.1), *is Lagrangian symmetric and given explicitly by*

$$\ell^n_{\mathrm{L},k}[y](t) = \sum_{j=0}^{n} (-1)^j (c_j(n,k)(1-t^2)^j y^{(j)}(t))^{(j)}, \tag{4.16}$$

where $c_j(n,k)$ is defined in (4.2) *and* (4.3). *Moreover, for $p, q \in \mathscr{P}$, the following identity is valid*:

$$\int_{-1}^{1} \ell^n_{\mathrm{L},k}[p](t)\bar{q}(t)\,\mathrm{d}t = \sum_{j=0}^{n} c_j(n,k) \int_{-1}^{1} p^{(j)}(t)\bar{q}^{(j)}(t)(1-t^2)^j\,\mathrm{d}t. \tag{4.17}$$

Proof. We first establish the identity in (4.17). Since the Legendre polynomials $\{P_m\}_{m=0}^{\infty}$ form a basis for $\mathscr{P}$, it suffices to show (4.17) is valid for $p = P_m(t)$ and $q = P_r(t)$, for arbitrary $m, r \in \mathbb{N}_0$. From the identity

$$\ell^n_{\mathrm{L},k}[P_m](t) = (m(m+1)+k)^n P_m(t) \quad (m \in \mathbb{N}_0),$$

it follows from (3.4), with this particular choice of p and q, that the left-hand side of (4.17) reduces to

$$(m(m+1)+k)^n \delta_{m,r}. \tag{4.18}$$

On the other hand, from (3.11), we see that the right-hand side of (4.17) yields

$$\sum_{j=0}^{n} c_j(n,k) \int_{-1}^{1} \frac{\mathrm{d}^j(P_m(t))}{\mathrm{d}t^j} \frac{\mathrm{d}^j(P_r(t))}{\mathrm{d}t^j} (1-t^2)^j \,\mathrm{d}t$$

$$= \sum_{j=0}^{n} c_j(n,k) \frac{(m+j)!}{(m-j)!} \delta_{m,r}. \tag{4.19}$$

Comparing (4.18) with (4.19), we see from (4.1) that the identity in (4.17) is valid.

To prove (4.16), define the differential expression

$$m_{\mathrm{L}}[y](t) := \sum_{j=0}^{n} (-1)^j (c_j(n,k)(1-t^2)^j y^{(j)}(t))^{(j)} \quad (-1<t<1). \tag{4.20}$$

For $p, q \in \mathscr{P}$, integration by parts yields

$$\int_{-1}^{1} m_{\mathrm{L}}[p](t)\bar{q}(t)\,\mathrm{d}t$$

$$= \left[\sum_{j=0}^{n} (-1)^j c_j(n,k) \sum_{r=1}^{j} (-1)^{r+1} (p^{(j)}(t)(1-t^2)^j)^{(j-r)} \bar{q}^{(r-1)}(t) \right]_{-1}^{+1}$$

$$+ \sum_{j=0}^{n} c_j(n,k) \int_{-1}^{1} p^{(j)}(t)\bar{q}^{(j)}(t)(1-t^2)^j \,\mathrm{d}t.$$

Now, for any $p \in \mathscr{P}$ and integer r with $1 \leqslant r \leqslant j$, $(p^{(j)}(t)(1-t^2)^j)^{(j-r)} = p_{j,r}(t)(1-t^2)$ for some $p_{j,r} \in \mathscr{P}$; in particular,

$$\lim_{t\to\pm 1} (p^{(j)}(t)(1-t^2)^j)^{(j-r)} \bar{q}^{(r-1)}(t) = 0 \quad (p,q \in \mathscr{P};\ r,j \in \mathbb{N},\ r \leqslant j).$$

Consequently, we see that

$$\int_{-1}^{1} m_{\mathrm{L}}[p](t)\bar{q}(t)\,\mathrm{d}t = \sum_{j=0}^{n} c_j(n,k) \int_{-1}^{1} p^{(j)}(t)\bar{q}^{(j)}(t)(1-t^2)^j \,\mathrm{d}t \quad (p,q \in \mathscr{P}). \tag{4.21}$$

Hence, from (4.17) and (4.21), we see that for all polynomials p and q, we have

$$(\ell_{\mathrm{L},k}^n[p] - m_{\mathrm{L}}[p], q) = 0,$$

where $(\cdot,\cdot)$ is the usual inner product in $L^2(-1,1)$ defined in (3.2). From the density of $\mathscr{P}$ in $L^2(-1,1)$, it follows that

$$\ell^n_{\mathrm{L},k}[p](t) = m_{\mathrm{L}}[p](t) \quad (t\in(-1,1);\ p\in\mathscr{P}). \tag{4.22}$$

This latter identity implies that the expression $\ell^n_{\mathrm{L},k}[\cdot]$ has the form given in (4.16). □

For example, we see from this theorem that

$$\ell^2_{\mathrm{L},k}[y](t) = ((1-t^2)^2y'')'' - ((2k+2)(1-t^2)y')' + k^2y,$$

$$\ell^3_{\mathrm{L},k}[y](t) = -((1-t^2)^3y''')''' + ((3k+8)(1-t^2)^2y'')''$$
$$-((3k^2+6k+4)(1-t^2)y')' + k^3y$$

and

$$\ell^4_{\mathrm{L},k}[y](t) = ((1-t^2)^4y^{(4)})^{(4)} - ((4k+20)(1-t^2)^3y''')''' + ((6k^2+32k+52)(1-t^2)^2y'')''$$
$$-((4k^3+12k^2+16k+8)(1-t^2)y')' + k^4y.$$

The following corollary lists some additional properties of the Legendre differential expression $\ell_{\mathrm{L},k}[\cdot]$.

Corollary 4.1. *Let $n\in\mathbb{N}$. Then*

(a) *the nth power of the classical Legendre differential expression*

$$\ell_{\mathrm{L},0}[y](t) := -(1-t^2)y''(t) + 2ty'(t) = -((1-t^2)y'(t))'$$

is symmetrizable with symmetry factor $w(t)=1$ and has the Lagrangian symmetric form

$$\ell^n_{\mathrm{L},0}[y](t) := \sum_{j=1}^{n}(-1)^j(PS_n^{(j)}(1-t^2)^jy^{(j)}(t))^{(j)},$$

where $PS_n^{(j)}$ is defined in (4.4),

(b) *the bilinear form $(\cdot,\cdot)_{n,k}$, defined on $\mathscr{P}\times\mathscr{P}$ by*

$$(p,q)_{n,k} := \sum_{j=0}^{n} c_j(n,k)\int_{-1}^{1} p^{(j)}(t)\bar{q}^{(j)}(t)(1-t^2)^j\,\mathrm{d}t \quad (p,q\in\mathscr{P}) \tag{4.23}$$

is an inner product when $k>0$ and satisfies

$$(\ell^n_{\mathrm{L},k}[p],q) = (p,q)_{n,k} \quad (p,q\in\mathscr{P}), \tag{4.24}$$

(c) *the Legendre polynomials* $\{P_m\}_{m=0}^{\infty}$ *are orthogonal with respect to the inner product* $(\cdot,\cdot)_{n,k}$; *in fact*,

$$(P_m, P_r)_{n,k} = \sum_{j=0}^{n} c_j(n,k) \int_{-1}^{1} \frac{\mathrm{d}^j(P_m(t))}{\mathrm{d}t^j} \frac{\mathrm{d}^j(P_r(t))}{\mathrm{d}t^j} (1-t^2)^j \, \mathrm{d}t = (m(m+1)+k)^n \delta_{m,r}. \quad (4.25)$$

Proof. The proof of (a) follows immediately from Theorem 4.2 and $k=0$. The proof of (b) is clear since all the numbers $\{c_j(n,k)\}_{j=0}^{n}$ are positive when $k>0$. The identity in (4.24) follows from (4.21) and (4.22). Lastly, (4.25) follows from (4.1) and (4.19). □

Definition 4.1. For each $n \in \mathbb{N}$ and $j \in \{1,2,\ldots,n\}$, the number $PS_n^{(j)}$, given in Theorem 4.1, is called the Legendre–Stirling number of order (n,j).

Remark 4.1. Since the Legendre polynomials ($\alpha=\beta=0$) are a special case of the more general Jacobi polynomials $\{P_n^{(\alpha,\beta)}\}_{n=0}^{\infty}$, we use the notation $PS_n^{(j)} := P^{(0,0)}S_n^{(j)}$ in anticipation of a more general double sequence $\{P^{(\alpha,\beta)}S_n^{(j)}\}$ of numbers, which we call Jacobi–Stirling numbers, that we believe exist and are connected to the Jacobi differential equation in the same way that the Legendre–Stirling numbers are connected to the Legendre differential equation. More specifically, we feel that there are results analogous to Theorems 4.1 and 4.2 and Corollary 4.1 relating these general Jacobi–Stirling numbers to powers of the classical second-order Jacobi differential equation. This connection will be considered in the near future by these authors.

These Legendre–Stirling numbers $\{PS_n^{(j)}\}$ are the analogues of the classical Stirling numbers of the second kind $\{S_n^{(j)}\}$ (see [1, pp. 824–825], [4, Chapter V]), which appear in the integral composite powers of both the classical Laguerre and Hermite differential expressions. Indeed, in [9], Littlejohn and Wellman show that the integral composite powers of the Laguerre differential expression, defined by

$$\begin{aligned}\ell_{\mathrm{La},k}[y](t) &:= t^{-\alpha}\exp(t)[-(t^{\alpha+1}\exp(-t)y'(t))' + kt^{\alpha}\exp(-t)y(t)] \\ &= -ty''(t) + (t-1-\alpha)y'(t) + ky(t)\end{aligned}$$

are given by

$$\ell_{\mathrm{La},k}^{n}[y](t) = t^{-\alpha}\exp(t)\sum_{j=0}^{n}(-1)^j(a_j(n,k)t^{\alpha+j}\exp(-t)y^{(j)}(t))^{(j)},$$

where

$$a_0(n,k) = \begin{cases} 0 & \text{if } k=0 \\ k^n & \text{if } k>0 \end{cases} \quad \text{and} \quad a_j(n,k) = \begin{cases} S_n^{(j)} & \text{if } k=0 \\ \displaystyle\sum_{s=0}^{n-1}\binom{n}{s} S_{n-s}^{(j)} k^s & \text{if } k>0 \end{cases} \quad (j=1,2,\ldots,n).$$

With regards to these Laguerre powers, it is worth noting two facts. First, the Stirling numbers of the second kind $\{S_n^{(j)}\}$ can be defined (see [1, pp. 824–825]) as the coefficient of t^{n-j} in the Taylor series expansion of

$$g_j(t) := \prod_{r=1}^{j} \frac{1}{1-rt} \quad \left(|t| < \frac{1}{j}\right), \tag{4.26}$$

second, notice that the principal part of the eigenvalue parameter $\lambda_r = r + k$, namely r, that produces the Laguerre polynomial eigenfunction $L_r^{\alpha}(t)$ in the equation

$$\ell_{\mathrm{La},k}[y](t) = \lambda_r y(t)$$

appears in the denominator of $g_j(t)$.

In [7], the authors compute the integral composite powers of the Hermite differential expression $\ell_{\mathrm{H}}[\,\cdot\,]$, given by

$$\begin{aligned}\ell_{\mathrm{H},k}[y](t) &:= \exp(t^2)[-(\exp(-t^2)y'(t))' + k\exp(-t^2)y(t)] \\ &= -y''(t) + 2ty'(t) + ky(t).\end{aligned}$$

They show that, for each $n \in \mathbb{N}$,

$$\ell^n_{\mathrm{H},k}[y](t) = \exp(t^2)\sum_{j=0}^{n} (-1)^j (b_j(n,k)\exp(-t^2)y^{(j)}(t))^{(j)},$$

where

$$b_0(n,k) = \begin{cases} 0 & \text{if } k = 0 \\ k^n & \text{if } k > 0 \end{cases} \quad \text{and} \quad b_j(n,k) = \begin{cases} 2^{n-j}S_n^{(j)} & \text{if } k = 0 \\ \displaystyle\sum_{s=0}^{n-1} \binom{n}{s} 2^{n-j-s}S_{n-s}^{(j)}k^s & \text{if } k > 0 \end{cases} \quad (j = 1,2,\ldots,n).$$

In this case, we note that the number $2^{n-j}S_n^{(j)}$ is the coefficient of t^{n-j} in the Taylor series expansion of

$$h_j(t) := \prod_{r=1}^{j} \frac{1}{1-2rt} \quad \left(|t| < \frac{1}{2j}\right). \tag{4.27}$$

Again, two facts are worth mentioning with regards to this Hermite equation. Indeed, the principal part of the eigenvalue parameter $\lambda_r = 2r + k$ in the Hermite equation $\ell_{\mathrm{H},k}[y](t) = \lambda_r y(t)$ is $2r$ and this term appears in the denominator of $h_j(t)$.

Notice that this phenomenon continues with the Legendre equation. Indeed, as noted in (3.1), the principal part of the eigenvalue parameter for the Legendre differential expression (1.1) is $r(r+1)$ and this term appears in the denominator of $f_j(t)$, the generating function for the Legendre–Stirling numbers $\{PS_n^{(j)}\}_{n=0}^{\infty}$, defined in (4.5). Preliminary research indicates that this phenomenon also holds for the general Jacobi polynomials $\{P_n^{(\alpha,\beta)}\}_{n=0}^{\infty}$.

These are intriguing results between the principal parts of the eigenvalue parameters for the Legendre, Laguerre, and Hermite expressions and the generating functions (4.5), (4.26), and (4.27) for the powers of these differential expressions. There is also some mystery concerning this connection.

Indeed, in the case of the Legendre expression, the Glazman–Krein–Naimark theory implies that there is an uncountable number of self-adjoint operators in $L^2(-1,1)$ generated by the Legendre expression $\ell_{L,k}[\cdot]$, each of which has a discrete spectrum. It would appear that only one of these self-adjoint operators, namely the operator $A(k)$ defined in (3.13), has spectrum $\{r(r+1)+k \mid r\in\mathbb{N}_0\}$; however, this property is not established in this paper. Why does $f_j(t)$ *involve* the eigenvalues $r(r+1)$ of this operator $A(k)$ over the eigenvalues of one of these other self-adjoint operators? The answer could be that this is a new remarkable property of these classical orthogonal polynomials and the second-order differential equations that they satisfy.

5. The left-definite theory for the Legendre equation

For the results that follow in this section, we assume $k>0$, where k is the parameter in the Legendre expression (1.1). It is also convenient to introduce the following notation; for $n\in\mathbb{N}$, let

$$\mathrm{AC}^{(n-1)}_{\mathrm{loc}}(-1,1):=\{f:(-1,1)\to\mathbb{C}\mid f,f',\ldots,f^{(n-1)}\in \mathrm{AC}_{\mathrm{loc}}(-1,1)\}.$$

Notice that if $f\in \mathrm{AC}^{(n-1)}_{\mathrm{loc}}(-1,1)$, then $f^{(n)}(t)$ exists for almost all $t\in(-1,1)$.

Definition 5.1. Let $k>0$. For each $n\in\mathbb{N}$, define (see (3.8) for the definition of $L^2_j(-1,1)$)

$$V_n:=\{f:(-1,1)\to\mathbb{C}\mid f\in \mathrm{AC}^{(n-1)}_{\mathrm{loc}}(-1,1);\ f^{(j)}\in L^2_j(-1,1)\ (j=0,1,\ldots,n)\} \tag{5.1}$$

and let $(\cdot,\cdot)_{n,k}$ and $\|\cdot\|_{n,k}$ denote, respectively, the inner product

$$(f,g)_{n,k}=\sum_{j=0}^{n}c_j(n,k)\int_{-1}^{1} f^{(j)}(t)\bar{g}^{(j)}(t)(1-t^2)^j\,\mathrm{d}t \quad (f,g\in V_n), \tag{5.2}$$

(see (4.23) and (4.24)) and the norm $\|f\|_{n,k}=(f,f)^{1/2}_{n,k}$, where the numbers $c_j(n,k)$ are defined in (4.2) and (4.3). Finally, let $W_n(k):=(V_n,(\cdot,\cdot)_{n,k})$.

The inner product $(\cdot,\cdot)_{n,k}$, defined in (5.2), is a Sobolev inner product and is more commonly called the *Dirichlet inner product* associated with the symmetric differential expression $\ell^n_{L,k}[\cdot]$ given in (4.16).

We remark that, for each $r>0$, the rth left-definite inner product $(\cdot,\cdot)_{r,k}$ is abstractly given by

$$(f,g)_{r,k}=\int_{\mathbb{R}}\lambda^r\,\mathrm{d}E_{f,g}(k) \quad (f,g\in V_r:=\mathscr{D}((A(k))^{r/2})),$$

where $E(k)$ is the spectral resolution of the identity for $A(k)$; see [9]. However, we are able to determine this inner product in terms of the differential expression $\ell^r_{L,k}[\cdot]$ only when $r\in\mathbb{N}$; see also Remark 2.1.

Our aim is to show (see Theorem 5.4) that $W_n(k)$ is the nth left-definite space associated with the pair $(L^2(-1,1),A(k))$, where $A(k)$ is the self-adjoint Legendre operator defined in (3.13). We remind the reader that $A(k)$ is bounded below in $L^2(-1,1)$ by kI; see (3.15). We begin by showing that $W_n(k)$ is a complete inner product space.

Theorem 5.1. *Let $k>0$. For each $n\in\mathbb{N}$, $W_n(k)$ is a Hilbert space.*

Proof. Let $n\in\mathbb{N}$. Suppose $\{f_m\}_{m=1}^{\infty}$ is Cauchy in $W_n(k)$. Since each of the numbers $c_j(n,k)$ is positive, we see that $\{f_m^{(n)}\}_{m=1}^{\infty}$ is Cauchy in $L_n^2(-1,1)$ (for this notation, see (3.8)) and hence there exists $g_{n+1}\in L_n^2(-1,1)$ such that

$$f_m^{(n)}\to g_{n+1} \quad \text{in } L_n^2(-1,1).$$

Fix t, $t_0\in(-1,1)$ (t_0 will be chosen shortly) and assume $t_0\leqslant t$. From Hölder's inequality, we see that as $m\to\infty$,

$$\begin{aligned}\int_{t_0}^{t}|f_m^{(n)}(t)-g_{n+1}(t)|\,\mathrm{d}t &= \int_{t_0}^{t}|f_m^{(n)}(t)-g_{n+1}(t)|(1-t^2)^{n/2}(1-t^2)^{-n/2}\,\mathrm{d}t\\ &\leqslant \left(\int_{t_0}^{t}|f_m^{(n)}(t)-g_{n+1}(t)|^2(1-t^2)^n\right)^{1/2}\left(\int_{t_0}^{t}(1-t^2)^{-n}\,\mathrm{d}t\right)^{1/2}\\ &= M(t_0,t)\left(\int_{t_0}^{t}|f_m^{(n)}(t)-g_{n+1}(t)|^2(1-t^2)^n\,\mathrm{d}t\right)^{1/2}\to 0.\end{aligned}$$

Moreover, since $f_m^{(n-1)}\in \mathrm{AC}_{\mathrm{loc}}(-1,1)$, we see that

$$f_m^{(n-1)}(t)-f_m^{(n-1)}(t_0)=\int_{t_0}^{t}f_m^{(n)}(t)\,\mathrm{d}t\to\int_{t_0}^{t}g_{n+1}(t)\,\mathrm{d}t \tag{5.3}$$

and, in particular, $g_{n+1}\in L^1_{\mathrm{loc}}(-1,1)$. Furthermore, from the definition of $(\cdot,\cdot)_{n,k}$, we see that $\{f_m^{(n-1)}\}_{m=0}^{\infty}$ is Cauchy in $L^2_{n-1}(-1,1)$; hence, there exists $g_n\in L^2_{n-1}(-1,1)$ such that

$$f_m^{(n-1)}\to g_n \quad \text{in } L^2_{n-1}(-1,1).$$

Repeating the above argument, we see that $g_n\in L^1_{\mathrm{loc}}(-1,1)$ and, for any $t,t_1\in(-1,1)$,

$$f_m^{(n-2)}(t)-f_m^{(n-2)}(t_1)=\int_{t_1}^{t}f_m^{(n-1)}(t)\,\mathrm{d}t\to\int_{t_1}^{t}g_n(t)\,\mathrm{d}t. \tag{5.4}$$

Moreover, from [15, Theorem 3.12], there exists a subsequence $\{f_{m_{k,n-1}}^{(n-1)}\}$ of $\{f_m^{(n-1)}\}_{m=1}^{\infty}$ such that

$$f_{m_{k,n-1}}^{(n-1)}(t)\to g_n(t) \quad \text{a.e. } t\in(-1,1).$$

Choose $t_0 \in \mathbb{R}$ in (5.3) such that $f^{(n-1)}_{m_{k,n-1}}(t_0) \to g_n(t_0)$ and then pass through this subsequence in (5.3) to obtain

$$g_n(t) - g_n(t_0) = \int_{t_0}^{t} g_{n+1}(t)\,\mathrm{d}t \quad (\text{a.e. } t \in (-1,1)).$$

That is to say,

$$g_n \in \mathrm{AC}_{\mathrm{loc}}(-1,1) \quad \text{and} \quad g_n'(t) = g_{n+1}(t) \text{ a.e. } t \in (-1,1). \tag{5.5}$$

Again, from the definition of $(\cdot,\cdot)_{n,k}$, we see that $\{f_m^{(n-2)}\}_{m=1}^{\infty}$ is Cauchy in $L^2_{n-2}(-1,1)$; consequently, there exists $g_{n-1} \in L^2_{n-2}(-1,1)$ such that

$$f_m^{(n-2)} \to g_{n-1} \text{ in } L^2_{n-2}(-1,1).$$

As above, we find that $g_{n-1} \in L^1_{\mathrm{loc}}(-1,1)$; moreover, for any $t, t_2 \in (-1,1)$

$$f_m^{(n-3)}(t) - f_m^{(n-3)}(t_2) = \int_{t_2}^{t} f_m^{(n-2)}(t)\,\mathrm{d}t \to \int_{t_2}^{t} g_{n-1}(t)\,\mathrm{d}t,$$

and there exists a subsequence $\{f^{(n-2)}_{m_{k,n-2}}\}$ of $\{f_m^{(n-2)}\}$ such that

$$f^{(n-2)}_{m_{k,n-2}}(t) \to g_{n-1}(t) \quad \text{a.e. } t \in (-1,1).$$

In (5.4), choose $t_1 \in (-1,1)$ such that $f^{(n-2)}_{m_{k,n-2}}(t_1) \to g_{n-1}(t_1)$ and pass through the subsequence $\{f^{(n-2)}_{m_{k,n-2}}\}$ in (5.4) to obtain

$$g_{n-1}(t) - g_{n-1}(t_1) = \int_{t_1}^{t} g_n(t)\,\mathrm{d}t \quad (\text{a.e. } t \in (-1,1)).$$

Consequently, $g_{n-1} \in \mathrm{AC}^{(1)}_{\mathrm{loc}}(-1,1)$ and $g''_{n-1}(t) = g'_n(t) = g_{n+1}(t)$ a.e. $t \in (-1,1)$. Continuing in this fashion, we obtain $n+1$ functions $g_{n-j+1} \in L^2_{n-j}(-1,1)$ $(j = 0,1,\ldots,n)$ such that

(i) $f_m^{(n-j)} \to g_{n-j+1}$ in $L^2_{n-j}(-1,1)$ $(j = 0,1,\ldots,n)$,
(ii) $g_1 \in \mathrm{AC}^{(n-1)}_{\mathrm{loc}}(-1,1)$; $g_2 \in \mathrm{AC}^{(n-2)}_{\mathrm{loc}}(-1,1)$; $\ldots$; $g_n \in \mathrm{AC}_{\mathrm{loc}}(-1,1)$,
(iii) $g'_{n-j}(t) = g_{n-j+1}(t)$ a.e. $t \in (-1,1)$ $(j = 0,1,\ldots,n-1)$,
(iv) $g_1^{(j)} = g_{j+1}$ $(j = 0,1,\ldots,n)$.

In particular, we see that $f_m^{(j)} \to g_1^{(j)}$ in $L^2_j(-1,1)$ for $j = 0,1,\ldots,n$ and $g_1 \in V_n$. Hence, we see that

$$\|f_m - g_1\|^2_{n,k} = \sum_{j=0}^{n} c_j(n,k) \int_{-1}^{1} |f_m^{(j)}(t) - g_1^{(j)}(t)|^2 (1-t^2)^j\,\mathrm{d}t \to 0 \quad \text{as } m \to \infty.$$

Thus $W_n(k)$ is complete and, consequently, so is the proof of this theorem. □

We next establish the completeness of the Legendre polynomials $\{P_m\}_{m=0}^{\infty}$ in each $W_n(k)$.

Theorem 5.2. *Let $k > 0$. The Legendre polynomials $\{P_m\}_{m=0}^{\infty}$ form a complete orthogonal set in the space $W_n(k)$. Equivalently, the space $\mathscr{P}$ of polynomials is dense in $W_n(k)$.*

Proof. Let $f \in W_n(k)$; in particular, $f^{(n)} \in L_n^2(-1,1)$. Consequently, from the completeness and orthonormality of the Gegenbauer polynomials $\{P_m^{(n,n)}\}_{m=0}^{\infty}$ in $L_n^2(-1,1)$, it follows that

$$\sum_{m=0}^{r} c_{m,n} P_m^{(n,n)} \to f^{(n)} \quad \text{as } r \to \infty \text{ in } L_n^2(-1,1),$$

where the numbers $\{c_{m,n}\}_{m=0}^{\infty} \subset \ell^2$ are the Fourier coefficients of $f^{(n)}$ defined by

$$c_{m,n} = \int_{-1}^{1} f^{(n)}(t) P_m^{(n,n)}(t)(1-t^2)^n \, dt \quad (m \in \mathbb{N}_0). \tag{5.6}$$

For $r \geqslant n$, define the polynomials

$$p_r(t) = \sum_{m=n}^{r} \frac{c_{m-n,n}((m-n)!)^{1/2}}{((m+n)!)^{1/2}} P_m(t). \tag{5.7}$$

Then, using the derivative formula (3.5) for the Legendre polynomials, we see that

$$p_r^{(j)}(t) = \sum_{m=n}^{r} \frac{c_{m-n,n}((m-n)!)^{1/2}((m+j)!)^{1/2}}{((m+n)!)^{1/2}((m-j)!)^{1/2}} P_{m-j}^{(j,j)}(t) \quad (j = 0, 1, \ldots, n) \tag{5.8}$$

and, in particular

$$p_r^{(n)} = \sum_{m=n}^{r} c_{m-n,n} P_{m-n}^{(n,n)} \to f^{(n)} \quad \text{in } L_n^2(-1,1) \quad (r \to \infty).$$

Furthermore, from [15, Theorem 3.12], there exists a subsequence $\{p_{r_j}^{(n)}\}$ of $\{p_r^{(n)}\}$ such that

$$p_{r_j}^{(n)}(t) \to f^{(n)}(t) \quad \text{a.e. } t \in (-1,1). \tag{5.9}$$

Returning to (5.8), observe that since

$$\frac{((m-n)!)^{1/2}((m+j)!)^{1/2}}{((m+n)!)^{1/2}((m-j)!)^{1/2}} \to 0 \quad \text{as } m \to \infty \quad \text{for } j = 0, 1, \ldots, n-1,$$

we see that

$$\left\{ \frac{c_{m-n,n}((m-n)!)^{1/2}((m+j)!)^{1/2}}{((m+n)!)^{1/2}((m-j)!)^{1/2}} \right\}_{m=n}^{\infty} \in \ell^2.$$

Hence, from the completeness of the Gegenbauer polynomials $\{P_m^{(j,j)}(t)\}_{m=0}^{\infty}$ in $L_j^2(-1,1)$ and the Riesz–Fischer theorem (see [15, Chapter 4, Theorem 4.17]), there exists $g_j \in L_j^2(-1,1)$ such that

$$p_r^{(j)} \to g_j \text{ in } L_j^2(-1,1) \quad \text{as } r \to \infty \ (j = 0, 1, \ldots, n-1). \tag{5.10}$$

Since, for a.e. $a, t \in (-1,1)$,

$$p_{r_j}^{(n-1)}(t) - p_{r_j}^{(n-1)}(a) = \int_a^t p_{r_j}^{(n)}(u)\,\mathrm{d}u \to \int_a^t f^{(n)}(u)\,\mathrm{d}u = f^{(n-1)}(t) - f^{(n-1)}(a) \quad (j \to \infty),$$

we see that, as $j \to \infty$,

$$p_{r_j}^{(n-1)}(t) \to f^{(n-1)}(t) + c_1 \quad (\text{a.e. } t \in (-1,1)), \tag{5.11}$$

where c_1 is some constant. From (5.10), with $j = n-1$, we deduce that

$$g_{n-1}(t) = f^{(n-1)}(t) + c_1 \quad (\text{a.e. } t \in (-1,1)).$$

Next, from (5.11) and one integration, we obtain

$$p_{r_j}^{(n-2)}(t) \to f^{(n-2)}(t) + c_1 t + c_2 \quad (j \to \infty),$$

for some constant c_2 and hence, from (5.10),

$$g_{n-2}(t) = f^{(n-2)}(t) + c_1 t + c_2 \quad (\text{a.e. } t \in (-1,1)).$$

We continue this process to see that, for $j = 0, 1, \ldots, n-1$,

$$g_j(t) = f^{(j)}(t) + q_{n-j-1}(t) \quad (\text{a.e. } t \in (-1,1)),$$

where q_{n-j-1} is a polynomial of degree $\leqslant n-j-1$ satisfying

$$q'_{n-j-1}(t) = q_{n-j-2}(t).$$

Combined with (5.10), we see that, as $r \to \infty$,

$$p_r^{(j)} \to f^{(j)} + q_{n-j-1} \quad \text{in } L_j^2(-1,1) \quad (j = 0, 1, \ldots, n).$$

For each $r \geqslant n$, define the polynomial

$$\pi_r(t) := p_r(t) - q_{n-1}(t),$$

and observe that, for $j = 0, 1, \ldots, n$,

$$\begin{aligned} \pi_r^{(j)} &= p_r^{(j)} - q_{n-1}^{(j)} \\ &= p_r^{(j)} - q_{n-j-1} \\ &\to f^{(j)} \quad \text{in } L_j^2(-1,1). \end{aligned}$$

Hence, as $r \to \infty$,

$$\|f - \pi_r\|_{n,k}^2 = \sum_{j=0}^{n} c_j(n,k) \int_{-1}^{1} |f^{(j)}(t) - \pi_r^{(j)}(t)|^2 (1-t^2)^j \,\mathrm{d}t \to 0.$$

This shows that $\mathscr{P}$ is dense in $W_n(k)$ and completes the proof of this theorem. $\square$

The next result, which gives a simpler characterization of the function space V_n, follows from ideas in the above proof of Theorem 5.2. Due to the importance of this theorem (which can be seen in the statement of Corollary 5.1), we provide the following proof.

Theorem 5.3. *For each* $n \in \mathbb{N}$,

$$V_n = \{f : (-1,1) \to \mathbb{C} \mid f \in \mathrm{AC}_{\mathrm{loc}}^{(n-1)}(-1,1);\, f^{(n)} \in L_n^2(-1,1)\}. \tag{5.12}$$

Proof. Let $n \in \mathbb{N}$ and recall the definition of V_n in (5.1). Define

$$V_n' = \{f : (-1,1) \to \mathbb{C} \mid f \in \mathrm{AC}_{\mathrm{loc}}^{(n-1)}(-1,1);\, f^{(n)} \in L_n^2(-1,1)\}.$$

It is clear that $V_n \subset V_n'$. Conversely, suppose $f \in V_n'$ so $f^{(n)} \in L_n^2(-1,1)$ and $f \in \mathrm{AC}_{\mathrm{loc}}^{(n-1)}(-1,1)$. As shown in Theorem 5.2, as $r \to \infty$,

$$\sum_{m=0}^{r} c_{m,n} P_m^{(n,n)} \to f^{(n)} \quad \text{in } L_n^2(-1,1),$$

where $c_{m,n}$ is the Fourier coefficient defined in (5.6).

For $r \geqslant n$, let $p_r(t)$ be the polynomial that is defined in (5.7). Then, for any $j \in \mathbb{N}_0$, the jth derivative of p_r is given in (5.8) and, as in Theorem 5.2,

$$p_r^{(n)} \to f^{(n)} \quad \text{as } r \to \infty \text{ in } L_n^2(-1,1),$$

moreover, for $j = 0, 1, \ldots, n-1$, there exists polynomials q_{n-j-1} of degree $\leqslant n-j-1$ satisfying $q_{n-j-1}'(t) = q_{n-j-2}(t)$ with

$$\begin{aligned} p_r^{(j)} &\to f^{(j)} + q_{n-j-1} \quad \text{as } r \to \infty \text{ in } L_j^2(-1,1) \\ &= f^{(j)} + q_{n-1}^{(j)}. \end{aligned}$$

Consequently, for each $j = 0, 1, \ldots, n-1$, $\{p_r^{(j)} - q_{n-1}^{(j)}\}_{r=n}^{\infty}$ converges in $L_j^2(-1,1)$ to $f^{(j)}$. From the completeness of $L_j^2(-1,1)$, we conclude that $f^{(j)} \in L_j^2(-1,1)$ for $j = 0, 1, \ldots, n-1$. That is to say, $f \in V_n$. This completes the proof. □

We are now in a position to prove the main result of this section.

Theorem 5.4. *For* $k > 0$, *let* $A(k) : \mathscr{D}(A(k)) \subset L^2(-1,1) \to L^2(-1,1)$ *be the Legendre self-adjoint operator, defined in* (3.13), *having the Legendre polynomials* $\{P_m\}_{m=0}^{\infty}$ *as eigenfunctions. For each* $n \in \mathbb{N}$, *let* V_n *be given as in* (5.1) *or* (5.12) *and let* $(\cdot,\cdot)_{n,k}$ *denote the inner product defined in* (5.2). *Then* $W_n(k) = (V_n, (\cdot,\cdot)_{n,k})$ *is the nth left-definite space for the pair* $(L^2(-1,1), A(k))$. *Moreover, the Legendre polynomials* $\{P_m\}_{m=0}^{\infty}$ *form a complete orthogonal set in* $W_n(k)$ *satisfying the orthogonality relation* (4.25). *Furthermore, define*

$$A_n(k) : \mathscr{D}(A_n(k)) \subset W_n(k) \to W_n(k)$$

by

$$A_n(k)f = \ell_{\mathrm{L},k}[f] \quad (f \in \mathscr{D}(A_n(k)) := V_{n+2}),$$

where $\ell_{\mathrm{L},k}[\,\cdot\,]$ *is the Legendre differential expression defined in* (1.1). *Then* $A_n(k)$ *is the nth left-definite operator associated with the pair* $(L^2(-1,1), A(k))$. *Furthermore, the Legendre polynomials* $\{P_m\}_{m=0}^{\infty}$ *are eigenfunctions of* $A_n(k)$ *and the spectrum of* $A_n(k)$ *is given by*

$$\sigma(A_n(k)) = \{m(m+1)+k \mid m \in \mathbb{N}_0\} = \sigma(A(k)).$$

Proof. To show that $W_n(k)$ is the nth left-definite space for the pair $(L^2(-1,1), A(k))$, we must show that the five conditions in Definition 2.1 are satisfied.

(i) $W_n(k)$ is complete: The proof of this is given in Theorems 5.1 and 5.3.

(ii) $\mathscr{D}((A(k))^n) \subset W_n(k) \subset L^2(-1,1)$: Let $f \in \mathscr{D}((A(k))^n)$. Since the Legendre polynomials $\{P_m\}_{m=0}^{\infty}$ form a complete orthonormal set in $L^2(-1,1)$, we see that

$$p_j \to f \quad \text{in } L^2(-1,1) \quad (j \to \infty), \tag{5.13}$$

where

$$p_j(t) := \sum_{m=0}^{j} c_m P_m(t)$$

and $\{c_m\}_{m=0}^{\infty}$ are the Fourier coefficients of f in $L^2(-1,1)$ defined by

$$c_m = (f, P_m) = \int_{-1}^{1} f(t) P_m(t)\,\mathrm{d}t \quad (m \in \mathbb{N}_0).$$

Since $(A(k))^n f \in L^2(-1,1)$, we see that

$$\sum_{m=0}^{j} \alpha_m P_m \to (A(k))^n f \quad \text{in } L^2(-1,1) \quad (j \to \infty),$$

where

$$\alpha_m = ((A(k))^n f, P_m) = (f, (A(k))^n P_m) = (m(m+1)+k)^n (f, P_m) = (m(m+1)+k)^n c_m,$$

that is to say,

$$(A(k))^n p_j \to (A(k))^n f \quad \text{in } L^2(-1,1) \quad (j \to \infty).$$

Moreover, from (4.24), we see that

$$\| p_j - p_r \|_{n,k}^2 = ((A(k))^n [p_j - p_r], p_j - p_r) \to 0 \quad \text{as } j, r \to \infty,$$

that is to say, $\{p_j\}_{j=0}^{\infty}$ is Cauchy in $W_n(k)$. From Theorem 5.1, we see that there exists $g \in W_n(k) \subset L^2(-1,1)$ such that

$$p_j \to g \quad \text{in } W_n(k) \quad (j \to \infty).$$

Furthermore, by definition of $(\cdot,\cdot)_{n,k}$ and the fact that $c_0(n,k) = k^n$ for $k > 0$, we see that

$$(p_j - g, p_j - g)_{n,k} \geqslant k^n (p_j - g, p_j - g),$$

hence

$$p_j \to g \quad \text{in } L^2(-1,1). \tag{5.14}$$

Comparing (5.13) and (5.14), we see that $f = g \in W_n(k)$; this completes the proof of (ii).

(iii) *$\mathscr{D}((A(k))^n)$ is dense in $W_n(k)$*: Since polynomials are contained in $\mathscr{D}((A(k))^n)$ and are dense in $W_n(k)$ (see Theorem 5.2), it is clear that (iii) is valid. Furthermore, from Theorem 5.2, we see that the Legendre polynomials $\{P_m\}_{m=0}^{\infty}$ form a complete orthogonal set in $W_n(k)$; see also (4.25).

(iv) *$(f,f)_{n,k} \geqslant k^n(f,f)$ for all $f \in V_n$*: This is clear from the definition of $(\cdot,\cdot)_{n,k}$, the positivity of the coefficients $c_j(n,k)$, and the fact that $c_0(n,k) = k^n$.

(v) *$(f,g)_{n,k} = ((A(k))^n f, g)$ for $f \in \mathscr{D}((A(k))^n)$ and $g \in V_n$*: Observe that this identity is true for any f, $g \in \mathscr{P}$; indeed, this is seen in (4.24). Let $f \in \mathscr{D}((A(k))^n) \subset W_n(k)$ and $g \in W_n(k)$; since polynomials are dense in both $W_n(k)$ and $L^2(-1,1)$ and convergence in $W_n(k)$ implies convergence in $L^2(-1,1)$, there exists sequences of polynomials $\{p_j\}_{j=0}^{\infty}$ and $\{q_j\}_{j=0}^{\infty}$ such that, as $j \to \infty$,

$$p_j \to f \quad \text{in } W_n(k),\ (A(k))^n p_j \to (A(k))^n f \quad \text{in } L^2(-1,1)$$

(see the proof of part (ii) of this theorem), and

$$q_j \to g \quad \text{in } W_n(k) \quad \text{and} \quad L^2(-1,1).$$

Hence, from (4.12),

$$((A(k))^n[f], g) = \lim_{j\to\infty} ((A(k))^n[p_j], q_j) = \lim_{j\to\infty} (p_j, q_j)_{n,k} = (f,g)_{n,k}.$$

This proves (v). The rest of the proof follows immediately from Theorems 2.1–2.3. □

The next corollary follows immediately from Theorems 2.1, 5.3 and 5.4. Remarkably, it characterizes the domain of each of the integral composite powers of $A(k)$. The characterization given below of the domain $\mathscr{D}(A(k))$ of the classical Legendre differential operator $A(k)$ was first obtained in [3] and later, using different means, in [6].

Corollary 5.1. *Let $k > 0$. For each $n \in \mathbb{N}$, the domain $\mathscr{D}((A(k))^n)$ of the nth composite power $(A(k))^n$ of the self-adjoint Legendre operator $A(k)$, defined in* (3.13), *is given by*

$$\mathscr{D}((A(k))^n) = V_{2n} = \{f : (-1,1) \to \mathbb{C} \mid f \in \mathrm{AC}_{\mathrm{loc}}^{(2n-1)}(-1,1);\ f^{(2n)} \in L_{2n}^2(-1,1)\}.$$

In particular,

$$\mathscr{D}(A(k)) = V_2 = \{f : (-\infty,\infty) \to \mathbb{C} \mid f \in \mathrm{AC}_{\mathrm{loc}}^{(1)}(-1,1);\ f'' \in L_2^2(-1,1)\}.$$

From Theorems 2.2 and 5.4, it follows that the domain of the first left-definite operator $A_1(k)$ is given by

$$\mathscr{D}(A_1(k)) = V_3 = \{f : (-1,1) \to \mathbb{C} \mid f \in \mathrm{AC}^{(2)}_{\mathrm{loc}}(-1,1); (1-t^2)^{3/2} f''' \in L^2(-1,1)\}.$$

This result is also proved in [3] and [6].

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